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# THE EIGENVALUES AND EIGENVECTORS OF THE 5D DISCRETE FOURIER TRANSFORM NUMBER OPERATOR REVISITED

### NATIG ATAKISHIYEV UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, ORCID ID: 0000-0002-8115-0574

ABSTRACT. A systematic analytic approach to the evaluation of the eigenvalues and eigenvectors of the 5D discrete number operator  $\mathcal{N}_5$  is formulated. This approach is essentially based on the use of the symmetricity of 5D discrete Fourier transform operator  $\Phi_5$  with respect to the discrete reflection operator  $P_d$ .

### 1. INTRODUCTION

Let me begin by recalling first that the eigenfunctions of the classical Fourier inegral transform (FIT), associated with the eigenvalues  $i^n$ , are explicitly given as

$$\psi_n(x) := H_n(x) \exp(-x^2/2), \quad n = 0, 1, 2, ...,$$
 (1.1)

where  $H_n(x)$  are Hermite polynomials. The functions  $\psi_n(x)$  are usually referred to as *Hermite functions* in the mathematical literature, whereas in quantum mechanics they emerge as eigenfunctions of the Hamiltonian for the linear harmonic oscillator, which is a self-adjoint differential operator of the second order (see, for example, [1]). It is well known that the functions  $\psi_n(x)$  are either symmetric or antisymmetric with respect to the reflection operator P, defined on the full real line  $x \in \mathbb{R}$  as P x = -x; that is,

$$P\psi_n(x) = \psi_n(-x) = (-1)^n \psi_n(x).$$
(1.2)

Recall also that the discrete (finite) Fourier transform (DFT) based on N points is represented by an  $N \times N$  unitary symmetric matrix  $\Phi$  with entries

$$\Phi_{kl} = N^{-1/2} q^{kl}, \qquad k, l \in \mathbb{Z}_N := \{0, 1, 2..., N-1\},$$
(1.3)

where  $q = \exp(2\pi i / N)$  is a primitive N-th root of unity and N is an arbitrary integer (see, for example, [2]-[7]). The discrete analogue of the above mentioned

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reflection operator P, associated with the DFT operator (1.3), is represented by the  $N \times N$  matrix

$$P_d := C^{\mathsf{T}} J_N \equiv J_N C \,, \tag{1.4}$$

where C is the basic circulant permutation matrix with entries  $C_{kl} = \delta_{k,l-1}$  and  $J_N$  is the  $N \times N$  'backward identity' permutation matrix with ones on the secondary diagonal (see [8], pages 26 and 28, respectively). Note that the matrix of the discrete reflection operator (1.4) can be partitioned as

$$P_d = \begin{bmatrix} 1 & 0_{N-1} \\ 0_{N-1}^{\mathsf{T}} & J_{N-1} \end{bmatrix}, \tag{1.5}$$

where  $0_m$  and  $0_n^{\intercal}$  are *m*-row and *n*-column zero vectors, respectively.

It is readily verified that the DFT operator (1.3) is  $P_d$ -symmetric, that is, the commutator  $[\Phi, P_d] := \Phi P_d - P_d \Phi = 0$ . Therefore, similar to the continuous case (1.2), the eigenvectors of the DFT operator  $\Phi$  should be either  $P_d$ -symmetric or  $P_d$ -antisymmetric.

The purpose of this work is to discuss some additional findings concerning symmetry properties of two finite-dimensional intertwining operators with the DFT matrix 1.3. These operators are represented by matrices A and  $A^{\intercal}$  of the same size  $N \times N$  such that the intertwining relations

$$A\Phi = i\Phi A, \qquad A^{\mathsf{T}}\Phi = -i\Phi A^{\mathsf{T}}, \tag{1.6}$$

are valid. The explicit form of the matrices A and  $A^{\intercal}$  is

$$A = X + iY = X + D$$
,  $A^{\intercal} = X - iY = X - D$ , (1.7)

where  $X = diag(\mathbf{s}_0, \mathbf{s}_1, ..., \mathbf{s}_{N-2}, \mathbf{s}_{N-1})$ ,  $\mathbf{s}_n := 2\sin(2\pi n/N)$ ,  $n \in \mathbb{Z}_N$ , and  $Y = -i\mathbf{D} = i(\mathbf{C}^{\intercal} - \mathbf{C})$ . The operators X and Y are Hermitian and play the role of finite-dimensional analogs of the operators of the coordinate and momentum in quantum mechanics, respectively.

The intertwining operators A and  $A^{\mathsf{T}}$  have emerged in a paper [9] devoted to the problem of finding the eigenvectors of the DFT operator  $\Phi$ . They can be interpreted as discrete analogs of the quantum harmonic oscillator lowering and raising operators  $\mathbf{a} = 2^{-1/2} \left( x + \frac{d}{dx} \right)$  and  $\mathbf{a}^{\dagger} = 2^{-1/2} \left( x - \frac{d}{dx} \right)$ ; their algebraic properties had been studied in detail in [10] -[12]. In particular, it was shown in [12] that the operators A and  $A^{\mathsf{T}}$  form a cubic algebra  $\mathcal{C}_q$  with q a root of unity. This algebra is intimately related to the two other well-known realizations of the cubic algebra: the Askey-Wilson algebra [13]–[16] and the Askey-Wilson-Heun algebra [17]. Note also that from the intertwining relations (1.6) it follows at once that the operator  $\mathcal{N} := A^{\mathsf{T}}A$  commutes with the DFT operator  $\Phi$ , that is,  $[\mathcal{N}, \Phi] = 0$ . The discrete number operator  $\mathcal{N}$  and the DFT operator  $\Phi$  thus have the same eigenvectors and one can employ the former for finding an explicit form of the eigenvectors of the latter (see [9] for a more detailed discussion of this point).

