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DEVELOPMENTAL PROCESS OF QUADRATIC EQUATIONS FROM PAST TO PRESENT AND REFLECTIONS ON TEACHING-LEARNING[*]

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ABSTRACT

The mathematical concept of quadratic equations is one of the important topics in algebra and has deep developmental process. It is also an inseparable component of history of mathematics and mathematics curriculum. In this study, it was aimed to present historical development of quadratic equations through periodic examples with reference to using history of mathematics that may help students to pay attention to the subject and improve meaningful understanding. Besides, the purpose of the current study was to examine the reflections of the developmental process on education and to produce implications for quadratic equations training. Data were presented in time sequence. The analysis showed that making sense of quadratic equations was difficult in terms of the students and they had various misconceptions. Therefore, this topic should be focused on it and some implications such as using historical examples and explaining developmental process may facilitate learning and teaching quadratic equations.

Keywords: *quadratic equations, algebra, history of mathematics*

GEÇMİŞTEN GÜNÜMÜZE İKİNCİ DERECEDEN DENKLEMLERİN GELİŞİMSEL SÜRECİ VE ÖĞRETME-ÖĞRENME ÜZERİNE YANSIMALARI

ÖZ

Matematik tarihinin ve matematik öğretim programının ayrılmaz bir parçası olan ikinci dereceden denklemler zengin gelişim sürecine sahip önemli cebir konularından birisidir. Bu çalışmada, matematik tarihinin kullanımı öğrencilerin konuya olan ilgilerini ve anlamlı öğrenmelerini arttırmada yardımcı olabilir düşüncesinden hareketle ikinci dereceden denklemlerin tarihsel gelişimine ilişkin örnekler yoluyla sunulması hedeflenmiştir. Bunun yanı sıra, çalışmada ikinci derece denklemlerdeki gelişimsel sürecin eğitim üzerindeki yansımalarını incelemek ve bu konunun öğretimine yönelik önerilerde bulunmak amaçlanmıştır. Bilgiler zaman sırasına uygun bir şekilde sunulmuştur. Veriler, ikinci dereceden denklemler konusunun anlamlandırılmasının öğrenciler açısından zor olduğunu ve öğrencilerin bu konuda çeşitli kavram yanılgılarının mevcut olduğunu göstermektedir. Dolayısıyla, üzerinde durulması gereken bu konunun öğretiminde ve öğreniminde tarihsel örneklerin kullanılması ve konunun gelişimsel sürecinin açıklanması faydalı olabilir.

Anahtar Kelimeler: *ikinci dereceden denklemler, cebir, matematik tarihi*

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1. INTRODUCTION

1.1. The History of Quadratic Equations

Quadratic equations have rich historical development process. The past documents showed that the origin of the quadratic equations was based on the concept of rectangle which had length and width. Depending on known quantities, the unknown such as length, width or area could be calculated. These easy problems led to constitution of quadratic equations and in time, they were transformed into complex structural forms (Gandz, 1937). For the first time, a solution of quadratic equation appeared in the Berlin papyrus (ca. 2160 -1700 BC) in Egypt (Smith, 1953, p. 443), however, the type of the examples in this period was pure quadratic equations (Gandz, 1940).

The Egyptian method included more general solution, Babylonian mathematics gained importance. Babylonian`s clay tablets (2000-1700 BC) included a set of quadratic problems and showed that they solved quadratic equations. These problems were mainly numerical, required adding an area or a side and finding such quantities as the length and width of rectangle with the help of the geometrical meaning of unknowns and using different methods. The essential method was completing the square (Katz, 2007). Although these concepts were not named as equations, the solution ways of problems set up the substructure of quadratic equations. In addition, the problems were usually used to degrade complex problems into simpler rather than finding an answer.

