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CHARACTERIZATION OF A PARASASAKIAN MANIFOLD ADMITTING BACH TENSOR

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Abstract. In the present article, our aim is to characterize Bach flat paraSasakian manifolds. It is established that a Bach flat paraSasakian manifold of dimension greater than three is of constant scalar curvature. Next, we prove that if the metric of a Bach flat paraSasakian manifold is a Yamabe soliton, then the soliton field becomes a Killing vector field. Finally, it is shown that a 3-dimensional Bach flat paraSasakian manifold is locally isometric to the hyperbolic space $H^{2n+1}(1)$.

1. Introduction

Adati and Matsumoto [1] introduced the concept of paraSasakian (briefly, P-Sasakian) manifolds, which are considered as a specific case of an almost paracontact manifold initiated by Sato [15]. Matsumoto and Mihai studied P-Sasakian manifolds that admit W_2 or E-Tensor fields and also some curvature conditions [17]. In ([18], [19]) the authors investigated P-Sasakian manifolds obeying certain curvature conditions. In another way, on a pseudo-Riemannian manifold M^{2n+1} Kaneyuki and Kozai [21] introduced the almost paracontact structure and set up the almost paracomplex structure on $M^{2n+1}\times\mathbb{R}$. The main difference between the almost paracontact metric manifold in the sense of Sato [15] and Kaneyuki et al [20] is the signature of the metric. In [27], Zamkovoy introduced paraSasakian manifolds as a normal paracontact manifold whose metric is pseudo-Riemannian and acquired

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a necessary and sufficient condition for which a paracontact metric manifold is a paraSasakian manifold. ParaSasakian manifolds have been investigated by many geometers such as De and De [5], Erken, Dacko and Murathan ([9], [10], [11]), Ghosh et al. [8], Zamkovoy [27] and many others.
On the other hand in [13], Hamilton introduced the idea of Yamabe soliton. In a complete Riemannian manifold (M^{2n+1}, g), the metric g is named a Yamabe soliton if it obeys

$$\pounds_Y g = (\lambda - r)g, \tag{1}$$

where Y is a smooth vector field and λ, $\pounds$ and r indicate a real number, the Lie-derivative operator and the scalar curvature, respectively. For further information about Yamabe solitons see ([4], [6], [16], [26]).

To initiate the investigation of the conformal relativity with regards to conformally Einstein spaces, Bach introduced a new tensor named Bach tensor [2]. We know that the Bach tensor is a trace-free tensor of rank 2 and is also conformally invariant in 4 dimensions [2]. Bach tensor was the single known conformally invariant tensor before 1968 which was algebraically independent of the Weyl tensor [25]. Therefore, as an alternative of the Hilbert-Einstein functional, one chooses the functional

$$\mathcal{W}(g) = \int_M \| W \|_g^2 \, d\mu_g, \tag{2}$$

for 4-dimensional manifolds, where W indicates the Weyl tensor defined by

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3}$$

where R and S indicate the Riemannian curvature tensor and the Ricci tensor, respectively and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

Critical points of the functional (2) are characterized by the vanishing of certain symmetric 2-tensor B, which is generally named as Bach tensor. Also, if $B = 0$, then the metric is called Bach flat. In a Riemannian manifold (M^{2n+1}, g), the Bach tensor B is defined by

$$\begin{aligned} B(X,Y) &= \frac{1}{2n-2}\sum_{k,j=1}^{2n+1}((\nabla_{e_k}\nabla_{e_j}W)(X,e_k)e_j, Y) \\ &\quad + \frac{1}{2n-1}\sum_{k,j=1}^{2n+1} S(e_k,e_j)W(X,e_k,e_j,Y), \end{aligned} \tag{4}$$

where $\{e_k\}_{k=1}^{2n+1}$ is a local orthonormal basis on M. Using the expression of Cotton tensor

$$\begin{aligned} C(X,Y)Z &= (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &\quad -\frac{1}{4n}[(Xr)g(Y,Z) - (Yr)g(X,Z)], \end{aligned} \tag{5}$$

and the Weyl tensor (3), the Bach tensor can be written as

$$B(X,Y) = \frac{1}{2n-1}\sum_{k=1}^{2n+1}[(\nabla_{e_k} C)(e_k, X)Y) + S(e_k, e_k)W(X, e_k, e_k, Y)]. \tag{6}$$