This idea that the discrete number operator  $\mathcal{N}$  is the one that really governs the eigenvectors of the DFT operator  $\Phi$ , was first successfully tested in [18] by considering the particular case of the 5D DFT operator  $\Phi_5$ . But the explicit form of the 4 nonzero eigenvalues  $\lambda_k, 1 \leq k \leq 4$ , of the discrete number operator  $\mathcal{N}_5$  have been found in [18] by using *Mathematica*. So it is the main goal of this work to formulate a systematic analytic approach to the evaluation of the above-mentioned eigenvalues  $\lambda_k$  without resorting to the help of any computer programs. The lay out of the paper is as follows. In section 2 a detailed account is given on how one can construct a  $P_d$ -symmetrized basis in the eigenspace  $\mathcal{H}_5$  of the discrete number operator  $\mathcal{N}_5$ , in terms of the eigenvectors of either the operator  $X_5$ , or the operator  $Y_5$ . In section 3 it is shown that the eigenspace  $\mathcal{H}_5$  with thus symmetrized basis splits into two 3D and 2D subspaces  $\mathcal{H}_3$  and  $\mathcal{H}_2$ ; this remarkable fact is used then to find desired explicit forms of the eigenvalues and eigenvectors of the discrete number operator  $\mathcal{N}_5$ . Finally, section 4 briefly outlines some further research directions of interest.

## 2. 5D operators $X_5$ and $Y_5$ in the $P_d$ -symmetrized basis

This section begins by a quotation from [12]: It is a remarkable fact that the operators X and Y are "classical" operators with nice spectral properties. For the 5D operator  $X_5 = diag(s_0, s_1, s_2, s_3, s_4)$ , it is obvious because the spectrum of  $X_5$  is

$$\lambda_n = \mathbf{s}_n = \mathbf{i}(q^{-n} - q^n), \qquad n \in \mathbb{Z}_5, \tag{2.1}$$

where  $q = \exp(2\pi i/5)$  and we introduced for brevity  $\mathbf{s}_n := 2\sin(2\pi n/5)$ . This indicates that the spectrum (2.1) belongs to the class of the Askey–Wilson spectra of the type

$$\lambda_n = C_1 q^n + C_2 q^{-n} + C_0 \,. \tag{2.2}$$

The eigenvectors of the operator  $X_5$  are represented by the Euclidean 5-column orthonormal vectors  $e_k$  with the components  $(e_k)_l = \delta_{kl}, k, l \in \mathbb{Z}_5$ , that is,

$$X_5 e_k = \mathbf{s}_k e_k \,. \tag{2.3}$$

The spectrum of the matrix  $Y_5$  belongs to the same Askey–Wilson family since the operators  $X_5$  and  $Y_5$  are unitary equivalent,  $Y_5 = \Phi X_5 \Phi^{\dagger}$ , and hence isospectral [12]. Note that the spectrum of  $X_5$  is simple, i.e., it is nondegenerate. Also, from the unitary equivalence of the operators  $X_5$  and  $Y_5$  it follows that the eigenvectors of the latter operator are of the form

$$Y_5 \epsilon_k = \mathbf{s}_k \epsilon_k , \qquad \epsilon_n := \Phi \, e_n = 5^{-1/2} \Big( 1, q^n, q^{2n}, q^{3n}, q^{4n} \Big)^{\mathsf{T}} . \tag{2.4}$$

Let me draw attention now to the remarkable symmetry between the operators  $X_5$ and  $Y_5$ : the operator  $X_5$  is two-diagonal in the eigenbasis of the operator  $Y_5$ ,

$$X_5 \epsilon_n = i \left( \epsilon_{n-1} - \epsilon_{n+1} \right), \tag{2.5}$$

whereas the operator  $Y_5$  is similarly two-diagonal in the eigenbasis of the operator  $X_5$ ,

$$Y_5 e_n = i \left( e_{n+1} - e_{n-1} \right). \tag{2.6}$$

**Remark.** It may also be worth mentioning here that the N-column eigenvectors of the operator Y for a general N,

$$\epsilon_n = \Phi \, e_n = \sum_{k=0}^{N-1} \Phi_{kn} \, e_k = N^{-1/2} \Big( 1, q^n, q^{2n}, \dots, q^{(N-1)n} \Big)^{\mathsf{T}}, \qquad (2.7)$$

form an orthonormal basis in the N-dimensional complex plane  $\mathbb{C}^N$  and are frequently used therefore as building blocks of the discrete Fourier transform in applications (see, for example, p.130 in [19], where the  $\epsilon_n$  referred to as discrete trigonometric functions).