In Babylonian, the purpose of solving quadratic equations was to upskill the students. Learning to determine the type and time of utilizing methods and degrading from complicated to simple one was important in terms of growing desirable individual for the future (Katz, 1997). It was suprising attitude in the past. There were eight types of Babylonian equations and first six of them were characterized as the Diophantine types since they were solved in accordance with Diophantus' method whereas the last two types were denoted as the Arabic types because Arabic Algebra of Al-Khuwarizmi first made them known and gave the solution for them. However, the origins of all of these types relied on Babylonian mathematics. The Babylonian types were as follows (Gandz, 1937, p. 405),

$$\text{I.}\quad x+y=a;\ xy=b. \qquad \left.\begin{matrix}x\\y\end{matrix}\right\}=\frac{a}{2}\pm\sqrt{\left(\frac{a}{2}\right)^2-b}$$

$$\text{II.}\quad x-y=a;\ xy=b. \qquad \left.\begin{matrix}x\\y\end{matrix}\right\}=\sqrt{\left(\frac{a}{2}\right)^2+b}\pm\frac{a}{2}$$

$$\text{III.}\quad x+y=a;\ x^2+y^2=b \qquad \left.\begin{matrix}x\\v\end{matrix}\right\}=\frac{a}{2}\pm\sqrt{\frac{b}{2}-\left(\frac{a}{2}\right)^2}$$

$$\text{IV.}\quad x-y=a;\ x^2+y^2=b \qquad \left.\begin{matrix}x\\y\end{matrix}\right\}=\sqrt{\frac{b}{2}-\left(\frac{a}{2}\right)^2}\pm\frac{a}{2}$$

$$\text{V.}\quad x+y=a;\ x^2-y^2=b. \qquad \left.\begin{matrix}x\\y\end{matrix}\right\}=\frac{a}{2}\pm\frac{b}{2a}=\frac{1}{2}\cdot\frac{a^2\pm b}{a}$$

$$\text{VI.}\quad x-y=a;\ x^2-y^2=b \qquad \left.\begin{matrix}x\\y\end{matrix}\right\}\frac{b}{2a}\pm\frac{a}{2}=\frac{1}{2}\cdot\frac{b\pm a^2}{a}$$

$$\text{VII.}\quad x^2+ax=b.\ (=\text{A I}).$$

$$\text{VIII.}\quad x^2-ax=b.\ (=\text{A III}).$$

$$\text{IX.}\quad x^2+b=ax.\ (=\text{A II}).$$

Figure 1. *Quadratic Equation Types of Babylonians*

Babylonians usually degraded complicated types of quadratic equations into Diophantine types as methods of solution, especially the first two elementary ones, Diophantine procedure was followed in order to solve equations and they avoided applying the Arabic types. This kind of tendency resulted from some deficiencies. If the deficient properties of Babylonians were mentioned, the concept of double root was not known, replacing the two unknown in equation could not be possible, and namely, whereas one of them referred to the length, the other one did the width in rectangle. In addition, x which represented the length had to be greater than y which represented the width due to that it was subtracted (Gandz, 1937). Besides, there was no perception of negative numbers in quadratic equations at this period, Babylonians always used positive numbers. Therefore, the improvement in equations and other mathematical subjects occurred after sense of negative numbers (Gallardo and Rojano, 1994; Gallardo, 2000). It was clearly understood that there were differences between their method and modern procedure but it created the origins of modern formulas. When we consider the quadratic equations in Greek algebra in which mathematics was geometry, it is possible to say that old Babylonians' theories were protected by the Greek mathematicians such as Diophantus, Euclid, Hero. Euclid (325-270BC) regarded the quadratic equations as the old Babylonian method; however, he differently operated on geometrical figures to solve quadratic equations (Gandz, 1937). Although there were many similarities between algebraic expressions which derived from geometric need in Babylonians and geometric manipulations in Greek mathematics, Babylonians developed algorithms and methods for solving quadratic equations whereas Greek preferred geometry (Katz, 2007). In addition, as it was in Babylonians, Euclid did not choose to use negative quantities and subtraction. It was seen that Euclid used the Babylonian problems and their solution methods and he transformed the Babylonians theorems into geometric forms. On the other hand, due to that Diophantus (210-290BC) did not have knowledge related to negative roots, he worked on only the greater values given by the positive roots in order to prevent an absurdity which negative roots caused (Gandz, 1937).

Babylonians types and methods were not sufficient to solve problems which required applying equations with one unknown which which were not degradable into simple Babylonian types, especially the first two types. It could be said that although Babylonians well knew three Arabic types of quadratic equations, we named before the first six types as Diophantine and the last three types as Arabic and all together as Babylonian, they avoided using these equations due to the fact that their procedure relied on manipulation with the special numbers and their tendency of transforming the equation types into the first two equation forms ($x+y=a$; $xy=b$). Because of transforming into these forms causing two solutions and two values, corresponding of two values to the same quantity led to ambiguity. Therefore, this situation explained the reasons of Babylonians' preference to avoid putting up an equation for y and for duplicity of solutions.