In the event that the manifold M is conformally related locally with an Einstein space, B needs to vanish. However, there exist Riemannian manifolds equipped with $B = 0$, that are not conformally related with Einstein spaces [14]. From the equation (6), it is not difficult to notice that Bach flatness is the inherent generalization of conformal and Einstein flatness. For additional insights concerning Bach tensor, we reffer to see ([3], [12], [23], [24], [25]).

In 2017, Ghosh and Sharma [23] initiated the study of purely transversal Bach tensor in Sasakian manifold. Specifically, they established that assuming a Sasakian manifold M^{2n+1} admitting a purely transversal Bach tensor, g has a constant scalar curvature $\geq 2n(2n-1)$ and S has a constant norm. It is also noticed that the previously stated equality holds if and only if the metric is Einstein. Likewise, they studied (k,μ)-contact manifolds with $B = 0$ and divergence-free Cotton tensor in [24]. The investigations of Ghosh and Sharma ([23], [24]) revolve our concentration to investigate Bach tensor in the context of certain classes of paracontact metric manifolds, in particular paraSasakian manifolds.

In this paper, we consider the Bach flat (2n+1)-dimensional paraSasakian manifolds and we establish the subsequent results.

Theorem 1. Let $M^{2n+1}(n>1)$ be a paraSasakian manifold. If the manifold admits a purely transversal Bach tensor, then the scalar curvature is constant.

Corollary 1. If the metric of a Bach flat paraSasakian manifold is a Yamabe soliton, then the soliton field becomes a Killing vector field.

Theorem 2. If a 3-dimensional paraSasakian manifold M admits a purely transversal Bach tensor, then M is locally isometric to the hyperbolic space $H^{2n+1}(1)$.

2. ParaSasakian Manifolds

Let M^{2n+1} be a differentiable manifold. If there exits a triplet (φ,ξ,η) , where $\boldsymbol{\varphi},\xi,\eta$ indicate a tensor field, a vector field and a 1-form, respectively on M^{2n+1} which obey the relation [15]

$$\varphi^2 = I - \eta\otimes\xi, \quad \eta(\xi) = 1, \ \varphi\xi = 0, \ \eta\circ\varphi = 0, \tag{7}$$

then we name the structure (φ,ξ,η) is an almost paracontact structure. Hence, M is an almost paracontact manifold.

Additionally, if M with the structure (φ,ξ,η) admits a pseudo-Riemannian or semi-Riemannian metric g which obeys the equation [21]

$$g(X,Y) = -g(\varphi X,\varphi Y) + \eta(X)\eta(Y), \tag{8}$$

then M has an almost paracontact metric structure (φ,ξ,η,g). Here, g is named a compatible metric having signature $(n+1,n)$.

In M, the fundamental 2-form is written by

$$\Phi(X,Y) = g(X,\varphi Y).$$

An almost paracontact metric structure reduces to a paracontact metric structure if

$$d\eta(X,Y) = g(X,\varphi Y)$$

for any vector fields X,Y, where

$$d\eta(X,Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X,Y])].$$

An almost paracontact structure is named normal if and only if $N_\varphi - 2d\eta\otimes\xi = 0$, where Nijenhuis tensor of φ is defined by: $N_\varphi(X,Y) = [\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y]$ [27]. A normal paracontact metric manifold is named as paraSasakian manifold. Let ∇ be the Levi-Civita connection with respect to the pseudo-Riemannian metric. Then from [27], it is noticed that an almost paracontact manifold is paraSasakian manifold if and only if

$$(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{9}$$

for any X,Y. From (9), we acquire

$$\nabla_X\xi = -\varphi X. \tag{10}$$

Besides, for M^{2n+1} ParaSasakian manifolds R and S satisfy [27]