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Since the operators X and Y generate a particular algebra, associated with the DFT operator  $\Phi$  for arbitrary integer values of N, one should use the eigenvectors of either the operator X, or the operator Y, as the most convenient basis for finding explicit forms of the eigenvectors of the operator  $\Phi$ . But we know that the eigenvectors of the operator  $\Phi$  should be either  $P_d$ -symmetric, or  $P_d$ -antisymmetric, whereas the eigenvectors of both the operators X and Y do not reveal any symmetry property of this type. The point is that the reflection operator  $P_d$  acts in the same way on both the eigenvectors  $e_n$  and  $\epsilon_m$ , that is,

$$P_d e_n = e_{N-n}, \quad e_N = e_0, \qquad P_d \epsilon_n = \epsilon_{N-n}, \quad \epsilon_N = \epsilon_0.$$
 (2.8)

Hence, the reflection operator  $P_d$  does not transform the eigenvectors  $e_0$  and  $\epsilon_0$ , and acts similarly by *cyclic permutation* on the other eigenvectors  $e_m$  and  $\epsilon_n$ , with  $1 \leq m, n \leq N - 1$ . To overcome this type of obstacle on the way of finding the eigenvectors of the operator  $\Phi$ , one thus needs to find first some  $P_d$ -symmetric bases, associated with both of the operators X and Y. This can be achieved as follows.

Returning now to the case of the 5D operators  $X_5$  and  $Y_5$ , let us consider first unit column-vectors  $\tilde{e}_n, n \in \mathbb{Z}_5$ , defined in terms of the eigenvectors  $e_n$  of the operator  $X_5$  as

$$\widetilde{e}_0 = e_0, \qquad \widetilde{e}_k = \frac{1}{\sqrt{2}} \left( e_k - e_{5-k} \right), \ k = 1, 2, \qquad \widetilde{e}_l = \frac{1}{\sqrt{2}} \left( e_l + e_{5-l} \right), \ l = 3, 4.$$
(2.9)

The explicit componentwise forms of the thus  $P_d$ -symmetrized column-vectors  $\tilde{e}_n$  are

$$\widetilde{e}_{0} = (1, 0, 0, 0, 0)^{\mathsf{T}}, \quad \widetilde{e}_{1} = \frac{1}{\sqrt{2}} (0, 1, 0, 0, -1)^{\mathsf{T}}, \quad \widetilde{e}_{2} = \frac{1}{\sqrt{2}} (0, 0, 1, -1, 0)^{\mathsf{T}}, \\ \widetilde{e}_{3} = \frac{1}{\sqrt{2}} (0, 0, 1, 1, 0)^{\mathsf{T}}, \quad \widetilde{e}_{4} = \frac{1}{\sqrt{2}} (0, 1, 0, 0, 1)^{\mathsf{T}}.$$
(2.10)

The interrelation (2.9) between the vectors  $\tilde{e}_n$  and the eigenvectors  $e_n$  of the operator  $X_5$  can be written in the compact form as  $\tilde{e}_k = Te_k, k \in \mathbb{Z}_5$ , where the unitary matrix T is

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & -1 & 0 & 0 & 1 \end{bmatrix},$$
  
$$T^{-1} = T^{\mathsf{T}}, \quad T T^{-1} = T^{-1} T = I_5. \tag{2.11}$$

Note that from the geometric point of view the matrix T represents simply a product of two rotations by the same angle  $\alpha = \pi/4$  in the 14- and 23-planes of the 5D-space, that is,  $T = R_{14}(\pi/4)R_{23}(\pi/4)$ , where

$$R_{14}(\pi/4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & 0 & 0 & \sin\frac{\pi}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\sin\frac{\pi}{4} & 0 & 0 & \cos\frac{\pi}{4} \end{bmatrix},$$

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$$R_{23}(\pi/4) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ 0 & 0 & -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} .$$
 (2.12)

Similarly, let us introduce now 5 orthonormal column-vectors  $\tilde{\epsilon}_n, n \in \mathbb{Z}_5$ , defined in terms of the eigenvectors  $\epsilon_n$  of the operator  $Y_5$  as  $\tilde{\epsilon}_k = T\epsilon_k, k \in \mathbb{Z}_5$ , with the same matrix T as in (2.11). Then the explicit forms of these  $P_d$ -symmetrized 5-vectors  $\tilde{\epsilon}_n$  are

$$\widetilde{\epsilon}_{0} = \epsilon_{0} = \frac{1}{\sqrt{5}} (1, 1, 1, 1, 1)^{\mathsf{T}}, \qquad \widetilde{\epsilon}_{1} = \frac{1}{\sqrt{2}} (\epsilon_{1} - \epsilon_{4}) = \frac{\mathrm{i}}{\sqrt{10}} (0, \mathsf{s}_{1}, \mathsf{s}_{2}, -\mathsf{s}_{2}, -\mathsf{s}_{1})^{\mathsf{T}}, \widetilde{\epsilon}_{2} = \frac{1}{\sqrt{2}} (\epsilon_{2} - \epsilon_{3}) = \frac{\mathrm{i}}{\sqrt{10}} (0, \mathsf{s}_{2}, -\mathsf{s}_{1}, \mathsf{s}_{1}, -\mathsf{s}_{2})^{\mathsf{T}}, \widetilde{\epsilon}_{3} = \frac{1}{\sqrt{2}} (\epsilon_{2} + \epsilon_{3}) = \frac{\mathrm{i}}{\sqrt{10}} (\mathsf{c}_{0}, \mathsf{c}_{2}, \mathsf{c}_{1}, \mathsf{c}_{1}, \mathsf{c}_{2})^{\mathsf{T}}, \widetilde{\epsilon}_{4} = \frac{1}{\sqrt{2}} (\epsilon_{1} + \epsilon_{4}) = \frac{\mathrm{i}}{\sqrt{10}} (\mathsf{c}_{0}, \mathsf{c}_{1}, \mathsf{c}_{2}, \mathsf{c}_{2}, \mathsf{c}_{1})^{\mathsf{T}},$$
(2.13)

where  $c_n := 2 \cos(2\pi n/5)$ .