There was resistance of Babylonians toward Arabic types because of ambiguity. However, these types stood out as needed and Al-Khwarizmi (825AD) realized this need and Babylonians` failure attitude. He attempted to solve same problems in Babylonian mathematics by using the three Arabic types as different from their solution methods (Gandz, 1937). In a way different from Babylonians' procedure, he applied algorithm to solve problems and utilized cut and paste geometry similar to Babylonians in order to justify it. He worked on abstract problems instead of length-width problems unlike Babylonians (Katz, 2007).

He believed that all Babylonian types were not necessary and all of the problems of Babylonians were to degrade into the three Arabic types. Therefore, the solution methods were standardized so that he introduced algebra that we know in present (Gandz, 1937). Al-Khwarizmi's book included manipulation of algebraic expression as well as quadratic equations and geometry was used to clarify the content. It showed his contribution on the developmental progress of algebra (Katz, 1997). However, in this book he neglected negative roots because of creating ambiguity like the others that we mentioned before and avoided to use the types of equations which included negative numbers. To sum up, the negative solutions of equations were not accepted by Babylonians and the Greek. Hindu mathematician Bhaskara introduced the negative roots in 12th century and he showed that the solution could be both real and imaginary (Katz, 1998, p. 226-227), before that time any idea, procedure or formula related to negative roots of quadratic equations were not encountered. In addition, Islamic mathematicians developed all of the procedures of polynomial algebra including both negative and positive roots by working on quadratic equations over a few centuries (Katz, 2007).

If the approaches of Diophantus, Euclid and Al-Khwarizmi are compared; whereas Euclid presented old Babylonian algebra by using advanced geometry, Al-Khwarizmi presented advanced algebra by using geometry of old Babylonians. In addition, Diophantus accepted and used old Babylonians' methods of solution while Al-Khwarizmi developed those methods and introduced modern forms.

In addition to these mathematicians, there were many mathematicians who contributed to the development of quadratic equations. Al-Karkhi, Savadorsa, Ibn Erza, Immanuel Bonfils maintained their studies by starting from the tradition of Babylonians and Egyptians (Gandz, 1937). The Hindu mathematician Āryabhata gave a rule for the sum of a geometric series that showed knowledge of the quadratic equations with both solutions (Smith 1951, p. 159; Smith 1953, p. 444), while Brahmagupta (628AD) appeared to have considered only one of them (Smith 1951, p. 159; Smith 1953, pp. 444-445). Similarly, Mahāvīra substantially had the modern rule for the positive root of a quadratic. Srīdhara gave the positive root of the quadratic formula, as stated by Bhāskara, Cardano allowed complex solutions for the quadratic equations, Viete discovered the relations between roots of a quadratic equation and the coefficients and a constant of the equation (Smith 1953, pp. 445-446).

In 17th century, Decartes was interested in solutions of equations. He found a complete solution for both positive and negative imaginary roots of quadratic equations (Katz, 1998, p.448). Prior to Decartes, it was thought that algebraic equations including two unknown could not be solved. The only technique was substituting chosen value for x then finding y depending on these values. However, Decartes presented a general solution for these types of equations. In his book, Geometry, Decartes made an important advance beyond the Greek mathematicians by allowing negative roots and imaginary roots in his structural analysis of equations relating to solid and supersolid problems and his book contained quadratic formula (Cooley, 1993, pp. 95-96).

There were four basic methods of solving quadratic equations throughout history: the square root method, completing square, quadratic formula and factorization. The Chinese composed a basic method for root extraction and developed it to use from quadratic equations to higher numerical equations. The procedures for root extraction were involved in a book whose name was Chiu Chang Suan Shu. In this book, there was a problem of finding the square root of 71824, the procedures of this problem were shown in detail and the geometrical origin of the method for solving numerical equations was illustrated. After finding the square root, the same procedure was adopted to the solutions of more general quadratic equations (Yong, 1970). Therefore it can be said that the square root method for quadratic equation was based on numerical equations. The first solution of quadratic equations was the one given in an ancient Egyptian papyrus (1300-1200BC). This document contained a problem: *"The area of a square of 100 is equal to that of two smaller squares; the side of one square is ¾ of the other. What are the sides of two unknown squares?*" This question presented the solution of quadratic equation by taking squares and square root (Harding and Engelrebcht, 2007). If quadratic equation involved a square and constant, the square was positioned on one side and the constant on the other side. Later, the roots were found by taking the square roots of both sides.