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{11}$$

$$R(\xi,X)Y = -g(X,Y) + \eta(Y)X, \tag{12}$$

$$S(X,\xi) = -2n\eta(X), \tag{13}$$

$$Q\xi = -2n\xi. \tag{14}$$

Zamkovoy [27] proved the subsequent proposition :

Proposition 2.1. In a paraSasakian manifold M^{2n+1}, we have

$$S(X,\varphi Y) = -S(\varphi X,Y) - g(X,\varphi Y). \tag{15}$$

3. Bach Flat ParaSasakian Manifolds

Before proving the main theorem we first present the subsequent lemma.

Lemma 1. *Let M^{2n+1} be a paraSasakian manifold. Then*
(i)

$$\sum_{k=1}^{2n+1} g((\nabla_X Q)\varphi e_k, e_k) = 0$$

and
(ii)

$$\sum_{k=1}^{2n+1} g((\nabla_{e_k} Q)Y, \varphi e_k) = (-4n^2 - r)\eta(Y) - \frac{1}{2}(\varphi Y)r$$

.

Proof. From Proposition 2.1. it follows

$$\varphi QX = Q\varphi X - \varphi X. \tag{16}$$

Now

$$g((\nabla_X Q)\varphi Y, Z) + g((\nabla_X Q)Y, \varphi Z) = g((\nabla_X Q\varphi Y - Q\nabla_X \varphi Y), Z) + g((\nabla_X QY - Q\nabla_X Y), \varphi Z). \tag{17}$$

Using the equation (9) and (16) in (17), we acquire
$g((\nabla_X Q)\varphi Y, Z) + g((\nabla_X Q)Y, \varphi Z) = g((\nabla_X \varphi)QY, Z) - g(Q(\nabla_X \varphi)Y, Z) + g(Q(\nabla_X \varphi)Y, Z)$.

Again using (9) and (13) in the above equation, we get

$$g((\nabla_X Q)\varphi Y, Z) + g((\nabla_X Q)Y, \varphi Z) = -g(X, QY)\eta(Z) + \eta(QY)g(X, Z) - (2n+1)g(X, Z)\eta(Y) - g(QX, Z)\eta(Y) + g(X, Z)\eta(Y). \tag{18}$$

Putting $Y = Z = e_k$ in the foregoing equation and summing over k $(1 \leq k \leq 2n+1)$, we obtain

$$\sum_{k=1}^{2n+1} g((\nabla_X Q)\varphi e_k, e_k) + \sum_{k=1}^{2n+1} g((\nabla_X Q)e_k, \varphi e_k) = 0.$$

That is,

$$\sum_{k=1}^{2n+1} g((\nabla_X Q)\varphi e_k, e_k) = 0.$$

This completes the proof of (i).

Again, substituting $X = Z = e_k$ in the equation (18) yields

$$\sum_{k=1}^{2n+1} g((\nabla_{e_k} Q)Y, \varphi e_k) = (-4n^2 - r)\eta(Y) - \frac{1}{2}(\varphi Y)r$$

.

This completes the proof of (ii). □

Proof of Theorem 1. Replacing ξ for Z in (5), we get

$$\begin{aligned} C(X,Y)\xi = g((\nabla_X Q)Y,\xi) - g((\nabla_Y Q)X,\xi) \\ -\frac{1}{4n}[(Xr)g(Y,\xi) - (Yr)g(X,\xi)]. \end{aligned} \tag{19}$$