It remains only to recall that if  $Z_{kl} := (e_k, Ze_l)$  represent the matrix elements of the operator (matrix) Z in the basis  $e_n$ , then the matrix  $\tilde{Z} := TZT^{-1}$  represents the matrix elements of the same operator Z in the basis  $\tilde{e}_n = Te_n$ . Hence the explicit forms of the operators  $X_5$  and  $Y_5$  in the  $P_d$ -symmetrized basis  $\tilde{e}_n$  are

where  $0_{nn}$  is the  $n \times n$  zero matrix.

Finally, from (2.14) it follows that the intertwining operators  $A_5$  and  $A_5^{\intercal}$  in the basis  $\tilde{e}_n$  can be partitioned as

$$\tilde{A}_5 = \tilde{X}_5 + \tilde{D}_5 = \begin{bmatrix} 0_{33} & -a_{32}(s) \\ a_{32}^{\mathsf{T}}(-s) & 0_{22} \end{bmatrix}, \qquad a_{32}(s) := \begin{bmatrix} 0 & \sqrt{2} \\ 1 & \mathsf{s}_1 \\ \mathsf{s}_2 - 1 & -1 \end{bmatrix},$$

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$$\tilde{A}_{5}^{\mathsf{T}} = \tilde{X}_{5} - \tilde{D}_{5} = \begin{bmatrix} 0_{33} & a_{32}(-s) \\ -a_{32}^{\mathsf{T}}(s) & 0_{22} \end{bmatrix}, \quad a_{32}(-s) := \begin{bmatrix} 0 & \sqrt{2} \\ 1 & -\mathsf{s}_{1} \\ -\mathsf{s}_{2} - 1 & -1 \end{bmatrix}.$$
(2.15)

To close this section, it may be worth drawing attention now to the particular manner in which the  $P_d$ -symmetrization modifies explicit forms of the DFT eigenvectors  $f_k$ ,  $k \in \mathbb{Z}_5$  in the basis  $\tilde{e}_n$ . It is obvious that the partitioning of the intertwining operators  $\tilde{A}_5$  and  $\tilde{A}_5^{\mathsf{I}}$  of the form (2.15) leads to the need to appropriately split the DFT eigenvectors  $f_k$ ,  $k \in \mathbb{Z}_5$  in the basis  $\tilde{e}_n$  into two components, that is, to represent the vectors  $\tilde{f}_k := Tf_k$  as

$$\widetilde{f}_k = (\eta_k, \xi_k)^{\mathsf{T}}, \qquad \eta_k := (x_0, x_1, x_2), \qquad \xi_k := (x_3, x_4).$$
(2.16)

Since every symmetric 5D DFT eigenvector  $f^{(s)}$  in the basis  $e_n$  is of the form  $f^{(s)} = (a, b, c, c, b)^{\mathsf{T}}$ , whereas every antisymmetric 5D DFT eigenvector  $f^{(a)}$  in the same basis  $e_n$  is of the form  $f^{(a)} = (0, b, c, -c, -b)^{\mathsf{T}}$ , it turns out that

$$\widetilde{f}^{(s)} := Tf^{(s)} = (a, \sqrt{2}b, \sqrt{2}c, 0, 0)^{\mathsf{T}}, \quad \widetilde{f}^{(a)} := Tf^{(a)} = -\sqrt{2}(0, 0, 0, c, b)^{\mathsf{T}}.$$
(2.17)

This means that all 5D DFT eigenvectors  $f_k$  in the basis  $\tilde{e}_n$  are either of the  $\eta$ -type (that is, with vanishing lower component  $\xi_k$ ), or of the  $\xi$ -type (with the upper component  $\eta_k = 0$ ).

### 3. DFT number operator $N_5$ in the $P_d$ -symmetrized basis

Having defined explicitly the matrices  $A_5$  and  $A_5^{\mathsf{T}}$  in the  $P_d$ -symmetrized basis  $\tilde{e}_n$ in the previous section, it is not hard to evaluate that the discrete number operator  $\mathcal{N}_5 = A_5^{\mathsf{T}} A_5$  in the same basis  $\tilde{e}_n$  is of the following form

$$\tilde{\mathcal{N}}_5 = \tilde{A}_5^{\mathsf{T}} \tilde{A}_5 = \begin{bmatrix} \mathcal{N}_3 & 0_{32} \\ 0_{32}^{\mathsf{T}} & \mathcal{N}_2 \end{bmatrix}, \tag{3.1}$$

where  $0_{32}$  is the  $3 \times 2$  zero matrix and  $\mathcal{N}_3$  and  $\mathcal{N}_2$  are  $3 \times 3$  and  $2 \times 2$  full Hermitian matrices,