For example,

$$x^2-16=0$$
$$x^2=16$$
$$\sqrt{x^2}=\pm\sqrt{16}$$
$$x=\pm 4,\ \textit{which means}\ x=4\ \textit{and}\ x=-4$$

$$2(x+3)^2-14=0$$
$$2(x+3)^2=14$$
$$(x+3)^2=7$$
$$\sqrt{(x+3)^2}=\pm\sqrt{7}$$
$$x+3=\pm\sqrt{7}$$
$$x=-3\pm\sqrt{7}$$

Figure 2. *Example of Taking the Square Root*

An equation in the form of $x^2+bx=c$ was developed by the Arab mathematician El-Khawarizmi. He referred to the area of square with the term of square and referred to the length of side with the term of root. Therefore, x^2 and x respectively corresponded to a square and a root in modern symbolic algebra. The following is an example of square and roots equal to numbers: a square and ten roots are equal to 39 units.

> *The manner of solving this type of equation is to take roots one half of the roots just mentioned. The roots in the problem are 10 so half of it is 5, which multiplied by itself gives 25, an amount which you add to 39, giving 64. Having taken the square this roots of which is 8, subtract from it the half of the roots 5, leaving 3. The number 3 therefore represents one root of square, which itself is 9, so 9 gives that square (Rosen, 1831, p.8).*

$x^2 + 10x = 39$

Solve for x^2

$\frac{1}{2} . 10 = 5$

$x^2 + 10x + 25 = 39 + 25$

$x^2 + 10x + 25 = 64$

$(x + 5)^2 = 64$

$x + 5 = 8$

$x + 5 - 3 = 8 - 3$

$x = 3$

$x^2 = 9$

Figure 3. *Solution Method of Quadratic Equation*

It showed a modern version of the same solution and it was called completing the square. In the example of El-Khawarizmi, the desired solution was the square, not the root. Solving quadratic equations was also built by the geometric foundation (Boyer & Merzbach, 2011). In 10th century, Abu Kamil developed a set of rules for calculation with the roots and justified this solution method with geometry as the following.

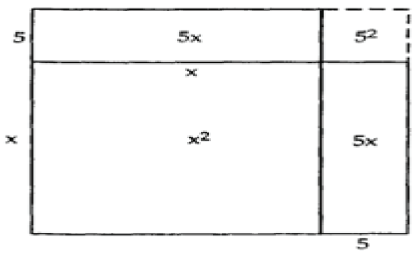

Figure 4. *Quadratic Equation Solution Method with Geometry*

In 17th century Decartes developed completing the square method into quadratic formula by the way of contrast to Abu Kamil. He allowed negative and positive coefficients and roots. The complete solution for all quadratic equations in the form of $ax^2 + bx + c = 0$ was given by quadratic formula as follows.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 5. *General Form of Quadratic Formula*

The concept of factoring numbers into primes came out since Euclid defined what primes were and the idea of unique factorization with the fundamental theorem of arithmetic. Due to the fact that there were no indications about any methods before the time of Fermat, the history of factoring integers started with Pierre de Fermat. In 1643, Fermat produced some interesting ideas for factoring integers and these ideas provided to form the fastest solution methods today. Fermat's idea was to write an integer as the difference between two square numbers, i.e. he tried to find integers x and y such that the composite integer n could be written as n = x^2-y^2 thereby revealing the factors (x+y) and (x−y). Legendre (1798) presented the idea that eventually would revolutionize factoring algorithms. Euler (1750) and Gauss (1800) made contribution to the development of factorization (Jensen, 2005). After recognizing distinct cases for different combinations of the signs for quadratics, the rule for factoring quadratics was reduced to a single case. It includes finding all pairs of factors of the constant term of quadratic then selecting the pair whose algebraic sum equals to coefficient of the linear term and writing the factorization as (x+c)(x+d) (Zhu & Simon, 1987).

