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Special Issue Paper

Superlinear elliptic hemivariational inequalities

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Dedicated to Prof. Stanislaw Migorski on the occasion of his 60th birthday

Abstract

We study a nonlinear nonhomogeneous Dirichlet problem with a nonsmooth potential which is superlinear but without satisfying the Ambrosetti-Rabinowitz condition. Using the nonsmooth critical point theory and critical groups we prove two multiplicity theorems producing three and five solutions respectively. In the second multiplicity theorem, we provide sign information for all the solutions and the solutions are ordered.

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1. Introduction

In this paper we study the following nonlinear, nonhomogeneous elliptic differential inclusion

$$\begin{cases} -\operatorname{div} a(Du(z)) \in \partial j(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

In this boundary value problem $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. The map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ involved in the definition of the differential operator is continuous, strictly monotone (thus maximal monotone too) and satisfies certain other regularity and growth conditions which are listed in hypotheses H_0 (see Section 2). Those conditions are general and incorporate in our framework differential operators that we encounter in applications such as the p-Laplacian and the (p, q)-Laplacian (double phase problems with balanced growth). The reaction (right hand side) of (1.1) is set-valued and it is the generalized (Clarke) subdifferential of the locally Lipschitz integrand $j(z, \cdot)$ (see Section 2). So, we are dealing with a hemivariational inequality (a variational inequality with a nonconvex superpotential). Such inequalities arise in problems of contact mechanics (see Migórski-Ochal-Sofonea [17], Panagiotopoulos [20]).

We assume that $j(z, \cdot)$ is "p-superlinear" but need not satisfy the Ambrosetti-Rabinowitz condition, which is common in the literature when dealing with superlinear problems. The

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Ambrosetti-Rabinowitz condition is convenient in order to verify that the energy functional of the problem satisfies a compactness condition which is essential in the minimax theorems of the critical point theory. This compactness condition is the Palais-Smale condition or the more general Cerami condition. The two are equivalent if the energy functional is bounded below. In the case of problem (1.1) the superlinearity of the potential function $j(z,\cdot)$ implies that the energy functional is not bounded from below. Here we replace the Ambrosetti-Rabinowitz condition by a less restrictive one, which allows also superlinear nonlinearities with slower growth as $x \to \pm\infty$ that fail to satisfy the Ambrosetti-Rabinowitz condition.

Our aim in this paper is to provide multiplicity results for problem (1.1), providing sign information for all the solutions produced. The first multiplicity theorem for superlinear problems was proved by Wang [30], for semilinear Dirichlet equation driven by the Laplacian and with a smooth potential (hence with a single-valued right hand side). Wang [30] obtained three nontrivial nonsmooth solutions, two of constant sign (positive and negative) and a third of undetermined sign. Wang [30] initiated the use of Morse theory (critical groups) in the search for multiple solutions of elliptic equations. Extensions to nonlinear equations can be found in the papers of Aizicovici-Papageorgiou-Staicu [1], Bartsch-Liu [3], Liu [16], Papageorgiou-Winkert [25]. For problems with a nonsmooth potential (hemivariational inequalities), we mention the work of Papageorgiou-Rădulescu-Repovš [22], who obtained nodal solutions but not for superlinear equations.

2. Mathematical background - hypotheses

The main function spaces in the analysis of problem (1.1) are the Sobolev space $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\}.$$

By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : \ u(z) \geqslant 0 \quad \forall z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\mathrm{int} C_+ = \left\{u \in C_+ : \ u(z) > 0 \quad \forall z \in \Omega, \ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0\right\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with n being the outward unit normal on $\partial\Omega$.

Consider a function $\eta \in C^1(0,\infty)$ which satisfies

$$0 < \widehat{c} \leqslant \frac{t\eta'(t)}{\eta(t)} \leqslant c_0 \quad \text{and} \quad c_1 t^{p-1} \leqslant \eta(t) \leqslant c_2(t^{s-1} + t^{p-1}) \quad \forall t \geqslant 0,$$

for some $c_1, c_2 > 0$ and $1 < s < p$. Using η we can introduce our hypotheses on the map a:

<u>H_0:</u> $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

- **(i):** $a_0 \in C^1(0,\infty)$, $t \mapsto a_0(t)t$ is strictly increasing, $\lim\limits_{t \to 0^+} a_0(t)t = 0$ and $\lim\limits_{t \to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1$;
- **(ii):** there exists $c_3 > 0$ such that $|\nabla a(y)| \leqslant c_3 \frac{\eta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$
- **(iii):** we have $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant \frac{\eta(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv): if $G_0(t) = \int_0^t a_0(s)s\,ds$ for $t \geqslant 0$, then

$$0 \leqslant pG_0(t) - a_0(t)t^2 \quad \forall t \geqslant 0$$

and there exists $q \in (1, p]$ such that $0 < \widehat{c}_0 \leqslant \liminf_{t \to 0^+} \frac{qG_0(t)}{t^q}$.

Remark 2.1. Hypotheses $H_0(i), (ii), (iii)$ come from the nonlinear regularity theory of Lieberman [15] and the nonlinear maximum principle of Pucci-Serrin [27]. These are hypotheses similar to those of problems defined on generalized Orlicz spaces. Hypothesis $H_0(iv)$ serves the particular needs of our problem, but it is mild and it is verified in all situations of interest (see the Examples below).

Note that the primitive G_0 is strictly increasing and strictly convex. Let $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then G is convex and we have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \forall y \in \mathbb{R}^N \setminus \{0\}.$$

Therefore G is the primitive of a. Then from the convexity of G and since $G(0) = 0$, we have

$$G(y) \leqslant (a(y), y)_{\mathbb{R}^N} \quad \forall y \in \mathbb{R}^N. \tag{2.1}$$

Hypotheses $H_0(i), (ii), (iii)$ lead to the following properties of the map a (see Papageorgiou-Rădulescu [21]).

Lemma 2.2. *If hypotheses $H_0(i), (ii), (iii)$ hold, then*
(a) *the map $y \mapsto a(y)$ is maximal monotone and strictly monotone;*
(b) $|a(y)| \leqslant c_4(|y|^{s-1} + |y|^{p-1})$ *for some $c_4 > 0$, all $y \in \mathbb{R}^N$;*
(c) $(a(y), y)_{\mathbb{R}^N} \geqslant \frac{c_1}{p-1}|y|^p$ *for all $y \in \mathbb{R}^N$.*

Using this lemma and (2.1), we infer the following bilateral growth condition for the primitive G.

Corollary 2.3. *If hypotheses $H_0(i), (ii), (iii)$ hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant c_5(1 + |y|^p) \quad \forall y \in \mathbb{R}^N,$$

for some $c_5 > 0$.

Remark 2.4. According to this corollary G exhibits balanced growth and this leads to a global (up to boundary of Ω) regularity theory (see Lieberman [15]). Naturally this enriches the set of tools available for the analysis of (1.1).

Example 2.5. The following maps $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfies hypotheses H_0.
(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$.
This map corresponds to the p-Laplace operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \forall u \in W_0^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$. This map corresponds to the (p, q)-Laplace operator (double phase problems with balanced growth) defined by

$$\Delta_p u + \Delta_q u = \operatorname{div}((|Du|^{p-2} + |Du|^{q-2})Du) \quad \forall u \in W_0^{1,p}(\Omega).$$

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y$ with $1 < p < \infty$.
This map corresponds to the generalized p-mean curvature operator defined by

$$\operatorname{div}\left((1 + |Du|^2)^{\frac{p-2}{2}}Du\right) \quad \forall u \in W_0^{1,p}(\Omega).$$

Such operators arise in problems of plasticity theory (see Roubíček [29]).

Let $V\colon W_0^{1,p} \longrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $(\frac{1}{p}+\frac{1}{p'}=1)$ be the nonlinear operator defined by

$$\langle V(u),h\rangle = \int_\Omega (a(Du),Dh)_{\mathbb{R}^N}\,dz \quad \forall u,h \in W_0^{1,p}(\Omega).$$

From Gasiński-Papageorgiou [13, Problem 2.192, p. 279], we know that this operator has the following properties:

Proposition 2.6. *If hypotheses $H_0(i),(ii),(iii)$ hold, then $V\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and it is of type $(S)_+$, that is "$u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup\limits_{n\to+\infty}\langle V(u_n),u_n-u\rangle \leqslant 0$ imply that $u_n \to u$ in $W_0^{1,p}(\Omega)$".*

The nonsmoothness of the potential function requires the use of the nonsmooth critical point theory which is based on the subdifferential theory of locally Lipschitz functions due to Clarke [5].