$$\mathcal{N}_{3} := \begin{bmatrix} 2 & -\sqrt{2} \mathbf{s}_{1} & -\sqrt{2} \\ -\sqrt{2} \mathbf{s}_{1} & 3 - \mathbf{c}_{2} & \mathbf{c}_{1} \mathbf{s}_{2} - 1 \\ -\sqrt{2} & \mathbf{c}_{1} \mathbf{s}_{2} - 1 & 2(\mathbf{s}_{2} + 2) - \mathbf{c}_{1} \end{bmatrix},$$
$$\mathcal{N}_{2} := \begin{bmatrix} 2(2 - \mathbf{s}_{2}) - \mathbf{c}_{1} & \mathbf{c}_{1} \mathbf{s}_{2} + 1 \\ \mathbf{c}_{1} \mathbf{s}_{2} + 1 & 5 - \mathbf{c}_{2} \end{bmatrix},$$
(3.2)

respectively. Thus the Fock space  $\mathcal{H}_5$  of all eigenvectors of the discrete number operator  $\mathcal{N}_5$  in the  $P_d$ -symmetrized basis  $\tilde{e}_n$  splits into two 3D and 2D subspaces  $\mathcal{H}_3$  and  $\mathcal{H}_2$ ; the operator  $\tilde{\mathcal{N}}_5$  represents in the eigenspace  $\mathcal{H}_5$  the direct sum of the operators  $\mathcal{N}_3$  and  $\mathcal{N}_2$ , that is,  $\tilde{\mathcal{N}}_5 = \mathcal{N}_3 \oplus \mathcal{N}_2$ .

One clarifying remark must be made at this point in connection with (3.1). The point is that this formula reveals that the discrete number operator  $\tilde{\mathcal{N}}_5$  in the  $P_d$ symmetrized basis  $\tilde{e}_n$  has 12 zero matrix elements, whereas its counterpart  $\mathcal{N}_5$  in the basis of the eigenvectors  $e_n$  is represented by a 5D matrix with 25 nonzero entries. Note that it was possible to formulate such a remarkable transformation of the full matrix  $\mathcal{N}_5$  into the sparse matrix  $\tilde{\mathcal{N}}_5$  only because of the  $P_d$ -symmetricity of the DFT operator  $\Phi_5$ . Recall then that the well-known Fast Fourier Transform algorithm of Cooley and Tukey is based essentially on a factorization of the Fourier matrix into a product of sparse matrices (see, for example, [19, 20]). Thus it becomes

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clear now that Cooley and Tukey had been able to construct so ingeniously their highly efficient implementation of the DFT only because of the  $P_d$ -symmetricity of the Fourier matrix, although they had never employed explicitly this fundamental symmetry property of the Fourier matrix.

From (3.1) it is evident that the eigenvectors and eigenvalues of the operator  $\tilde{\mathcal{N}}_5$ may be now defined in terms of the eigenvectors and eigenvalues of the operators  $\mathcal{N}_3$ and  $\mathcal{N}_2$  from the separate subspaces  $\mathcal{H}_3$  and  $\mathcal{H}_2$ , respectively. In order to proceed to this task under consideration, let me start first with the operator  $\mathcal{N}_2$ .

**Lemma 3.1.** An arbitrary  $2 \times 2$  Hermitian matrix of the form  $M = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$  can be written as a linear combination of the 2D identity matrix  $I_2$  and the  $2 \times 2$  traceless matrix M',

$$M = u I_2 + M' = u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} v & b \\ b & -v \end{bmatrix}, \qquad (3.3)$$

where 2u = a + d and 2v = a - d. The eigenvalues of the matrix M' are equal to

$$\lambda_{1,2} = \pm \left( -\det M' \right)^{1/2} = \pm (v^2 + b^2)^{1/2}, \qquad (3.4)$$

whereas the eigenvalues of the matrix M are

$$\mu_{1,2} = \lambda_{1,2} + (a+d)/2 = (a+d)/2 \pm (v^2 + b^2)^{1/2}.$$
(3.5)

Proof. Since

$$\det \left(M' - \lambda I_2\right) = \det \left[\begin{array}{cc} v - \lambda & b\\ b & -v - \lambda \end{array}\right] = \lambda^2 - v^2 - b^2, \qquad (3.6)$$

the eigenvalues  $\lambda_{1,2}$  of the matrix M' are roots of the quadratic equation  $\lambda^2 - v^2 - b^2 = 0$ ; hence  $\lambda_{1,2} = \pm (v^2 + b^2)^{1/2}$  and formula (3.4) is proved. Then from (3.3) it follows at once that the eigenvalues of the matrix M are equal to  $\mu_{1,2} = \lambda_{1,2} + (a+d)/2$  and formula (3.5) is proved as well.

Evidently, the matrix  $\mathcal{N}_2$  is of the same type as the matrix M from the lemma above, with  $a = 2(2 - \mathbf{s}_2) - \mathbf{c}_1$ ,  $b = \mathbf{c}_1\mathbf{s}_2 + 1$  and  $d = 5 - \mathbf{c}_2$ . This means that in this particular case  $u = 5 - \mathbf{s}_2$ ,  $v = \mathbf{c}_2 - \mathbf{s}_2 = \mathbf{c}_2 b$  and  $v^2 + b^2 = 2\sqrt{5}(\mathbf{s}_2 + 2\mathbf{c}_1) = (\mathbf{s}_1)^2 b^2$ . Thus from (3.5) it follows that the eigenvalues of the matrix  $\mathcal{N}_2$  are

$$\mu_1 = 5 - \mathbf{s}_2 + \mathbf{s}_1 \, b = 5 + \mathbf{s}_2 \, (\mathbf{s}_2 + \mathbf{c}_1), \qquad \mu_2 = 5 - \mathbf{s}_2 - \mathbf{s}_1 \, b = \mathbf{s}_1 \, (\mathbf{s}_1 + \mathbf{c}_2). \tag{3.7}$$