1.2. The Investigations Related to Quadratic Equations

Mathematical content related to quadratic equations in the textbooks is similar in all books (Sönnerhed, 2011). Secondary and college algebra introduces students to four techniques for quadratic equations and each of them has its specific strengths and weaknesses. If we tackle these aspects of the techniques, square root method is the easiest method to learn and perform but very few quadratics are in a form which is appropriate for this method. Completing the square provides to determine the solution of any quadratic equation, to graph quadratic equations using transformations and is useful when considering various aspects of conic sections but some students lack confidence or ability to perform the fractional arithmetic that is often necessary within this technique. Quadratic formula determines the solution of any quadratic equation and it is a friendly calculator technique but students lack confidence or ability to perform the radical arithmetic that is often necessary within this technique. Factorization is the quickest method when the quadratic equation is obviously factorable but the quadratic may be prime or difficult to factor, many quadratics equations may seem factorable but may not be and this method is likely to be time consuming (Bosse and Nandakumar, 2005).

The study related to students` errors in learning quadratic equations (Zakaria and Maat, 2010) separates them into types of error in factorization, completing the square and solving quadratic equation using quadratic formula. The most frequent errors made by the students in factorization are related to comprehension, transformation and process skill errors. Students generally misunderstand what the question wants and have difficulty in understanding the meaning of root word. We encounter with this type of error in computation during multiplication. In this process, students make mistakes related to positive and negative signs and replacing value that has a negative sign while writing algebraic expressions. The lowest and most average students get into difficulty in factorization and simplifying algebraic expressions and also performing algebraic operations (Norasiah, 2002; Parish and Ludwig, 1994; Roslina, 1997). These obstacles may result from the lack of emphasis of teachers in teaching the factorization method.

In completing the square, students usually have transformation and process skill errors. Students have difficulty in understanding and describing what is required by the question, thus, this situation causes these errors. Many students are not good at performing the completing the square method and according to this failure they can not solve the problem (Norasiah, 2002; Rahim, 1997). It is difficult for students to transform mathematical problems into mathematical forms and understanding mathematical terms. Because of that, teachers may not use the language of mathematics sufficiently and may not check whether the students have the basic skills for moving to new topics, these mistakes may occur (Zakaria and Maat, 2010). Therefore teachers should pay attention to overcome the deficiencies for learning new topics and also clarify the mathematical language in order to make students understand which concept it corresponds to.

In using quadratic formula while solving quadratic equation, computation and applying the quadratic formula is a common problem. Students have difficulties in operating addition, subtraction, multiplication and division and also replacing the positive and negative signs. In addition, carelessness and encoding may cause mistakes (Liew and Wan Muhammad, 1991; Zakaria and Maat, 2010). Taking notice of the algorithmic skills rather than the explanations related to mathematical concepts and principles may cause the mistakes.

These three common methods (Completing the square, Factorization, The quadratic formula) for solving quadratic equations are discussed in secondary mathematics educational fields in some previous studies (Allaire and Bradley, 2001; Bossé and Nandakumar, 2005; Hoffman, 1976; Leong et al., 2010; Kemp, 2010; Kennedy, Warshauer & Curtin, 1991; Nataraj & Thomas, 2006; Vaiyavutjamai and Clements, 2006; Vinogradova, 2007; Zhu and Simon, 1987). Some mathematics education articles give advice for mathematics teachers to simplify the method of completing the square based on geometrical ideas and then present the simplified version to students (Allaire and Bradley, 2001; Vinogradova, 2007). By starting with a concrete example in which students can understand the content, the teachers may relate mathematics quantity to physical objects in a visual way. Therefore, teachers are likely to facilitate transforming the relationships into algebraic expressions.

Factorization of the quadratic equations is a common didactic topic at secondary level in mathematics education (Bossé and Nandakumar, 2005; Hoffman, 1976; Leong et al., 2010; Kemp, 2010; Kennedy et al., 1991; Nataraj and Thomas, 2006; Zhu and Simon, 1987). There are many methods for doing factorization such as polynomial form, cross multiplication method (Kemp, 2010), the use of concrete algebra tiles which includes changing from algebra tiles to a rectangle diagram (Leong et al., 2010). In the last method, students understand concrete and visual geometrical representation of factorization and improve their awareness of the link to symbolic algebra (Leong et al., 2010).