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle_X$ we denote the duality brackets for the pair (X,X^*). A function $\varphi\colon X\to\mathbb{R}$ is said to be "locally Lipschitz", if for every $u\in X$, we can find a neighbourhood U of u and a constant $k_U>0$ such that

$$|\varphi(y)-\varphi(v)| \leqslant k_U\|y-v\|_X \quad \forall y,v\in U.$$

So, let $\varphi\colon X\longrightarrow\mathbb{R}$ be a locally Lipschitz function. The "generalized directional derivative" of φ at $u\in X$ in the direction h, denoted by $\varphi^0(x;h)$ is defined by

$$\varphi^0(u;h) = \limsup_{\substack{u'\to u\\ \lambda\to 0^+}} \frac{\varphi(u'+\lambda h)-\varphi(u')}{\lambda}.$$

This map has the following properties
(a) $\varphi^0(u;\cdot)$ is sublinear and Lipschitz continuous.
(b) $(u,h)\mapsto\varphi^0(u;h)$ is upper semicontinuous.
(c) $\varphi^0(u;-h) = (-\varphi)^0(u;h)$ for all $u,h\in X$.

From **(a)** above and the Hahn-Banach theorem, we see that we can define the set

$$\partial\varphi(u) = \{u^*\in X^*:\ \langle u^*,h\rangle_X \leqslant \varphi^0(u;h) \text{ for all } h\in X\}.$$

This multifunction $u\mapsto\partial\varphi(u)$ is known as the "generalized (or Clarke) subdifferential" of φ. If $\varphi\in C^1(X)$, then φ is locally Lipschitz and we have

$$\partial\varphi(u) = \{\varphi'(u)\}.$$

We know that if $\varphi\colon X\longrightarrow\mathbb{R}$ is continuous convex, then φ is locally Lipschitz and the generalized subdifferential $\partial\varphi(u)$ coincides with the subdifferential in the sense of convex analysis $\partial_c\varphi(u)$ defined by

$$\partial_c\varphi(u) = \{u^*\in X^*:\ \langle u^*,v-u\rangle \leqslant \varphi(v)-\varphi(u) \text{ for all } v\in X\}.$$

The generalized subdifferential has the following properties:
(a) $\partial\varphi(u)\neq\emptyset$ and it is convex and w^*-compact.
(b) If $\varphi,\psi\colon X\longrightarrow\mathbb{R}$ are locally Lipschitz, then

$$\partial(\varphi+\psi)(u) \subseteq \partial\varphi(u)+\partial\psi(u) \quad \forall u\in X$$

and equality holds if and only if one of them is a singleton.
(c) $\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u)$ for all $\lambda\in\mathbb{R}$;
(d) The multifunction $u\mapsto\partial\varphi(u)$ is upper semicontinuous from X with the norm topology into X^* with the weak topology (denoted by $X^*_{w^*}$) that is, for all open sets $U\subseteq X^*_{w^*}$, the set $\partial\varphi^+(C) = \{u\in X:\ \partial\varphi(u)\subseteq U\}$ is norm open.
(e) If u is local extremum of φ, then $0\in\partial\varphi(u)$ (Fermat's law).

Generalizing the notion of critical points, we say that $u \in X$ is a critical point of the locally Lipschitz function φ, if $0 \in \partial\varphi(u)$. Then K_φ denotes the set of critical points of φ, that is, $K_\varphi = \{u \in X : \ 0 \in \partial\varphi(u)\}$. We set

$$m_\varphi(u) = \inf\left\{\|u^*\|_{X^*} : \ u^* \in \partial\varphi(u)\right\} \quad \forall u \in X.$$

Since $\|\cdot\|_{X^*}$ is w^*-lower semicontinuous and $\partial\varphi(u) \subseteq X^*$ is w^*-compact, from the Weierstrass-Tonelli theorem, we infer that the infimum in the above definition is actually attained.

Definition 2.7. We say that the locally Lipschitz function satisfies the "nonsmooth Cerami condition", if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that the sequence $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|_X)m_\varphi(u_n) \longrightarrow 0 \quad \text{as } n \to \infty,$$

admits a strongly convergent subsequence.

Remark 2.8. Evidently, if $\varphi \in C^1(X)$, then this definition coincides with the classical Cerami condition (see Gasiński-Papageorgiou [12, p. 611]).

This is a compactness-type condition on the functional φ which compensates for the fact that the ambient space need not be locally compact (being infinite dimensional). It leads to a deformation theorem from which follow the minimax theorems of the nonsmooth critical point theory (see Chang [4], Gasiński-Papageorgiou [11]). We will need the following nonsmooth version of the well-known mountain pass theorem of Ambrosetti-Rabinowitz [2].

Theorem 2.9. *If X is a reflexive Banach space, $\varphi \colon X \longrightarrow \mathbb{R}$ is locally Lipschitz which satisfies the nonsmooth Cerami condition, there exist $u_0, u_1 \in X$ and $r > 0$ such that*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = r\} = m, \ \|u_1 - u_0\| > r$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)),$$

with $\Gamma = \{\gamma \in C([0,1]; X) : \ \gamma(0) = u_0, \ \gamma(1) = u_1\}$,
then $m \leqslant c$ and c is a critical value of φ (that is, there exists $u \in K_\varphi$ such that $\varphi(u) = c$). Moreover if $c = m$, then

$$K_\varphi \cap \{u \in X : \ \|u - u_0\| = r\} \neq \emptyset.$$

In the analysis of (1.1) we will also use the nonsmooth Morse theory (critical groups). This is an extension of the classical (smooth) Morse theory and can be found in Corvellec [6]. For the smooth theory and the relevant notions from singular homology, we refer to Papageorgiou-Rădulescu-Repovš [23, Chapter 6].

Let $B \subseteq A \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(A, B)$ we denote the k-th singular homology group with real coefficient. Hence $H_k(A, B)$ are actually linear spaces. With this choice of coefficients we avoid torsion phenomena. Let $\varphi \colon X \longrightarrow \mathbb{R}$ be a locally Lipschitz functional which satisfies the nonsmooth Cerami condition (see Definition 2.7). Let $u \in K_\varphi$ be isolated and let $c = \varphi(u)$. We define $\overline{\varphi}^c = \{u \in X : \ \varphi(u) \leqslant c\}$. Then the k-th critical group of φ at u is defined by

$$C_k(\varphi, u) = H_k(\overline{\varphi}^c \cap U, \overline{\varphi}^c \cap U \setminus \{u\})$$

with U being the neighbourhood of u such that $K_\varphi \cap U = \{u\}$ (recall that u is isolated). The excision property of singular homology implies that this definition is independent of the choice of the isolating neighbourhood U.

As in the proof of Proposition 6.2.16 of Papageorgiou-Rădulescu-Repovš [23, p. 486], using this time the nonsmooth second order deformation lemma of Corvellec [7], we have the following decomposition result in terms of the critical groups.

Proposition 2.10. *If $a < b \leqslant \infty$, $\varphi(K_\varphi) \cap [a, b] = \{c\}$ with $a < c < b$ and $K_\varphi^c = \{u \in K_\varphi :\ \varphi(u) = c\}$ is finite, then for all $k \in \mathbb{N}_0$, we have*

$$H_k(\overline{\varphi}^b, \overline{\varphi}^a) = \bigoplus_{u \in K_\varphi^c} C_k(\varphi, u)$$

(if $b = +\infty$, then $\overline{\varphi}^b = X$).

This result leads to the so-called "Morse relation" (see Papageorgiou-Rădulescu-Repovš [23, Theorem 6.2.20, p.489]). First let us introduce the critical groups of φ at infinity. We assume that $\inf \varphi(K_\varphi) > -\infty$. Then the critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \overline{\varphi}^c) \quad \forall k \in \mathbb{N}_0,$$

with $c < \inf \varphi(K_\varphi)$. The nonsmooth second deformation theorem of Corvellec [7] guarantees that this definition is independent of the choice of the level c.

We assume that K_φ is finite and introduce the following polynomials

$$\begin{aligned} M(t, u) &= \sum_{k \in \mathbb{N}_0} \dim C_k(\varphi, u) t^k \quad \forall u \in K_\varphi, \\ P(t, u) &= \sum_{k \in \mathbb{N}_0} \dim C_k(\varphi, \infty) t^k. \end{aligned}$$

Then the Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \forall t \in \mathbb{R}, \tag{2.2}$$

with $Q(t)$ being a polynomial in $t \in \mathbb{R}$ with nonnegative integer coefficients (see Papageorgiou-Rădulescu-Repovš [23, p. 492]).