**Remark.** The equation which is solved to find eigenvalues of  $n \times n$  matrix M is usually interpreted as the equation for finding roots of the characteristic polynomial in  $\lambda$  of degree n,

$$p_n(\lambda) := \det \left(\lambda I - M\right) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_{n-1} \lambda + c_n, \qquad (3.8)$$

I is the  $n \times n$  identity matrix and the coefficient  $c_k$  is  $(-1)^k$  times the sum of the determinants of all of the principal  $k \times k$  minors of M (in particular,  $c_1 =$  $-\operatorname{trace}(M)$  and  $c_n = (-1)^n \det M$ ). The lemma 3.1 has been employed in order to reduce the characteristic equation  $p_2(\lambda) = \lambda^2 + c_1 \lambda + c_2 = 0$  for the matrix  $\mathcal{N}_2$  to the readily solvable equation for the matrix  $\mathcal{N}'_2$ , which is of the form  $p_2(\lambda) = \lambda^2 + c_2 = 0$ .

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Having defined the eigenvalues  $\mu_1$  and  $\mu_2$  of the matrix  $\mathcal{N}_2$ , it is not hard to find eigenvectors of  $\mathcal{N}_2$ , associated with those eigenvalues (3.7). Indeed, note first that

$$\mathcal{N}_2 = (5 - \mathbf{s}_2) I_2 + \mathcal{N}'_2, \qquad \mathcal{N}'_2 = (1 + \mathbf{c}_1 \, \mathbf{s}_2) \begin{bmatrix} \mathbf{c}_2 & 1\\ 1 & -\mathbf{c}_2 \end{bmatrix},$$
(3.9)

where the eigenvalues of the matrix  $\begin{bmatrix} c_2 & 1 \\ 1 & -c_2 \end{bmatrix}$  are  $\pm s_1$ . Therefore to find the eigenvectors of the matrix  $\mathcal{N}_2$ , it is sufficient to determine the eigenvectors of the much simpler matrix  $\begin{bmatrix} c_2 & 1 \\ 1 & -c_2 \end{bmatrix}$ . So one readily derives that

$$\begin{bmatrix} \mathbf{c}_2 & 1\\ 1 & -\mathbf{c}_2 \end{bmatrix} \begin{pmatrix} \mathbf{c}_1\\ 1+\mathbf{s}_2 \end{pmatrix} = \mathbf{s}_1 \begin{pmatrix} \mathbf{c}_1\\ 1+\mathbf{s}_2 \end{pmatrix},$$
$$\begin{bmatrix} \mathbf{c}_2 & 1\\ 1 & -\mathbf{c}_2 \end{bmatrix} \begin{pmatrix} 1+\mathbf{s}_2\\ -\mathbf{c}_1 \end{pmatrix} = -\mathbf{s}_1 \begin{pmatrix} 1+\mathbf{s}_2\\ -\mathbf{c}_1 \end{pmatrix}. \tag{3.10}$$

Thus explicit forms of the two linearly independent eigenvectors of the operator  $\mathcal{N}_2$ , associated with the eigenvalues (3.7), are

$$\varphi_1 := (\mathbf{c}_1, 1 + \mathbf{s}_2)^{\mathsf{T}}, \qquad \varphi_2 := (1 + \mathbf{s}_2, -\mathbf{c}_1)^{\mathsf{T}},$$
(3.11)

respectively. Note that the vectors  $\varphi_1$  and  $\varphi_2$  are essentially the same as the down-components of the antisymmetric eigenvectors  $\tilde{f}_1 = Tf_1$  and  $\tilde{f}_3 = Tf_3$  of the discrete number operator  $\tilde{\mathcal{N}}_5$  in the  $P_d$ -symmetrized basis  $\tilde{e}_n$ , where  $f_1$  and  $f_3$  have been already derived in [18] by employing *Mathematica*; that is

$$\widetilde{f}_1 = (0_3, \varphi_1)^{\mathsf{T}}, \qquad \widetilde{f}_3 = (0_3, \varphi_1)^{\mathsf{T}}.$$
 (3.12)

Turning now to the case of the matrix  $\mathcal{N}_3$ , one may likewise employ the polynomial  $p_3(\lambda) = \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$  in order to find first the eigenvalues of  $\mathcal{N}_3$ . It turns out that the determinant of the matrix  $\mathcal{N}_3$  is equal to zero,

$$\det \mathcal{N}_{3} = \begin{vmatrix} 2 & -\sqrt{2} \, \mathbf{s}_{1} & -\sqrt{2} \\ -\sqrt{2} \, \mathbf{s}_{1} & 3 - \mathbf{c}_{2} & \mathbf{c}_{1} \mathbf{s}_{2} - 1 \\ -\sqrt{2} & \mathbf{c}_{1} \mathbf{s}_{2} - 1 & 2(\mathbf{s}_{2} + 2) - \mathbf{c}_{1} \end{vmatrix}$$
$$= \begin{vmatrix} 2 & 0 & 0 \\ -\sqrt{2} \, \mathbf{s}_{1} & \mathbf{s}_{1} - 2 \, \mathbf{c}_{2} & -\mathbf{s}_{2} - 1 \\ -\sqrt{2} & 3(\mathbf{c}_{2} - \mathbf{s}_{1}) - \mathbf{c}_{1} & (\mathbf{s}_{2} + 1)^{2} \end{vmatrix}$$
$$= 2 \begin{vmatrix} \mathbf{s}_{1} - 2 \, \mathbf{c}_{2} & -\mathbf{s}_{2} - 1 \\ 3(\mathbf{c}_{2} - \mathbf{s}_{1}) - \mathbf{c}_{1} & (\mathbf{s}_{2} + 1)^{2} \end{vmatrix} = 2(\mathbf{c}_{2} - \mathbf{s}_{1}) \begin{vmatrix} -1 & -1 \\ \mathbf{s}_{2} + 1 & \mathbf{s}_{2} + 1 \end{vmatrix} = 0. \quad (3.13)$$