In the USA, quadratic formula is accepted as standard method for solving quadratic equations (Obermeyer, 1982). When students use quadratic formula, manipulating the parameters, coefficients and symbols in quadratic equations become obstacles for students (Olteanu, 2007). Stover (1978) and Olteanu (2007) suggest using a graph of quadratic function in order to solve the quadratic equation as an alternative method. Although graphical approach is not so common, using algebraic approach for quadratic functions is common. Quadratic equations and functions are considered different from each other and are defined insufficiently but in order to prevent confusion considering them together and comparing is important (Sönnerhed, 2011).

The research of Sönnerhed (2011) shows that there is relationship between elementary algebra and geometry in mathematical history. It provides the knowledge of quadratic equations in five teaching sub-trajectories which develop from basic to more complicated algebra. He supports the fact that the geometrical models can be very useful in teaching quadratic equations in classroom if teachers realize the importance of these models and learn the history of algebra. In addition, sub-trajectories may be useful in terms of forming the environment of teaching quadratic equations.

The studies show that solving and simplifying quadratic equations are discussed as teaching ideas and strategies by mathematics educators in different countries. In order to teach quadratic equations, the perspective involves using geometric approaches (Allaire and Bradley, 2001); utilizing completing squares (Vinogradova, 2007); factoring quadratics (Leong et al., 2010; Kotsopoulos, 2007; Rauff, 1994) and using factorization, completing the square and graphical methods to solve quadratic equations (Bossé and Nandakumar, 2005; MacDonald, 1986; Vaiyavutjamai and Clements, 2006). Only a few of these studies present the results of research studies (e.g. Bossé and Nandakumar 2005; Leong et al., 2010; Vaiyavutjamai and Clements 2006) rather than teaching ideas.

The research of Vaiyavutjamai and Clements (2006) shows that students have misconceptions related to variables in terms of understanding quadratic equation and have difficulties in realizing $x^2 - 8x + 15 = 0$ and $(x - 3)(x - 5) = 0$ are actually equivalent. In addition, although many students obtain correct solutions, it is hard to understand what quadratic equations and their meanings are. According to the researchers the traditional approach to teach quadratic equations may provide development in performance skills but it is not enough to help students understand the relations of quadratic equations. They suggest that quadratic equations may be taught in functions using modern technology such as graphic calculators.

According to Olteanu (2007), students try to understand the parameters, the unknown quantity and function because of seeing the relationships among quadratic formula, a quadratic equation and a quadratic function. She defends using graphical representations in order to make students understand and solve quadratic equations as well as some research but some others advice to make more research related to this topic (Bossé and Nandakumar, 2005, Olteanu, 2007). Vaiyavutjamai and Clements (2006) supports that solving quadratic equations does not mean understanding of quadratics; therefore, teachers should emphasize the explanations of them for students.

There are three forms of quadratic relations in many curriculums (NCTM, 2000; Ontario Ministry of Education (OME), 2005). These are

1. factored form, $y = a(x - r)(x - s)$
2. standard form, $y = ax^2 + bx + c$
3. vertex form, $y = a(x - h)^2 + k$.

These multiple representations of the quadratic relations may cause confusion and difficulties and also the changes in quadratic forms may cause misunderstanding. Thus, students should be encouraged to make connections and to encounter broader multiple representations of quadratic equations (Kotsopouslos, 2007).

The study of Yakes and Star (2011) shows that while teachers represent different solution methods, students are not allowed to have conversations about these solution methods and to have a choice to determine which method is used to solve a given problem by their teachers. It emphasizes the importance of teachers` flexibility in algebra teaching. The researchers think that having side-by-side comparisons of solution methods might help students to see that elimination is often more efficient when the least common multiple of either the coefficients of the x terms (or the y terms respectively) is easy to find. It is important to encourage students to construct their own solution methods or to choose the method that they feel more comfortable. Finally, the results significantly suggest that more time should be spent on questioning and discourse strategies in the classroom, both to encourage teachers to have comparison discussions and to provide them with pedagogical tools to do so.

Abramovich and Norton (2006) address technology based approaches that facilitate teaching of quadratic equation with parameters. It is thought that it can be used as a tool in order to form a quadratic equation formula since the use of computer technology helps students to understand invariance. According to the results, meaning can be constructed with technology through the activities of visualizing, manipulating and analyzing graphs of quadratic equations. Besides, the study emphasizes the development of these teaching activities.