We will also need some facts about the spectrum of the Dirichlet q-Laplacian ($1 < q < p$). So, we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_q u(z) = \widehat{\lambda} |u(z)|^{q-2} u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.3}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue, if (2.3) has a nontrivial solution $\widehat{u} \in W_0^{1,q}(\mathbb{R})$ known as an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. Acting on (2.3) with $\widehat{u}$ we see that $\widehat{\lambda} \geqslant 0$. In fact, (2.3) has a smallest eigenvalue $\widehat{\lambda}_1(q)$ such that

- $\widehat{\lambda}_1(q) > 0$ and it is isolated and simple (that is, if $\widehat{u}_1$, $\widehat{u}_2$ are eigenfunctions corresponding to $\widehat{\lambda}_1(q)$, then $\widehat{u}_1 = k \widehat{u}_2$ with $k \in \mathbb{R}$).
- $\widehat{\lambda}_1(q) = \inf \left\{ \dfrac{\|Du\|_q^q}{\|u\|_q^q} :\ u \in W_0^{1,q}(\Omega),\ u \neq 0 \right\}$.

The infimum in the above formula is realized on the corresponding one-dimensional eigenspace, the elements of which do not change sign. In fact $\widehat{\lambda}_1(q) > 0$ is the only eigenvalue with eigenfunctions that have a constant sign. All the other eigenvalues have eigenfunctions which are nodal functions.

The above properties, lead to the following lemma (see Mugnai-Papageorgiou [18, Lemma 4.11]).

Lemma 2.11. *If $\vartheta_0 \in L^\infty(\Omega)$, $\vartheta_0(z) \leqslant \widehat{\lambda}_1(q)$ for almost all $z \in \Omega$ and $\vartheta_0 \not\equiv \widehat{\lambda}_1(q)$, then there exists $\widehat{c} \in (0, 1)$ such that*

$$\widehat{c} \|Du\|_q^q \leqslant \|Du\|_q^q - \int_\Omega \vartheta(z) |u|^q \, dz \quad \forall u \in W_0^{1,q}(\Omega).$$

Let $u \colon \Omega \longrightarrow \mathbb{R}$ be a measurable function. We define

$$u^\pm(z) = \max\{\pm u(z), 0\} \quad \forall z \in \Omega.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$. If $u, v\colon \Omega \longrightarrow \mathbb{R}$ are measurable functions and $u(z) \leqslant v(z)$ for all $z \in \Omega$, then we define

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : \ u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega\}.$$

Also by $\operatorname{int}_{C_0^1(\Omega)}[u, v]$ we denote the interior in $C_0^1(\overline{\Omega})$ of $[u, v] \cap C_0^1(\overline{\Omega})$. Finally by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad N \leqslant p. \end{cases}$$

We introduce the hypotheses on the potential function $j(z, x)$.

$\underline{H_1}$: $j\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that for all $x \in \mathbb{R}$, $z \mapsto j(z, x)$ is measurable, for a.a. $z \in \Omega$, $x \mapsto j(z, x)$ is locally Lipschitz, $j(z, 0) = 0$ for a.a. $z \in \Omega$ and

- **(i):** $|u^*| \leqslant a(z)(1+|x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $u^* \in \partial j(z, x)$ with $a \in L^\infty(\Omega)$, $p < r < p^*$;
- **(ii):** $\displaystyle\lim_{x \to \pm\infty} \frac{j(z, x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
- **(iii):** there exists $\mu \in \left((r-p)\max\left\{\frac{N}{p}, 1\right\}, p^*\right)$ such that

$$0 < \beta_0 \leqslant \liminf_{x \to \pm\infty} \min_{u^* \in \partial j(z,x)} \frac{u^* x - p j(z, x)}{|x|^\mu}$$

  uniformly for a.a. $z \in \Omega$;
- **(iv):** if $\widehat{c}_0 > 0$ and $q \in (1, p]$ are as in hypothesis $H_0(iv)$, then there exists $\vartheta \in L^\infty(\Omega)$ such that

$$\vartheta(z) \leqslant \widehat{c}_0 \widehat{\lambda}_1(q) \quad \text{for a.a. } z \in \Omega, \quad \vartheta \neq \widehat{c}_0 \widehat{\lambda}_1(q),$$

$$\limsup_{x \to 0} \frac{q j(z, x)}{|x|^q} \leqslant \vartheta(z)$$

  uniformly for a.a. $z \in \Omega$;
- **(v):** for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto u^* + \widehat{\xi}_\varrho |x|^{p-2} x,$$

  for all $u^* \in \partial j(z, x)$, is nondecreasing on $[-\varrho, \varrho]$.

Remark 2.12. From hypotheses $H_1(ii), (iii)$ it follows that

$$\lim_{x \to \pm\infty} \left(\min_{u^* \in \partial j(z,x)} \frac{u^*}{|x|^{p-2} x} \right) = +\infty,$$

uniformly for a.a. $z \in \Omega$.

So, the multivalued right hand side of problem (1.1) is $(p-1)$-superlinear. Note though, that we do not employ the Ambrosetti-Rabinowitz condition, which in the present nonsmooth setting, has the following form: "There exist $\tau > p$ and $M > 0$ such that

$$0 < \tau j(z, x) \leqslant u^* x$$

for a.a. $z \in \Omega$, all $|x| \geqslant M$, all $u^* \in \partial j(z, x)$ and

$$\operatorname*{ess\,inf}_{\Omega} j(\cdot, x) > 0 \quad \text{for all } |x| \geqslant M''.$$

Integrating this condition, we obtain the weaker requirement

$$c_6 |x|^\tau \leqslant j(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geqslant M,$$

with some $c_6 > 0$. Here we replace the Ambrosetti-Rabinowitz condition with the weaker hypothesis $H_1(iii)$ $(p < \tau)$. The following function satisfies hypotheses H_1 but fails to satisfy the Ambrosetti-Rabinowitz condition. For the sake of simplicity we drop the z-dependence

$$j(x) = \begin{cases} \frac{\vartheta}{q}|x|^q & \text{if } \ |x| \leqslant 1, \\ \frac{|x|^p}{p}(\ln|x| - \frac{1}{p}) + c|x|^\mu & \text{if } \ 1 < |x|, \end{cases}$$

with $1 < \mu$, $q \leqslant p$, $\vartheta < \widehat{c}_0\widehat{\lambda}_1(q)$ and $c = \frac{\vartheta}{q} + \frac{1}{p^2}$.

Let $\varphi\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \int_\Omega G(Du)\,dz - \int_\Omega j(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Also, we introduce the positive and negative truncation of φ, namely the functionals

$$\varphi_\pm(u) = \int_\Omega G(Du)\,dz - \int_\Omega j(z,\pm u^\pm)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

From Clarke [5, p.83], we know that the functionals φ, $\varphi_\pm$ are locally Lipschitz.

3. Three solutions

In this section, using hypotheses H_0 and H_1, we will prove a multiplicity theorem producing three nontrivial smooth solutions. We provide sign information for the two, the sign of the third is undetermined. Our result here extends the multiplicity theorem of Wang [30] (semilinear equations driven by the Laplacian with a smooth potential) and of Aizicovici-Papageorgiou-Staicu [1] (nonlinear Neumann problems driven by the p-Laplacian and with a smooth potential).

First we show that the functionals φ and $\varphi_\pm$ satisfy the nonsmooth compactness condition (see Definition 2.7).

Proposition 3.1. *If hypotheses H_0, H_1 hold, then the functionals φ and $\varphi_\pm$ satisfy the nonsmooth Cerami condition.*

Proof. First we do the functionals φ.