Now using (10) and (14), we have

$$(\nabla_X Q)\xi = 2n\varphi X + Q\varphi X. \tag{20}$$

From the above equation it follows that

$$g((\nabla_X Q)Y,\xi) = 2ng(\varphi X, Y) + g(Q\varphi X, Y). \tag{21}$$

Using (21) in (19) implies

$$\begin{aligned} C(X,Y)\xi = 2ng(\varphi X,Y) + g(Q\varphi X,\varphi Y) - 2ng(\varphi Y,X) - g(Q\varphi Y,X) \\ +g(QY,\varphi X) + g(Y,\varphi X) - \frac{1}{4n}[(Xr)\eta(Y) - (Yr)\eta(X)]. \end{aligned} \tag{22}$$

Differentiating (22) along Z, provides

$$\begin{aligned} (\nabla_Z C)(X,Y)\xi = \nabla_Z C(X,Y)\xi - C(\nabla_Z X,Y)\xi \\ -C(X,\nabla_Z Y)\xi - C(X,Y)\nabla_Z\xi. \end{aligned} \tag{23}$$

Using (10) and (22) in (23) and after some calculations, we obtain

$$\begin{aligned} (\nabla_Z C)(X,Y)\xi = 2ng((\nabla_Z\varphi)X,Y) - g((\nabla_Z Q)X,\varphi Y) \\ -g(QX,(\nabla_Z\varphi)Y) - g(X,(\nabla_Z\varphi)Y) - 2ng((\nabla_Z\varphi)Y,X) \\ +g((\nabla_Z Q)Y,\varphi X) + g(QY,(\nabla_Z\varphi)X) + g(Y,(\nabla_Z\varphi)X) \\ -\frac{1}{4n}[g(\nabla_Z Dr,X)\eta(Y) - g(\nabla_Z Dr,Y)\eta(X) \\ -g(\varphi Z,Y)(Xr) + g(\varphi Z,X)(Yr)]. \end{aligned} \tag{24}$$

Now we calculate the 2nd term of right hand side of (23), which follows from (22) as

$$\begin{aligned} C(\nabla_Z X,Y)\xi = 2ng(\varphi\nabla_Z X,Y) - g(Q\nabla_Z X,\varphi Y) \\ -g(\nabla_Z X,\varphi Y) - 2ng(\varphi Y,\nabla_Z X) + g(QY,\varphi\nabla_Z X) \\ +g(Y,\varphi\nabla_Z X) - \frac{1}{4n}[((\nabla_Z X)r)\eta(Y) - (Yr)\eta(\nabla_Z X)]. \end{aligned} \tag{25}$$

Similarly from (22), it follows that

$$\begin{aligned} C(X,\nabla_Z Y)\xi = 2ng(\varphi X,\nabla_Z Y) - g(QX,\varphi\nabla_Z Y) \\ -g(X,\varphi\nabla_Z Y) - 2ng(\varphi\nabla_Z Y,X) + g(Q\nabla_Z Y,\varphi X) \\ +g(\nabla_Z Y,\varphi X) - \frac{1}{4n}[(Xr)\eta(\nabla_Z Y) - ((\nabla_Z Y)r)\eta(X)]. \end{aligned} \tag{26}$$

Again from (5), we have

$$C(X,Y)\nabla_Z\xi = (\nabla_X S)(Y,\varphi Z) - (\nabla_Y S)(X,\varphi Z) \tag{27}$$
$$-\frac{1}{4n}[(Xr)g(Y,\varphi Z) - (Yr)g(X,\varphi Z)].$$