1.3. Implications for Teaching and Learning

When we look over the findings of the investigations related to quadratic equations, we see that there have been many suggestions for teaching and learning. Most of them insist on the difficulties that the students have about quadratic equations and teaching ideas. Developmental levels and knowledge competency of students and also connections between quadratic equations and the other mathematical subjects should be taken into consideration initially and then depending on these factors content of quadratic equations should be presented. We should follow a way in teaching quadratic equations from basic terms to structural forms. Sönnerhed (2011) suggests an order for algebra content which supports this idea: introduction of different polynomials and essential terms, the value of the variable in the polynomial, parenthesis rules, multiplication of two binominals, the difference-of-two squares formula and square rules, factorization by using the difference-of-two squares formula and square rules inversely, using the square root method and null-factor law to solve simple quadratic equations, using the approach of completing the square, using the general formula called quadratic formula.

The history of mathematical ideas should be an important aspect of mathematics teaching. Algebra content like completing the square and factorization are likely to be illustrated by historical examples so historically related pedagogy can be used in presenting quadratic equations (Sönnerhed, 2011). Thus, students realize the reasons behind the use of quadratic equations and methods by mentioning the origins of them. Sharing information related to history engages students` attention, promotes them to apply and make better understanding (NCTM, 2000). Therefore, some word problems such as Chinese classical problem and Babylonian clay tablet problem can be presented. In addition, students do not believe and feel the necessity of algebra as in other subjects of mathematics (Usiskin, 1995). The key point here is making them aware of why they need to learn and the reasons behind what they do. Thus, teachers should illustrate to students by concrete daily life and historical development examples.

In order to prevent the constitution of confusion related to quadratic equations in students mind, the concepts, expressions and their meanings need to be clarified. If teachers make detailed definitions of concepts such as variable, coefficient and explain the reasons of operations, students are going to understand the subject and it may prevent misconceptions. For example, minus can be comprehended as both sign and subtraction so teachers should help students to realize the meaning in equations by explaining. Due to the fact that students make mistakes since the mathematical terminology is not used, teachers must balance between the mathematical concepts and arithmetic skills (Intaku, 2003; Norasiah, 2002; Rahim, 1997; Roslina, 1997). Teachers generally focus on arithmetic skills and finding the answers correctly so that they usually miss the point related to the importance of what they said. Since students usually trust teachers and their construction of knowledge occurs depending on the teachers' explanations and definitions, teachers should pay attention to use mathematical language, especially correct language, to clear up students' questions in their minds and to prevent misconceptions.

Using geometrical models provides to connect with algebra and geometrical images. Thus, in order to teach quadratic equations, teachers should apply geometrical images, modeling as in cut paste geometry of Babylonians and El-Khwarazmi. Allaire and Bradely (2001), Leong et al. (2010) and Sönnerhed (2011) support this idea as well. According to Sönnerhed (2011) geometrical modeling may be applied to illustrate the abstract content of formulization and quadratic algebra and also the multi-functions of the geometrical models are likely to be useful and powerful alternative approach in teaching quadratics. In addition, using the geometrical figures provides opportunities for the student to understand the process of approaching the method of completing the square visually in terms of didactic perspective and facilitates transforming the relationships into algebraic expressions.

Quadratic equations and functions are interrelated subjects and they should not be considered as one apart from the other. So teachers can utilize functions in teaching quadratic equations to get students make different meanings of equations and to present a different form. Using functions can be useful because students have difficulty in recognizing different meanings and relationships in equations. Vaiyavutjamai and Clements (2006), Olteanu (2007) and Sönnerhed (2011) suggest teaching quadratic equations with functions and support these two should not be treated as separate subjects and after giving detailed definitions, comparing them can prevent conceptual confusion. Because these two subjects complete each other and learning one of them affects the other, teachers should pay attention to using the relationships between them.

López, Robles and Martínez-Planell (2016) indicate that students do not solve quadratic equations as a process of reversion because they do not generally understand the basic properties of the square root. Therefore, Lopez et al. (2016) suggest that "more time and attention should be given to the design of activities and exploration of the basic properties given by (1) = |x| and (2) if = a then x = ±" (p. 570). They also emphasize that "students should be given the opportunity to interiorize the 'do the same thing on both sides' step–by-step process that relates these properties: if a ≥ 0 and = a then = , so that |x| = , and hence x = ±" (p. 570).