We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\varphi(u_n)| \leqslant c_7 \quad \forall n \in \mathbb{N}, \tag{3.1}$$

for some $c_7 > 0$ and

$$(1 + \|u_n\|)m_\varphi(u_n) \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{3.2}$$

The w-compactness of $\partial\varphi(u_n) \subseteq W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ and the weak lower semicontinuity of the norm functional $\|\cdot\|_*$ of $W^{-1,p'}(\Omega)$ imply that we can find $u_n^* \in \partial\varphi(u_n)$ such that $m_\varphi(u_n) = \|u_n^*\|_*$. Then from (3.1), we have

$$|\langle u_n^*, h\rangle| \leqslant \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

with $\varepsilon_n \to 0^+$. Note that $u_n^* = V(u_n) - g_n^*$ with $g_n^* \in S^{r'}_{\partial j(\cdot,u_n(\cdot))} = \{g \in L^{r'}(\Omega) : \ g(z) \in \partial j(z,u_n(z))$ for a.a. $z \in \Omega\}$ (see Clarke [5, p. 83]). Then

$$\left|\langle V(u_n), h\rangle - \int_\Omega g_n^* h\,dz\right| \leqslant \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \quad \forall n \in \mathbb{N}. \tag{3.3}$$

In (3.3) we use the test function $h = u_n \in W_0^{1,p}(\Omega)$ and obtain

$$-\int_\Omega (a(Du_n), Du_n)_{\mathbb{R}^N}\,dz + \int_\Omega g_n^* u_n\,dz \leqslant \varepsilon_n \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Also from (3.1), we have

$$\int_\Omega pG(Du_n)\,dz - \int_\Omega pj(z,u_n)\,dz \leqslant pc_7 \quad \forall n \in \mathbb{N}. \tag{3.5}$$

We add (3.4) and (3.5) we obtain

$$\int_\Omega (pG(Du_n) - (a(Du_n), Du_n)_{\mathbb{R}^N})\, dz + \int_\Omega (g_n^* u_n - pj(z, u_n))\, dz \leqslant c_8 \quad \forall n \in \mathbb{N},$$

for some $c_8 > 0$, so

$$\int_\Omega (g_n^* u_n - pj(z, u_n))\, dz \leqslant c_8 \quad \forall n \in \mathbb{N} \tag{3.6}$$

(see hypothesis $H_1(iv)$). Hypotheses $H_1(i), (iii)$ imply that there exist $\beta_1 \in (0, \beta_0)$ and $c_9 > 0$ such that

$$\beta_1 |u_n(z)|^\mu - c_9 \leqslant g_n^*(z) u_n(z) - pj(z, u_n(z)) \quad \text{for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}. \tag{3.7}$$

We return to (3.6) and use (3.7). We obtain

$$\beta_1 \|u_n\|_\mu^\mu \leqslant c_{10} \quad \forall n \in \mathbb{N},$$

for some $c_{10} > 0$, so

$$\text{the sequence } \{u_n\}_{n\in\mathbb{N}} \subseteq L^\mu(\Omega) \text{ is bounded.} \tag{3.8}$$

From hypothesis $H_1(iii)$ it is clear that we can always assume that $\mu < r < p^*$. Let $t \in (0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{p^*}. \tag{3.9}$$

The interpolation inequality (see Papageorgiou-Winkert [24, p. 116]) implies that

$$\|u_n\|_r \leqslant \|u_n\|_\mu^{1-t} \|u_n\|_{p^*}^t,$$

so

$$\|u_n\|_r^r \leqslant c_{11} \|u_n\|^{tr} \quad \forall n \in \mathbb{N}, \tag{3.10}$$

for some $c_{11} > 0$. Here we have used (3.8) and the fact that embedding $W_0^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$ is continuous if $p \neq N$ (Sobolev embedding theorem).

So, first we assume that $p \neq N$ and we have

$$p^* = \frac{Np}{N-p} \text{ if } p < N \quad \text{and} \quad p^* = +\infty \text{ if } N < p.$$

From (3.9), we have

$$tr = \frac{p^*(r-\mu)}{p^* - \mu} \text{ if } p < N \quad \text{and} \quad tr = r - \mu \text{ if } N < p,$$

so

$$tr < p \tag{3.11}$$

(see hypothesis $H_1(iii)$).

In (3.3) we use the test function $h = u_n \in W_0^{1,p}(\Omega)$ and have

$$\int_\Omega (a(Du_n), Du_n)_{\mathbb{R}^N}\, dz \leqslant \varepsilon_n + \int_\Omega g_n^* u_n\, dz \leqslant c_{12}(1 + \|u_n\|_r^r) \quad \forall n \in \mathbb{N},$$

for some $c_{12} > 0$ (see hypothesis $H_1(i)$), so

$$\frac{c_1}{p-1} \|Du_n\|_p^p \leqslant c_{13}(1 + \|u_n\|^{tr}) \quad \forall n \in \mathbb{N}$$

for some $c_{13} > 0$ (see (3.10) and Lemma 2.2), so

$$\text{the sequence } \{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded} \tag{3.12}$$

(see (3.11)).

Next let $p = N$. In this case $p^* = +\infty$, but the embedding $W_0^{1,p}(\Omega) \subseteq L^s(\Omega)$ is continuous (in fact compact) for all $1 \leqslant s < \infty$. So, in the previous argument, we need to replace p^* by $s > r > \mu$. As before let $t \in (0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{s},$$

so

$$tr = \frac{s(r-\mu)}{s-\mu} \longrightarrow r - \mu < p \quad \text{as } s \to +\infty$$

(see hypothesis $H_1(iii)$). Choosing $s > r$ big we have $tr < p$ and so again we have (3.12).

Because of (3.12), we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega), \quad u_n \longrightarrow u \quad \text{in } L^r(\Omega). \tag{3.13}$$

In (3.3) we use the test function $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.13). We obtain

$$\lim_{n\to+\infty} \langle V(u_n), u_n - u\rangle = 0,$$

so

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p}(\Omega)$$

(see Proposition 2.6), thus φ satisfies the Cerami condition.

Now we do the proof for the functional φ_+.

Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ be a Cerami sequence. As before, we have

$$|\langle \widehat{u}_n^*, h\rangle| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall n \in \mathbb{N},\ h \in W_0^{1,p}(\Omega),$$

with $\varepsilon_n \longrightarrow 0^+$ and for some $\widehat{u}_n^* \in \partial\varphi(u_n)$. We have

$$\widehat{u}_n^* = V(u_n) - \widehat{g}_n^*,$$

with $\widehat{g}_n^* \in S^{r'}_{\partial j(\cdot,u_n^+(\cdot))}$ and so

$$\Big|\langle V(u_n), h\rangle - \int_\Omega \widehat{g}^* h\,dz\Big| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall h \in W_0^{1,p}(\Omega),\ n \in \mathbb{N}.$$

Choosing $h = -u_n^- \in W_0^{1,p}(\Omega)$ and using Lemma 2.2 and hypothesis $H_1(i)$, we obtain

$$\frac{c_1}{p-1}\|Du_n^-\|_p^p \leqslant c_{14} \quad \forall n \in \mathbb{N},$$

for some $c_{14} > 0$, so

$$\text{the sequence } \{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.14}$$

Then we continue as we did for the functional φ using this time the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$ and we show that the sequence $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. It follows that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded (see (3.14)). From this and using Proposition 2.6, we conclude that φ_+ satisfies the Cerami condition.

Similarly for the functional φ_-. ☐

The next proposition shows that the functionals φ and $\varphi_\pm$ satisfy the mountain pass geometry.

Proposition 3.2. *If hypotheses H_0, H_1 hold, then $u = 0$ is a local minimizer for φ and $\varphi_\pm$.*

Proof. Hypotheses $H_0(iv)$ and $H_1(iv)$ imply that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\begin{cases} G(y) \geqslant \frac{1}{q}(\widehat{c}_0 - \varepsilon)|y|^q & \forall |y| \leqslant \delta, \\ j(z,x) \leqslant \frac{1}{q}(\vartheta(z) + \varepsilon)|x|^q & \text{for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta. \end{cases} \tag{3.15}$$

Let $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$. Using (3.15), we have

$$\begin{aligned} \varphi(u) &\geqslant \frac{1}{q}(\widehat{c}_0 - \varepsilon)\|Du\|_q^q - \frac{1}{q}\int_\Omega (\varphi(z) + \varepsilon)|u|^q \, dz \\ &= \frac{1}{q}\left(\widehat{c}_0\|Du\|_q^q - \int_\Omega \vartheta(z)|u|^q \, dz\right) - \frac{\varepsilon}{q}\left(1 + \frac{1}{\widehat{\lambda}_1(q)}\right)\|u\|_{1,q}^q \\ &\geqslant c_{15}\|u\|_{1,q}^q - \varepsilon c_{16}\|u\|_{1,q}^q, \end{aligned}$$

for some $c_{15}, c_{16} > 0$ (here $\|\cdot\|_{1,q} = \|\cdot\|_{W_0^{1,q}(\Omega)}$ and we used Lemma 2.11). Choosing $\varepsilon \in (0, \frac{c_{15}}{c_{16}})$, we see that

$$\varphi(u) \geqslant 0 = \varphi(0) \quad \forall \|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta,$$

so

$$u = 0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi$$

thus

$$u = 0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi$$

(see Gasiński-Papageorgiou [10, Proposition 2.6]).