Using (24), (25), (26) and (27) in (23) we have,

$$(\nabla_Z C)(X,Y)\xi = 2ng((\nabla_Z\varphi)X,Y) - g((\nabla_Z Q)X,\varphi Y) \tag{28}$$
$$-g(QX,(\nabla_Z\varphi)Y) - g(X,(\nabla_Z\varphi)Y) - 2ng((\nabla_Z\varphi)Y,X)$$
$$+g((\nabla_Z Q)Y,\varphi X) + g(QY,(\nabla_Z\varphi)X) + g(Y,(\nabla_Z\varphi)X)$$
$$-\frac{1}{4n}[g(\nabla_Z Dr,X)\nabla(Y) - g(\nabla_Z Dr,Y)\eta(X) - g(\varphi Z,Y)(Xr)$$
$$+g(\varphi Z,X)(Yr)] - 2ng(\varphi\nabla_Z X,Y) + g(Q\nabla_Z X,\varphi Y)$$
$$+g(\nabla_Z X,\varphi Y) + 2ng(\varphi Y,\nabla_Z X) - g(QY,\varphi\nabla_Z X) - g(Y,\varphi\nabla_Z X)$$
$$+\frac{1}{4n}[((\nabla_Z X)r)\eta(Y) - (Yr)\eta(\nabla_Z X)] - 2ng(\varphi X,\nabla_Z Y)$$
$$+g(QX,\varphi\nabla_Z Y) + g(X,\varphi\nabla_Z Y) + 2ng(\varphi\nabla_Z Y,X)$$
$$-g(Q\nabla_Z Y,\varphi X) - g(\nabla_Z Y,\varphi X) + \frac{1}{4n}[(Xr)\eta(\nabla_Z Y)$$
$$-((\nabla_Z Y)r)\eta(X)] - (\nabla_X S)(Y,\varphi Z) + (\nabla_Y S)(X,\varphi Z)$$
$$+\frac{1}{4n}[(Xr)g(Y,\varphi Z) - (Yr)g(X,\varphi Z)].$$

Putting $X = Z = e_k$ in (28) and summing over k $(1 \le k \le (2n+1))$, we have,

$$\sum_{k=1}^{2n+1}(\nabla_{e_k}C)(e_k,Y)\xi = \sum_{k=1}^{2n+1}[2ng(e_k,Y)\eta(e_k) \tag{29}$$
$$+g((\nabla_{e_k}Q)\varphi e_k,Y) + g(Qe_k,Y)\eta(e_k)$$
$$-\frac{1}{4n}\{g(\nabla_{e_k}Dr,e_k)\eta(Y) - g(\nabla_{e_k}Dr,Y)\eta(e_k)\}.$$

Applying Lemma 3.1 into the foregoing equation yields

$$\sum_{k=1}^{2n+1}(\nabla_{e_k}C)(e_k,Y)\xi = (-4n^2 - r)\eta(Y) - \frac{1}{2}(\varphi Yr) \tag{30}$$
$$-\frac{1}{4n}[(divDr)\eta(Y) - g(\nabla_\xi Dr,Y)].$$

Replacing Z by ξ in (3) we infer

$$W(X,Y)\xi = R(X,Y)\xi - \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y \tag{31}$$
$$+\eta(Y)QX - \eta(X)QY] + \frac{r}{2n(2n-1)}[\eta(Y)X - \eta(X)Y].$$

Using the equation (11) and (13) in (31), we acquire

$$QW(X,Y)\xi = [1 - \frac{2n}{2n-1} + \frac{r}{2n(2n-1)}](\eta(Y)QX - \eta(X)QY) \quad (32)$$
$$-\frac{1}{2n-1}(\eta(Y)Q^2X - \eta(X)Q^2Y).$$

Now taking inner product with U in (32) and then putting $Y = U = e_k$ and summing over $k(1 \leq k \leq 2n+1)$, we obtain

$$\sum_{k=1}^{2n+1} g(QW(X,e_k)\xi, e_k) = -\frac{r^2 - 4n^2}{2n(2n-1)}\eta(X) \quad (33)$$
$$+\frac{1}{2n-1}[\frac{\mid Q \mid^2 - 4n^2}{2n-1}].$$

Now

$$g(Qe_k, e_j)g(W(X,e_k)e_j), Y) \quad (34)$$
$$= -g(W(X,e_k)Y, Qe_k) = -g(QW(X,e_k)Y, e_k).$$

Using (4) and (34) we have

$$B(X,Y) = \frac{1}{2n-1}[\sum_{i=1}^{2n+1}(\nabla_{e_k}C)(e_k,X,Y) - \sum_{i=1}^{2n+1} g(QW(X,e_k)Y,e_k)]. \quad (35)$$

By hypothesis, $B(Y,\xi) = 0$.