Using graphical representations helps students to understand quadratic equations. It is useful in terms of concretization and visualization. Students can realize in which point there are intersections and roots and also see that quadratic equation correspond to which type of mathematical object. Therefore, students can make some generalizations about quadratic equations by understanding and interpreting the relationships between expression and graph. Olteanu (2007) suggests that the extreme point of a quadratic function could be handled by using the derivate, it may be more comprehensible for students. Besides, graphical methods to solve quadratic equations are also supported by some other reseachers (Bossé and Nandakumar, 2005; MacDonald, 1986; Vaiyavutjamai and Clements, 2006).

As Bosse and Nandakumar (2005) say, four basic solution methods of quadratic equations have strong and weakness aspects. Whereas one student prefers factorization, the other may choose to use completing the square. It changes from one to another since their levels of feeling comfortable with methods are different. Moreover, decision of which method to use may be hard for students depending on the questions. According to Bosse and Nandakumar

(2005), the solving method depends on the type of quadratic equations, the coefficients and constants in a quadratic equation. Hence, teachers should pay attention to teaching each method in detail and allow students to use whichever they prefer or to develop alternative approaches to solve quadratic equations. It presents opportunities for students to get deep insight into structure so choices must be made (Kendal and Stacey, 2004, p. 345).

The mathematics exercises should be organized for different cognitive areas and they should include quadratic equations structure sense, mathematical proofs, application of quadratic equations, operational rules and relational understanding of variables and parameters of quadratic equations. Different kinds of exercises, problems and activities may be useful for teachers to organize and help students in their own learning (Sönnerhed, 2011). Therefore, choosing appropriate exercises in terms of need and level of students is likely to consolidate students' knowledge related to quadratic equations.

Different ways of representing and formulating quadratic equations make it comprehensible for the students (Shulman, 1986), on the other hand, it may be confusing (Kotsopouslos, 2007). Therefore, teachers should present different forms and methods related to quadratic equations to help students gain alternative perspectives to solve quadratic equations but, at this process, they should help students to make connection among these multiple representations or they should explain relationships.

Technology can be used in teaching quadratic equations. By applying technology completing the square, factorization and quadratic formula methods are likely to be shown, students understand the relationships visually, see mathematical object which is relevant to quadratic equation and make some changes on these figures and simultaneously observe the changes in quadratic expression. The solutions obtained with technology present new ways of thinking about quadratic equations and their graphs. Many researchers support the use of technology in teaching quadratic equations (Abramovich and Norton, 2006; Sangwing, 2007; Stols, 2004; Vaiyavutjamai and Clements 2006).

When the studies related to quadratic equations are considered, it is seen that there is research that examines the strengths and weaknesses of solution methods (Bosse and Nandakumar, 2005; Leong et al., 2010), misconceptions (Vaiyavutjamai and Clements, 2006) and types of error in these methods (Norasiah, 2002; Parish and Ludwig, 1994; Roslina, 1997; Zakaria and Maat, 2010). Some of them reveal the difficulties that students face in learning quadratic equations (Norasiah, 2002; Olteanu, 2007; Rahim, 1997; Vaiyavutjamai and Clements, 2006) and some of them make suggestions to improve teaching of this subject (Abramovich and Norton, 2006; Allaire and Bradley, 2001; Kotsopouslos, 2007; Olteanu, 2007; Sönnerhed, 2011; Stover, 1978; Vinogradova, 2007; Yakes and Star, 2011). Although there is some research, it is suggested to make more research related to this topic (Bossé and Nandakumar, 2005, Olteanu, 2007).

If we summarize the implications in teaching and learning of quadratic equations, some suggestions can be made. They are ordering from basic terms to structural forms, sharing the history of mathematical ideas, illustrating the methods to solve quadratic equations in detail, making definition of concepts, explanation of procedure, clarifying the meanings of expressions, using mathematical language correctly, applying geometrical representations, utilizing the relationships between quadratic equations and functions, using graphical representations, providing flexibility in preferring the solution method, presenting different exercises of quadratic equations, making connections between multiple representations and using technology.

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