Hence the characteristic equation for the matrix  $\mathcal{N}_3$  reduces to the form

$$\lambda(\lambda^2 + c_1\lambda + c_2) = 0. \qquad (3.14)$$

Consequently, one of the eigenvalues of the matrix  $\mathcal{N}_3$  is  $\lambda_0 = 0$ , whereas the two remaining eigenvalues of  $\mathcal{N}_3$  are roots of the quadratic equation

$$\lambda^2 + c_1 \lambda + c_2 = 0, \qquad (3.15)$$

where the coefficient  $c_1 = -\operatorname{trace}(\mathcal{N}_3) = -2(5 + s_2)$  and the coefficient  $c_2$ , which represents the sum of the determinants of the three principal  $2 \times 2$  minors of  $\mathcal{N}_3$ , is readily evaluated to be  $c_2 = 10 + (4 s_2 + 3)(s_1)^2$ . So one concludes that

$$\lambda_{1,2} = -\frac{\mathsf{c}_1}{2} \pm \sqrt{\frac{(\mathsf{c}_1)^2}{4} - \mathsf{c}_2} = 5 + \mathsf{s}_2 \pm (\mathsf{c}_1 \, \mathsf{s}_2 - 1) \, \mathsf{s}_1, \tag{3.16}$$

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upon taking into account the readily verified identity  $2(2 - s_1) = (s_2 + c_2)^2$ .

Quite similar to the case of the matrix  $\mathcal{N}_2$ , the knowledge of the explicit forms of the eigenvalues for the matrix  $\mathcal{N}_3$  essentially simplifies the task of defining the appropriate eigenvectors of  $\mathcal{N}_3$  for each of those eigenvalues. Indeed, by looking for solutions of the equation  $\mathcal{N}_3 \phi_\lambda = \lambda \phi_\lambda$  in the form  $\phi_\lambda = (x_0, x_1, x_2)^{\mathsf{T}}$ , one arrives simply at a system of three homogeneous equations

for the components of the 3D column-vector  $\phi_{\lambda}$ .

1°. In the case of  $\lambda_0 = 0$  the system (3.17) reduces to

$$\begin{array}{rcrcrcrcrc}
\sqrt{2} x_0 & - & \mathbf{s}_1 x_1 & - & x_2 & = 0, \\
-\sqrt{2} \mathbf{s}_1 x_0 & + & (3 - \mathbf{c}_2) x_1 & + & (\mathbf{c}_1 \mathbf{s}_2 - 1) x_2 & = 0, \\
-\sqrt{2} x_0 & + & (\mathbf{c}_1 \mathbf{s}_2 - 1) x_1 & + & [2(\mathbf{s}_2 + 2) - \mathbf{c}_1] x_2 & = 0.
\end{array}$$
(3.18)

Eliminating the component  $x_0$  by adding to the second equation in (3.18) the first one, multiplied by  $s_1$ , one arrives at the relation  $x_1 = (1 + s_2) x_2$ . Substituting this relation back into the first equation enables one to express the component  $x_0$  via the component  $x_2$  as

$$\sqrt{2} x_0 = \mathbf{s}_1 x_1 + x_2 = (\mathbf{s}_1 - 2 \mathbf{c}_2) x_2.$$

Taking into account that the system (3.17) defines the eigenvector  $\phi_0$  up to the multiplication by an arbitrary constant factor, one thus concludes that

$$\phi_0 = \left(\mathsf{s}_1 - 2\,\mathsf{c}_2, \sqrt{2}\,(1+\mathsf{s}_2), \sqrt{2}\,\right)^\mathsf{T}.\tag{3.19}$$

2°. In the case of  $\lambda_1 = 5 + s_2 + (c_1 s_2 - 1) s_1 = 5 + s_2 (s_2 - c_1)$ , the system of equations (3.17) reduces to

$$\begin{bmatrix} \mathsf{c}_1(\mathsf{s}_2+1) - 5 \end{bmatrix} x_0 - \sqrt{2} \,\mathsf{s}_1 \,x_1 - \sqrt{2} \,x_2 &= 0, \\ -\sqrt{2} \,\mathsf{s}_1 \,x_0 &+ \begin{bmatrix} \mathsf{c}_1(\mathsf{s}_2+2) - 3 \end{bmatrix} x_1 &+ \begin{pmatrix} \mathsf{c}_1\mathsf{s}_2 - 1 \end{pmatrix} x_2 &= 0, \\ -\sqrt{2} \,x_0 &+ \begin{pmatrix} \mathsf{c}_1\mathsf{s}_2 - 1 \end{pmatrix} x_1 - \begin{pmatrix} \mathsf{c}_2\mathsf{s}_1 + 3 \end{pmatrix} x_2 &= 0, \\ \end{bmatrix}$$
(3.20)