Similarly for the functionals $\varphi_\pm$. □

Next we localize the critical sets of the functionals $\varphi_\pm$.

Proposition 3.3. *If hypotheses H_0, H_1 hold, then $K_{\varphi_+} \subseteq \{0\} \cup \text{int}C_+$ and $K_{\varphi_-} \subseteq \{0\} \cup (-\text{int}C_+)$.*

Proof. Let $u \in K_{\varphi_+}$. We have

$$V(u) = \widehat{g}^* \quad \text{in } W^{-1,p'}(\Omega), \tag{3.16}$$

with $\widehat{g}^* \in S^{r'}_{\partial j(\cdot, u^+(\cdot))}$. From the nonsmooth chain rule of Clarke [5, p. 42], we have

$$\partial j(z, u^+(z)) \begin{cases} = \{0\} & \text{if } \ u(z) < 0, \\ \subseteq \text{conv}\{\lambda \partial j(z,0) : \ 0 \leqslant \lambda \leqslant 1\} & \text{if } \ u(z) = 0, \\ = \partial j(z, u(z)) & \text{if } \ 0 < u(z). \end{cases} \tag{3.17}$$

If in (3.16) we act with $-u^- \in W_0^{1,p}(\Omega)$ and use (3.17) and Lemma 2.2, we obtain

$$\frac{c_1}{p-1}\|Du^-\|_p^p \leqslant 0,$$

so $u \geqslant 0$.

From (3.16) and the nonlinear Green's identity (see Papageorgiou-Winkert [24, p. 211]), we have

$$\begin{cases} -\text{div}\, a(Du) = \widehat{g}^*(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

A standard application of Moser's iteration technique (see Gasiński-Papageorgiou [12]), implies that $u \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [15] implies that $u \in C_+$.

Suppose that $u \in K_{\varphi_+} \setminus \{0\}$ and let $\varrho = \|u\|_\infty$ and $\widehat{\xi}_\varrho > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$-\text{div}\, a(Du) + \widehat{\xi}_p u^{p-1} = \widehat{g}^*(z) + \widehat{\xi}_\varrho u^{p-1} \geqslant 0 \quad \text{in } \Omega,$$

so

$$\text{div}\, a(Du) \leqslant \widehat{\xi}_\varrho u^{p-1} \quad \text{in } \Omega,$$

and thus $u \in \text{int}C_+$ (see Pucci-Serrin [27, p. 120]).

Similarly for K_{φ_-}. □

The elements of $K_{\varphi_\pm}$ are the positive and negative solutions of (1.1) and so we may assume that these sets are finite. Similarly K_φ consists of the solutions of (1.1) and so we may assume that K_φ is finite too. Therefore we can compute $C_k(\varphi,\infty)$ and $C_k(\varphi_\pm,\infty)$ for all $k\in\mathbb{N}_0$.

We follow the reasoning of Aizicovici-Papageorgiou-Staicu [1, Proposition 3.6] modified appropriately in order to accommodate the fact that φ is not C^1 and it is only locally Lipschitz.

Proposition 3.4. *If hypotheses H_0, H_1 hold, then $C_k(\varphi,\infty)=0$ for all $k\in\mathbb{N}_0$.*

Proof. First we show that if $u\in W_0^{1,p}(\Omega)\setminus\{0\}$, we have

$$\varphi(tu)\longrightarrow-\infty \quad \text{as } t\to+\infty. \tag{3.18}$$

To this end note that hypotheses $H_1(i),(ii)$ imply that given $\eta>0$, we can find $c_{17}=c_{17}(\eta)>0$ such that

$$j(z,x)\geqslant\eta|x|^p-c_{17} \quad \text{for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R}. \tag{3.19}$$

Let $t>1$. We have

$$\begin{aligned}\varphi(tu) &= \int_\Omega G(D(tu))\,dz-\int_\Omega j(z,tu)\,dz\\ &\leqslant c_{18}t^p(1+\|u\|^p)-\eta t^p\|u\|_p^p+c_{19}\\ &\leqslant (c_{20}-\eta c_{21})t^p+c_{19},\end{aligned}$$

for some $c_{18},c_{19},c_{20},c_{21}>0$ (see Corollary, (3.16) and recall that $t>1$).

Recall that $\eta>0$ is arbitrary. Choosing $\eta>\frac{c_{20}}{c_{21}}$, we infer that (3.18) holds.

Fix $u\in W_0^{1,p}(\Omega)\setminus\{0\}$ and let $\vartheta\colon\mathbb{R}\longrightarrow W_0^{1,p}(\Omega)$ be defined by

$$\vartheta(t)=tu.$$

We set $\xi(t)=(\varphi\circ\vartheta)(t)$. Then this is a locally Lipschitz function and from the second chain rule of Clarke [5, p. 45], we have

$$\partial\xi(t)\subseteq\{\langle u^*,u\rangle : u^*\in\partial\varphi(tu)\}=\frac{1}{t}\{\langle u^*,tu\rangle : u^*\in\partial\varphi(tu)\} \tag{3.20}$$

Hypotheses $H_1(i),(ii)$ imply that we can find $\beta_1\in(0,\beta_0)$ and $c_{22}>0$ such that

$$pj(z,x)-h^*x\leqslant c_{22}-\beta_1|x|^\mu \quad \text{for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R},\ h^*\in\partial j(z,x). \tag{3.21}$$

We know that $u^*=V(tu)-g^*$, with $g^*\in S^r_{\partial j(\cdot,tu(\cdot))}$ (see Clarke [5, pp. 38, 83]). So, we have

$$\begin{aligned}\frac{1}{t}\langle u^*,u\rangle &= \frac{1}{t}\Big(\langle V(tu),tu\rangle-\int_\Omega g^*(tu)\,dz\Big)\\ &\leqslant \frac{1}{t}\Big(\int_\Omega pG(D(tu))\,dz+c_{22}|\Omega|_N-\int_\Omega pj(z,tu)\,dz-\beta_1\|tu\|_\mu^\mu\Big)\\ &\leqslant \frac{1}{t}(\varphi(tu)+c_{23})\end{aligned}$$

with $c_{23}=c_{22}|\Omega|_N>0$ (see hypothesis $H_1(iv)$ and (3.21)), so

$$\frac{1}{t}\langle u^*,tu\rangle\leqslant\varphi(tu)+c_{23}.$$

On account of (3.18), we see that for $t>0$ large we have

$$\frac{1}{t}\langle u^*,u\rangle\leqslant\gamma<0, \tag{3.22}$$

with $\gamma<\min\{-\frac{c_{23}}{p},m\}$, where $m=\inf\{\varphi(u):\ u\in K_\varphi\cup\partial B_1\}$.

At this point our proof changes drastically with respect to that of Aizicovici-Papageorgiou-Staicu [1, Proposition 3.6], since φ is not smooth and we cannot use the implicit function theorem. We need to come up with a substitute of it.

Let $u \in \partial B_1$. We will show that there exists a unique $\widetilde{t}_u > 1$ such that $\varphi(\widetilde{t}_u u) = \gamma$. From the choice of γ, we have $\gamma < \varphi(u)$. This fact combined with (3.18), via Bolzano's theorem, implies that we can find $\widehat{t}_u > 1$ such that $\varphi(\widehat{t}_u u) = \gamma$. So, we need to show the uniqueness of $\widehat{t}_u > 1$. We argue indirectly. So, suppose that we can find $1 < \widehat{t}_u^1 < \widehat{t}_u^2$ such that $\varphi(\widehat{t}_u^k u) = \gamma$ for $k = 1, 2$. From (3.17), we have

$$\partial\xi(\widehat{t}_u^1) \subseteq (-\infty, \widehat{\gamma}),$$

with $\widehat{\gamma} \in (\gamma, 0)$.