Then equation (30) and (33) together reveal

$$(4n - 4n^2 + r)\eta(Y) - \frac{1}{2}(\varphi Y r) - \frac{1}{4n}[(divDr)\eta(Y) - g(\nabla_\xi Dr, Y)] \quad (36)$$
$$+\frac{r^2 - 4n^2}{2n(2n-1)}\eta(Y) - \frac{1}{2n-1}[\frac{\mid Q \mid^2 - 4n^2}{2n-1}]\eta(Y).$$

Replacing Y by φY in the above equation provides

$$\nabla_\xi Dr = 2n\varphi Dr. \quad (37)$$

As ξ is a Killing vector field, we get

$$\pounds_\xi r = 0 \quad (38)$$

Taking exterior derivative d on it we can obtain

$$\pounds_\xi dr = 0,$$

which implies

$$\pounds_\xi Dr = 0. \quad (39)$$

Using (10) in (39), we have

$$\pounds_\xi Dr = -\varphi Dr. \tag{40}$$

Finally, using the equation (37) and (40) yields $\varphi Dr = 0$, that is, $Dr = 0$. Hence, r, the scalar curvature is constant.

This finishes the proof. □

Proof of Corollary 1. Since r =constant, the equation (1) becomes

$$\pounds_Y g = 2cg,$$

where $c = \frac{\lambda - r}{2} = constant.$

Therefore, Y, the soliton vector field becomes a homothetic vector field [7]. For a homothetic vector field Y, we get

$$\pounds_Y r = -2cr. \tag{41}$$

Since r =constant, it follows from the above equation $c = 0$. Thus the soliton fields turn into a Killing vector field. □

Remarks: Recently Erken [11] proved that if the metric of a 3-dimensional paraSasakian manifold is a Yamabe soliton then the soliton field is Killing and the scalar curvature is constant.

Therefore, Corollary 1 is an improvement of the result of Erken.

4. 3-Dimensional Bach Flat ParaSasakian manifolds

In a 3-dimensional paraSasakian manifold the Riemannian curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{42}$$

Substituting $X = Z = \xi$ in (42) and making use of (12), (13) and (14) implies

$$QY = (-3 - \frac{r}{2})\eta(Y)\xi + (1 + \frac{r}{2})Y. \tag{43}$$

From the forgoing equation it is quite clear that

$$Q\varphi = \varphi Q. \tag{44}$$

Now we establish the subsequent lemma:

Lemma 2. *Let M be a 3-dimensional paraSasakian manifold. Then*
(i)

$$\sum_{k=1}^{3} g((\nabla_X Q)\varphi e_k, e_k) = 0$$

and
(ii)

$$\sum_{k=1}^{3} g((\nabla_{e_k} Q)Y, \varphi e_k) = (r-2)\eta(Y) - \frac{1}{2}(\varphi Y)r$$

.

Proof. Using (44), we get
$g((\nabla_X Q)\varphi Y, Z) + g((\nabla_X Q)Y, \varphi Z) = g((\nabla_X \varphi)QY, Z) + g(Q(\nabla_X \varphi)Y, Z)$.
Again using (9) and (44) in the above equation yields

$$\begin{aligned} &g((\nabla_X Q)\varphi Y, Z) + g((\nabla_X Q)Y, \varphi Z) = -g(X, QY)\eta(Z) \\ &-2g(X, Z)\eta(Y) + 2g(X, Y)\eta(Z) + g(QX, Z)\eta(Y). \end{aligned} \tag{45}$$

Putting $Y = Z = e_k$ in the previous equation and taking summation over $k(1 \leq k \leq 3)$, we have

$$\sum_{k=1}^{3} g((\nabla_X Q)\varphi e_k, e_k) + \sum_{k=1}^{3} g((\nabla_X Q)e_k, \varphi e_k) = 0.$$

That is,

$$\sum_{k=1}^{3} g((\nabla_X Q)\varphi e_k, e_k) = 0.$$

This completes the proof of (i).
On the other hand substituting $X = Z = e_k$ in (45) yields

$$\sum_{k=1}^{3} g((\nabla_{e_k} Q)Y, \varphi e_k) = (r-2)\eta(Y) - \frac{1}{2}(\varphi Y)r.$$

This completes the proof of (ii).