As in the previous case of  $\lambda_0 = 0$ , one eliminates the component  $x_0$  by adding to the second equation in (3.20) the third one, multiplied by  $-\mathbf{s}_1$ . This leads to the relation

$$a x_1 + b x_2 = 0,$$
  $a = \sqrt{5} s_2 + 3 c_1 - 5,$   $b = (4 - c_1) s_1 + 3 c_2 - 2,$  (3.21)

interconnecting the components  $x_1$  and  $x_2$ . It turns out that the coefficients a and b in the relation (3.21) have a common factor,

$$a = \epsilon (2 \mathbf{s}_2 - 2\mathbf{c}_1 + 3), \qquad b = \epsilon (2 \mathbf{s}_2 + 1), \qquad \epsilon = -\mathbf{c}_2 (\mathbf{c}_2 \mathbf{s}_1 + 3).$$
 (3.22)

Eliminating this common factor from the relation (3.21) reduces it to the simpler form,

$$(2s_2 - 2c_1 + 3)x_1 + (2s_2 + 1)x_2 = 0, (3.23)$$

from which it follows at once that  $x_1 = -(2s_2 + 1)$  and  $x_2 = 2(s_2 - c_1) + 3$ . Substituting these values of  $x_1$  and  $x_2$  into the first equation in (3.20), one finally finds that the component  $x_0 = \sqrt{2} c_1$ . Thus the eigenvector  $\phi_1$ , associated with the eigenvalue  $\lambda_1$ , has the form

$$\phi_1 = \left(\sqrt{2}\,\mathsf{c}_1, -2\,\mathsf{s}_2 - 1, \, 2\,(\mathsf{s}_2 - \mathsf{c}_1) + 3\,\right)^\mathsf{T}.\tag{3.24}$$

3°. Finally, in the case of  $\lambda_2 = 5 + s_2 - (c_1 s_2 - 1) s_1 = s_1 (s_1 - c_2)$ , the system of equations (3.17) reduces to

As in the previous case of  $\lambda_1$ , one eliminates the component  $x_0$  by adding to the second equation in (3.25) the third one, multiplied by  $-s_1$ . This leads to the relation  $x_1 = x_2$ , which then enables one to find from the first equation in (3.25) that  $x_0 = -\sqrt{2} c_1 x_1$ . Thus the eigenvector  $\phi_2$ , associated with the eigenvalue  $\lambda_2$ , has the form

$$\phi_2 = \left( -\sqrt{2} \, \mathsf{c}_1, \, 1, \, 1 \right)^\mathsf{T}. \tag{3.26}$$

It remains only to add that the vectors  $\phi_0$ ,  $\phi_1$  and  $\phi_2$ , associated with the eigenvalues  $\lambda_0$ ,  $\lambda_1$  and  $\lambda_2$ , are essentially the same as the up-components of the three symmetric eigenvectors  $\tilde{f}_0 = Tf_0$ ,  $\tilde{f}_4 = Tf_4$  and  $\tilde{f}_2 = Tf_2$  of the discrete number operator  $\tilde{\mathcal{N}}_5$  in the  $P_d$ -symmetrized basis  $\tilde{e}_n$ , where  $f_0$ ,  $f_2$  and  $f_4$  have been already derived in [18] by employing *Mathematica*; that is,

$$\widetilde{f}_0 = (\phi_0, 0_2)^{\mathsf{T}}, \qquad \widetilde{f}_2 = (\phi_2, 0_2)^{\mathsf{T}}, \qquad \widetilde{f}_4 = (\phi_1, 0_2)^{\mathsf{T}}.$$
 (3.27)

### 4. Concluding Remarks

To conclude this work, the following should be recalled first. Recently it has been proved that the 'position' and 'momentum' DFT operators X and Y form a special case of the Askey-Wilson algebra AW(3) [12]. So it would be appropriate to use the eigenvectors of either X, or Y, as a basis in the eigenspace of the discrete number operator  $\mathcal{N}$ , that governs the eigenvectors of the DFT operator  $\Phi$ . In this work it is shown that in the case of DFT this technique of employing the 'position'  $e_n$  and 'momentum'  $\epsilon_n$  eigenvectors for resolving an eigenvalue problem for the discrete number operator  $\mathcal{N}$  is not applicable, unless those eigenvectors are being symmetrized with respect to the discrete reflection operator  $P_d$ . Therefore the  $P_d$ symmetrization operator T is found and a remarkable fact is established: it turns out that the matrix of the discrete number operator  $\mathcal{N}_5$  in the  $P_d$ -symmetrized basis  $\tilde{e}_n = Te_n$  has only half of the number of the nonzero entries of the same matrix in the initial basis  $e_n$ . This sparse alization of the discrete number operator  $\mathcal{N}_5$  is shown to be essentially helpful for finding explicit forms of the eigenvalues and eigenvectors of the operator  $\mathcal{N}_5$ . Finally, I believe that just a bit more time is needed now to resolve an eigenvalue problem for the DFT number operator  $\mathcal N$  of a general dimension N.

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NATIG ATAKISHIYEV,

INSTITUTO DE MATEMÁTICAS, UNIDAD CUERNAVACA, CUERNAVACA, 62210, MORELOS, MÉXICO, ORCID ID: 0000-0002-8115-0574

Email address: natig@im.unam.mx