The upper semicontinuity (as a multifunction) of the Clarke subdifferential, implies that we can find $\delta > 0$ such that

$$\partial\xi(t) \subseteq (-\infty, \widehat{\gamma}) \quad \forall t: \ |t - \widehat{t}_u^1| < \delta.$$

Recall that $t \mapsto \xi(t) = (\varphi \circ \vartheta)(t)$ is locally Lipschitz, hence differentiable almost everywhere (Rademacher Theorem; see Gasiński-Papageorgiou [12, p. 56]). Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of points of differentiability of ξ such that $t_n \to \widehat{t}_u^1$. From Chang [4, p. 108], we have

$$\frac{d\xi}{dt}(t_n) < \widehat{\gamma} \quad \forall n \in \mathbb{N},$$

so

$$\varphi((t_n + h)u) < \psi(t_n u) + \frac{\widehat{\gamma}}{2} \quad \forall h \in (-\delta, \delta), \ n \in \mathbb{N}$$

(choosing $\delta \in (0, 1)$), so

$$\varphi((\widehat{t}_u^1 + h)u) < \gamma \quad \forall h \in (-\delta, \delta) \tag{3.23}$$

(since $t_n \longrightarrow \widehat{t}_u^1$ and $\widehat{\gamma} < 0$).

Similarly by choosing $\delta \in (0, 1)$ even smaller if necessary, we have

$$\varphi((\widehat{t}_u^2 + h)u) < \gamma \quad \forall h \in (-\delta, \delta). \tag{3.24}$$

We can assume that $\widehat{t}_u^2 > \widehat{t}_u^1$ is the first time instant $t > \widehat{t}_u^1$ for which we have $\varphi(tu) = \gamma$. Then $\varphi(\widehat{t}_u^2 u) = \gamma$ and

$$\varphi(tu) < \gamma \quad \forall t \in (\widehat{t}_u^1, \widehat{t}_u^2 + \delta), \quad t \neq \widehat{t}_u^2$$

(see (3.20), (3.21)). From this we infer that $\widehat{t}_u^2$ is a local maximizer of locally Lipschitz function ξ. It follows that

$$0 \in \partial\xi(\widehat{t}_u^2).$$

This means that we can find $u^* \in \partial\varphi(\widehat{t}_u^2 u)$ such that

$$\langle u^*, u \rangle = 0$$

(see (3.20)). But this contradicts (3.22). Therefore $\widehat{t}_u > 1$ is unique.

Consider the map $l \colon \partial B_1 \longrightarrow (1, \infty)$ $(\partial B_1 = \{u \in W_0^{1,p}(\Omega) : \ \|u\| = 1)$ defined by

$$l(u) = \widehat{t}_u.$$

We show that this map is continuous. Let $\varepsilon > 0$ be small and let $t \in (\widehat{t}_u - \varepsilon, \widehat{t}_u)$, $s \in (\widehat{t}_u, \widehat{t}_u + \varepsilon)$. We have

$$\varphi(su) < \gamma < \varphi(tu),$$

so

$$\varphi(sv) < \gamma < \varphi(tv) \quad \forall v \in B_\varrho(u) \cap \partial B_1$$

(recall that φ is continuous and $B_\delta = \{v \in W_0^{1,p}(\Omega) : \ \|v - u\| < \delta\|)$, so

$$\widehat{t}_v \in (t, s),$$

thus

$$|\widehat{t}_v - \widehat{t}_u| < \varepsilon$$

and so the map $u \mapsto l(u) = \widehat{t}_u$ is continuous.

Now that we have the continuous map l, we can follow again the argument of Aizicovici-Papageorgiou-Staicu [1, Proposition 3.6]. For $u \in \partial B_1$ and $t \geqslant 1$, we have

$$\varphi(tu) \begin{cases} > \gamma & \text{if} \quad t < \widehat{t}_u, \\ = \gamma & \text{if} \quad t = \widehat{t}_u, \\ < \gamma & \text{if} \quad \widehat{t}_u < t. \end{cases} \tag{3.25}$$

For $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we define

$$\widehat{l}(u) = \frac{1}{\|u\|} l\Big(\frac{u}{\|u\|}\Big).$$

The continuity of l implies the continuity of $\widehat{l}$. We have

$$\varphi(\widehat{l}(u)u) = \gamma$$

(see (3.25)), so

$$\varphi(u) = \gamma \implies \widehat{l}(u) = 1.$$

Therefore, if we define

$$\widehat{l}_0(u) = \begin{cases} 1 & \text{if} \quad \varphi(u) \leqslant \gamma, \\ \widehat{l}(u) & \text{if} \quad \gamma < \varphi(u), \end{cases} \tag{3.26}$$

then $\widehat{l}_0 \colon W_0^{1,p}(\Omega) \setminus \{0\} \longrightarrow \mathbb{R}$ is continuous.

Consider the homotopy

$$\widehat{h}(t,u) = (1-t)u + t\widehat{l}_0(u)u \quad \forall t \in [0,1],\ u \in W_0^{1,p}(\Omega) \setminus \{0\}.$$

The continuity of $\widehat{l}_0$, implies the continuity of $\widehat{h}(\cdot,\cdot)$. We have

$$\widehat{h}(0,u) = u, \quad \widehat{h}(1,u) \subseteq \overline{\varphi}^{\gamma} \quad \forall u \in W_0^{1,p}(\Omega) \setminus \{0\}.$$

Moreover,

$$u \in \overline{\varphi}^{\gamma} \Rightarrow h(t,u) = u \quad \forall t \in [0,1]$$

(see (3.26)). Then it follows that

$$\overline{\varphi}^{\gamma} \text{ is a strong deformation retract of } W_0^{1,p}(\Omega) \setminus \{0\}. \tag{3.27}$$

Using the radial retraction, we see that

$$\partial B_1 \text{ is a retract of } W_0^{1,p}(\Omega) \setminus \{0\}. \tag{3.28}$$

From (3.27) and (3.28) it follows that

$$\overline{\varphi}^{\gamma} \text{ and } \partial B_1 \text{ are homotopy equivalent}$$

(see Papageorgiou-Rădulescu-Repovš [23, p. 460]), so

$$H_k(W_0^{1,p}(\Omega), \varphi^{\gamma}) = H_k(W_0^{1,p}(\Omega), \partial B_1) \quad \forall k \in \mathbb{N}_0 \tag{3.29}$$

(see Papageorgiou-Rădulescu-Repovš [23, p. 462]).

Since $W_0^{1,p}(\Omega)$ is infinite dimensional, we know that ∂B_1 is contractible. Hence

$$H_k(W_0^{1,p}(\Omega), \partial B_1) = 0 \quad \forall k \in \mathbb{N}_0$$

(see Papageorgiou-Rădulescu-Repovš [23, p. 469]), so

$$H_k(W_0^{1,p}(\Omega), \varphi^{\gamma}) = 0 \quad \forall k \in \mathbb{N}_0$$

(see (3.29)) and finally

$$C_k(\varphi, \infty) = 0 \quad \forall k \in \mathbb{N}_0$$

(recall the choice of γ). □

We can have the same result for the critical groups at infinity of the functionals $\varphi_{\pm}$ (see also Papageorgiou-Rădulescu [21]).

Proposition 3.5. *If hypotheses H_0, H_1 hold, then*

$$C_k(\varphi_\pm,\infty)=0 \quad \forall k\in\mathbb{N}_0.$$

Proof. The proof is similar to that of Proposition 3.4. Let $\partial B_1^+=\{u\in\partial B_1:\ u^+\neq 0\}$. As in the proof of Proposition 3.4, exploiting the p-superlinearity of $F(z,\cdot)$, we show that for all $u\in\partial B_1^+$, we have

$$\varphi_+(tu)\longrightarrow -\infty \quad \text{as } t\to+\infty.$$

Choose $\gamma<\min\{0,\inf\limits_{K_{\varphi_+}\cup\partial B_1}\varphi\}$ such that

$$\langle h^*,h\rangle\leqslant\gamma<0 \quad \forall h\in\varphi_+^{-1}(\gamma),\ h^*\in\partial\varphi_+(h).$$

As in the proof of Proposition 3.4, we have a continuous map $l_+\colon\partial B_1^+\longrightarrow(1,\infty)$ such that

$$\varphi_+(tu)\begin{cases} >\gamma & \text{if } \ t<l_+(u),\\ =\gamma & \text{if } \ t=l_+(u),\\ <\gamma & \text{if } \ l_+(u)<t.\end{cases}$$

We see that $\overline{\varphi}_+^\gamma=\{tu:\ u\in\partial B_1^+,\ t\geqslant l_+(u)\}$. If we set $E_+=\{tu:\ u\in\partial B_1^+,\ t\geqslant 1\}$, then $\overline{\varphi}_+^\gamma\subseteq E_+$. Consider the deformation $h_+\colon[0,1]\times E_+\longrightarrow E_+$ defined by

$$h_+(s,tu)=\begin{cases}(1-s)tu+sl_+(u)u & \text{if } \ t\in[1,l_+(u)],\\ tu & \text{if } \ l_+(u)<t.\end{cases}$$