□

Proof of Theorem 2. Using (10) and (43), we infer that

$$(\nabla_X Q)\xi = Q\varphi X. \tag{46}$$

From (19) and (46) we have

$$C(X, Y)\xi = -2g(Q\varphi X, Y) - \frac{1}{4}[(Xr)\eta(Y) - (Yr)\eta(X)]. \tag{47}$$

Using (5), (9), (43) and (47) in (23) yields

$$\begin{aligned} (\nabla_X C)(Y, Z)\xi = &\, g((\nabla_Y Q)Z, \varphi X) - g((\nabla_Z Q)Y, \varphi X) \\ &+2g((\nabla_X Q)\varphi Y, Z) + 4g(X, Y)\eta(Z) + 2S(QX, Z)\eta(Y) \end{aligned}$$

$$+\frac{1}{4}[g(Z,\varphi X)(Yr)-g(\nabla_X Dr,Y)\eta(Z)$$
$$-g(\varphi X,Z)(Y)-g(\nabla_X Dr,Z)\eta(Y)]. \tag{48}$$

Putting $X=Y=e_k$ in the equation (48) and summing over $k(1\leq k\leq 3)$, we get

$$\begin{aligned}(\nabla_{e_k}C)(e_k,Z)\xi &= g((\nabla_{e_k}Q)Z,\varphi e_k)-g((\nabla_Z Q)e_k,\varphi e_k)\\ &+2g((\nabla_{e_k}Q)\varphi e_k,Z)+12\eta(Z)+2S(Qe_k,Z)\eta(e_k)\\ &+\frac{1}{4}[g(Z,\varphi e_k)(e_k r)-g(\nabla_{e_k}Dr,e_k)\eta(Z)\\ &-g(\varphi e_k,Z)(e_k)-g(\nabla_{e_k}Dr,Z)\eta(e_k)].\end{aligned} \tag{49}$$

Applying Lemma 4.1 and using (43) in (49) implies

$$\begin{aligned}(\nabla_{e_k}C)(e_k,Z)\xi &= 3(r+6)\eta(Z)-\frac{3}{2}g(\varphi Z,Dr)\\ &+\frac{1}{4}[(divDr)\eta(Z)-g(\nabla_\xi Dr,Z)].\end{aligned} \tag{50}$$

Since in a 3-dimensional paraSasakian manifold Weyl curvature tensor vanishes, so equation (6) reduces to

$$B(X,Y)=\sum_{k=1}^{3}[(\nabla_{e_k}C)(e_k,X)Y)]. \tag{51}$$

Replacing Y by ξ in (51) and use the the hypothesis, along with equation (50) provides

$$\begin{aligned}&3(r+6)\eta(X)-\frac{3}{2}g(\varphi X,Dr)\\ &+\frac{1}{4}[(divDr)\eta(X)-g(\nabla_\xi Dr,X)]=0.\end{aligned} \tag{52}$$

Replacing X by φX in (52) implies

$$\nabla_\xi Dr=-6(\varphi Dr). \tag{53}$$

From (40) and (53), we have $Dr=0$, that is r is constant. Then from (52), it follows that $r=-6$. Putting $r=-6$ in (43) yields

$$QY=-2Y. \tag{54}$$

Hence, the manifold is an Einstein manifold. Therefore, using $r=-6$ and the equation (54) in (42), we acquire

$$R(X,Y)Z=-[g(Y,Z)X-g(X,Z)Y].$$

Hence, the manifold is locally isometric to the hyperbolic space $H^{2n+1}(1)$ (p. 228, [22]). □

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