Then $h_+(0,tu)=tu$, $h_+(1,tu)\in\overline{\varphi}_+^\gamma$ and $h(s,\cdot)|_{\overline{\varphi}_+^\gamma}=\mathrm{id}|_{\overline{\varphi}_+^\gamma}$. This means that $\overline{\varphi}_+^\gamma$ is a strong deformation retractor of E_+. So, we have

$$H_k(W_0^{1,p}(\Omega),\overline{\varphi}_+^\gamma)=H_k(W_0^{1,p}(\Omega),E_+) \quad \forall k\in\mathbb{N}_0. \tag{3.30}$$

(see Papageorgiou-Rădulescu-Repovš [23, p. 462]). The set E_+ is contractible. Indeed, let $\widehat{u}\in\partial B_1\cap\mathrm{int}C_+$ and consider the deformation

$$\widehat{h}_+(t,u)=\frac{(1-t)u+t\widehat{u}}{\|(1-t)u+t\widehat{u}\|} \quad \forall t\in[0,1],\ u\in E_+.$$

This continuous deformation collapses E_+ to $\frac{\widehat{u}}{\|\widehat{u}\|}\in\partial B_1^+\subseteq E_+$ and so E_+ is contractible. It follows that

$$H_k(W_0^{1,p}(\Omega),E_+)=0 \quad \forall k\in\mathbb{N}_0$$

(see Papageorgiou-Rădulescu-Repovš [23, p. 469]), so

$$H_k(W_0^{1,p}(\Omega),\overline{\varphi}_+^\gamma)=0 \quad \forall k\in\mathbb{N}_0$$

(see (3.30)) and thus

$$C_k(\varphi_+,0)=0 \quad \forall k\in\mathbb{N}_0$$

(recall the choice of γ). ∎

Now we can have the first multiplicity theorem for problem (1.1).

Theorem 3.6. *If hypotheses H_0, H_1 hold, then problem* (1.1) *has at least three nontrivial solutions $u_0\in\mathrm{int}C_+$, $v_0\in-\mathrm{int}C_+$, $y_0\in C_0^1(\overline{\Omega})$.*

Proof. Recall that without any loss of generality we assume that $K_{\varphi_+}\subseteq\{0\}\cup\mathrm{int}C_+$ is finite. From Proposition 3.2 we know that $u=0$ is a local minimizer of φ_+. Then as in the proof of Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [23, p. 449] we can find $\varrho\in(0,1)$ small such that

$$\varphi_+(0)=0<\inf\{\varphi_+(u):\ \|u\|=\varrho\}=m_+. \tag{3.31}$$

Hypothesis $H_1(ii)$ implies that if $u\in\mathrm{int}C_+$, then

$$\varphi_+(tu)\longrightarrow-\infty \quad \text{as } t\to+\infty. \tag{3.32}$$

Moreover, from 3.1, we have that

$$\varphi_+ \text{ satisfies the nonsmooth Cerami condition.} \tag{3.33}$$

On account of (3.31), (3.32) and (3.33) we can apply the nonsmooth mountain pass theorem (see Theorem 2.9). So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_{\varphi_+} \subseteq \{0\} \cup \operatorname{int} C_+, \quad \varphi_+(0) = 0 < m_+ \leqslant \varphi_+(u_0) \tag{3.34}$$

(see Proposition 3.3 and (3.31)).

From (3.34) it follows that

$$u_0 \in \operatorname{int} C_+ \text{ is a positive solution of (1.1).}$$

Similarly working with φ_-, we produce a negative solution $v_0 \in -\operatorname{int} C_+$ of (1.1).

Suppose $K_{\widehat{\varphi}} = \{0, u_0, v_0\}$. We will show that

$$C_k(\varphi_+, u_0) = C_k(\varphi_-, v_0) = \delta_{k,1}\mathbb{R} \quad \forall k \in \mathbb{N}_0. \tag{3.35}$$

To this end, let $\tau, \sigma \in \mathbb{R}$ be such that

$$\tau < \varphi_+(0) = 0 < \sigma < \varphi_+(u_0) \tag{3.36}$$

(see (3.36)). We introduce the following triple of sets

$$\varphi_+^\tau \subseteq \varphi_+^\sigma \subseteq W_0 = W_0^{1,p}(\Omega).$$

For this triple of sets, we consider the corresponding long exact sequence of singular homological groups

$$\dots H_k(W_0, \overline{\varphi}_+^\tau) \xrightarrow{i_*} H_k(W_0, \overline{\varphi}_+^\sigma) \xrightarrow{\partial_*} H_k(\overline{\varphi}_+^\sigma, \overline{\varphi}_+^\tau) \dots \tag{3.37}$$

with i_* being the group homomorphism induced by the inclusion $i\colon (W_0, \varphi_+^\tau) \longrightarrow (W_0, \varphi_+^\sigma)$ and ∂_* is the boundary homomorphism (see Papageorgiou-Rădulescu-Repovš [23, p. 466]).

Since we have assumed that $K_\varphi = \{0, u_0, v_0\}$, we have that $K_{\varphi_+} = \{0, u_0\}$. Then from (3.36) it follows that

$$H_k(W_0, \varphi_+^\tau) = C_k(\varphi, \infty) = 0 \quad \forall k \in \mathbb{N}_0 \tag{3.38}$$

(see Proposition 3.5).

Note that

$$\varphi_+(K_{\varphi_+}) \cap (\sigma, \infty) = \eta_0 = \varphi_+(u_0).$$

Then Proposition 2.10 implies that

$$H_k(W_0, \varphi_+^\sigma) = C_k(\varphi_+, u_0) \quad \forall k \in \mathbb{N}_0. \tag{3.39}$$

Similarly, we have

$$H_{k-1}(\varphi_+^\sigma, \varphi_+^\tau) = C_{k-1}(\varphi_+, 0) = \delta_{k-1,0}\mathbb{R} = \delta_{k,1}\mathbb{R} \quad \forall k \in \mathbb{N}_0 \tag{3.40}$$

(see Proposition 3.2). Recall that (3.37) is exact. So, from (3.38), (3.39) and (3.40) we infer that ∂_* is an isomorphism and

$$C_k(\varphi_+, u_0) = \delta_{k,1}\mathbb{R} \quad \forall k \in \mathbb{N}_0. \tag{3.41}$$

Consider the homotopy $\widetilde{h}_+(t, u)$ defined by

$$\widetilde{h}(t, u) = t\varphi_+(u) + (1-t)\varphi(u) \quad \forall (t, u) \in [0, 1] \times W_0.$$

Suppose that we can find two sequences $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, 1]$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0$ such that

$$t_n \longrightarrow t, \quad u_n \longrightarrow u_0 \quad \text{and} \quad (\widetilde{h}_+)'_u(t_n, u_n) = 0 \quad \forall n \in \mathbb{N}. \tag{3.42}$$

From the equation in (3.42), we infer that

$$-\operatorname{div} a(Du_n) = t_n \widehat{g}_n^* + (1 - t_n) g_n^*,$$

with $\widehat{g}_n^* \in S^{r'}_{\partial j(\cdot,u_n^+(\cdot))}$, $g_n^* \in S^{r'}_{\partial j(\cdot,u_n(\cdot))}$. The nonlinear regularity theory, implies that there exist $\alpha \in (0,1)$ and $c_{24} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}),\ \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_{24} \quad \forall n \in \mathbb{N}. \tag{3.43}$$

Recall that the embedding $C_0^{1,\alpha}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$ is compact. So, from (3.42) and (3.43), it follows that

$$u_n \longrightarrow u_0 \quad \text{in } C_0^1(\overline{\Omega}).$$

We know that $u_0 \in \mathrm{int}C_+$. So, we can find $n_0 \in \mathbb{N}$ such that

$$u_n \in C_+ \setminus \{0\} \quad \forall n \geqslant n_0,$$

thus $\{u_n\}_{n\geqslant n_0} \subseteq K_{\varphi_+} = \{0, u_0\}$, a contradiction. So, (3.42) cannot be true and the homotopy invariance of critical groups for nonsmooth functionals due to Corvellec-Hantoute [8], implies that

$$C_k(\varphi, u_0) = C_k(\varphi_+, u_0) \quad \forall k \in \mathbb{N}_0,$$

so

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{R} \quad \forall k \in \mathbb{N}_0.$$

Similarly, working this time with the pair $\{\varphi_-, v_0\}$ we obtain that

$$C_k(\varphi, v_0) = \delta_{k,1}\mathbb{R} \quad \forall k \in \mathbb{N}_0.$$

Therefore (3.35) is true. From Proposition 3.3, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{R} \quad \forall k \in \mathbb{N}_0. \tag{3.44}$$

Moreover, from Proposition 3.4, we have

$$C_k(\varphi, \infty) = 0 \quad \forall k \in \mathbb{N}_0. \tag{3.45}$$

Recall that we have assumed that $K_\varphi = \{0, u_0, v_0\}$. So, from (3.35), (3.44), (3.45) and the Morse relation (see (2.2)) with $t = -1$, we have

$$(-1)^0 + 2(-1)^1 = 0,$$

a contradiction. So, there exists $y_0 \in K_\varphi$, $y_0 \notin \{0, u_0, v_0\}$. Then $y_0 \in C_0^1(\overline{\Omega})$ is the third nontrivial solution of (1.1). □

Remark 3.7. In the above multiplicity theorem, we do not provide any sign information for the third solution y_0.

4. Nodal solution

In this section we prove another multiplicity theorem for problem (1.1), producing this time five nontrivial smooth solutions, four of constant sign and the fifth nodal (sign-changing).

To do this we need to modify the hypotheses on the map a and the reaction $\partial j(z,\cdot)$, producing a new geometry near zero.

The new hypotheses on the data of problem (1.1) are the following:

$\underline{H_0'}$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, hypotheses $H_0'(i), (ii), (iii)$ are the same as the corresponding hypotheses $H_0(i), (ii), (iii)$ and

(iv): $pG_0(t) - a_0(t)t^2 \geqslant 0$ for all $t \geqslant 0$ and there exists $1 < q \leqslant p$ such that the map $t \mapsto G_0(t^{\frac{1}{q}})$ is convex, $\limsup\limits_{t\to 0^+} \frac{qG_0(t)}{t^q} \leqslant c^*$.

Remark 4.1. The examples given after hypotheses H_0, all satisfy the new conditions.

$\underline{H_1'}$: $j\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that for all $x \in \mathbb{R}$, $z \mapsto j(z,x)$ is measurable, for a.a. $z \in \Omega$, $x \mapsto j(z,x)$ is locally Lipschitz, $j(z,0) = 0$, $0 \in \partial j(z,0)$ for a.a. $z \in \Omega$ and

(i): $|u^*| \leqslant a(z)(1+|x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $u^* \in \partial j(z,x)$ and with $a \in L^\infty(\Omega)$, $p < r < p^*$;

(ii): $\lim_{x\to\pm\infty} \frac{j(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii): there exists $\mu \in \big((r-p)\max\{\frac{N}{p},1\}, p^*\big\}$ such that

$$0 < \beta_0 \leqslant \liminf_{x\to\pm\infty} \min_{u^*\in\partial j(z,x)} \frac{u^*x - pj(z,x)}{|x|^\mu},$$

uniformly for a.a. $z \in \Omega$.

(iv): there exists $\tau \in (1,q)$ ($q \in (1,p]$ as in hypothesis $H_1'(iv)$) such that

$$\lim_{x\to 0} \frac{\tau j(z,x)}{|x|^\tau} = \widehat{\eta}_0 > 0, \quad \liminf_{x\to 0} \min_{u^*\in\partial j(z,x)} \frac{u^*}{|x|^{\tau-2}x} \geqslant \eta_0 > 0$$

uniformly for a.a. $z \in \Omega$;

(v): there exist $\vartheta_- < 0 < \vartheta_+$ and $\widehat{\xi}_* > 0$ such that for a.a. $z \in \Omega$ we have $\max\{u^* : u^* \in \partial j(z,\vartheta_+)\} \leqslant -\widehat{\eta} < 0 \leqslant \widehat{\eta} = \min\{v^* : v^* \in \partial j(z,\vartheta)\}$ and for a.a. $z \in \Omega$, the function $x \mapsto j(z,x) + \frac{\widehat{\xi}_*}{p}|x|^p$ is strongly convex.

Remark 4.2. In hypothesis $H_1'(v)$ the strong convexity of $x \mapsto j(z,x)+\frac{\widehat{\xi}_*}{p}|x|^p$ is equivalent to saying that there exists $\sigma > 0$ such that $x \mapsto j(z,x) - \frac{\sigma}{2}x^2 + \frac{\widehat{\xi}_*}{p}|x|^p$ is convex. This in turn is equivalent to the strong monotonicity with constant σ of the multifunction $x \mapsto \partial j(z,x) + \widehat{\xi}_x|x|^{p-2}x$ (see Rockafellar-Wets [28, p. 565]).

The following function j satisfies hypotheses H_1'. For the sake of simplicity, we drop the z-dependence:

$$j(x) = \begin{cases} x^2 - c|x|^p & \text{if} \quad |x| \leqslant 1, \\ \frac{|x|^p}{p}\big(\ln|x| - \frac{1}{p}\big) + \widehat{c}x^2 & \text{if} \quad 1 < |x|, \end{cases}$$

with $2 < p$, $\frac{2}{p} < c < 1$, $\widehat{c} = (1-c) + \frac{1}{p^2} > 0$.

First we produce two constant sign smooth solutions (positive and negative) which are local minimizers of the energy functional φ.

Proposition 4.3. *If hypotheses H_0', H_1' hold, then problem* (1.1) *has two constant sign solutions $u_0 \in \operatorname{int}C_+$ and $v_0 \in -\operatorname{int}C_+$ which are local minimizers of the energy functional φ.*

Proof. We consider the following truncation of the potential function $j(z,\cdot)$:

$$\widehat{j}_+(z,x) = \begin{cases} j(z,x^+) & \text{if} \quad x \leqslant \vartheta_+, \\ j(z,\vartheta_+) & \text{if} \quad \vartheta_+ < x. \end{cases} \tag{4.1}$$

Evidently for all $x \in \mathbb{R}$, the map $z \mapsto \widehat{j}(z,x)$ is measurable and for a.a. $z \in \Omega$, the map $x \mapsto \widehat{j}_+(z,x)$ is locally Lipschitz. Using the first nonsmooth chain rule of Clarke [5, p. 42], we have

$$\partial\widehat{j}_+(z,x) \begin{cases} = \{0\} & \text{if} \quad x < 0, \\ \subseteq \operatorname{conv}\{t\partial j(z,0) : \ 0 \leqslant t \leqslant 1\} & \text{if} \quad x = 0, \\ = \partial j(z,x) & \text{if} \quad 0 < x < \vartheta_+, \\ \subseteq \operatorname{conv}\{s\partial j(z,\vartheta_+) : \ 0 \leqslant s \leqslant 1\} & \text{if} \quad x = \vartheta_+, \\ = \{0\} & \text{if} \quad \vartheta_+ < x. \end{cases} \tag{4.2}$$

We introduce the locally Lipschitz functional $\widehat{\varphi}_+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_+(u) = \int_\Omega G(Du)\,dz - \int_\Omega \widehat{j}(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

From Corollary 2.3 and (4.1), we see that the functional φ^+ is coercive. Also, using the Sobolev embedding theorem we see that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0) = \inf\{\widehat{\varphi}_+(u) : \; u \in W_0^{1,p}(\Omega)\}. \tag{4.3}$$

From hypotheses $H_0^1(iv)$ and $H_1'(iv)$, we see that we can find $c_1^* > c^*$, $\eta_1 \in (0, \eta_0)$ and $\delta > 0$ such that

$$\begin{cases} G(y) \leqslant \frac{c_1^*}{q}|y|^q & \text{for all } |y| \leqslant \delta, \\ u^* \geqslant \eta_1 x^{\tau-1} & \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta, \; u^* \in \partial j(z,x). \end{cases} \tag{4.4}$$

From the second inequality in (4.4) we obtain

$$j(z,x) \geqslant \frac{\eta_1}{\tau} x^\tau \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta. \tag{4.5}$$

Indeed, we know that $j(z,\cdot)$ is differentiable a.e. on $\mathbb{R}$ and $j_x'(z,x) \in \partial j(z,x)$ at every point of differentiability. Integrating we obtain (4.5). If $u \in \operatorname{int} C_+$, let $t \in (0,1)$ be small such that $0 \leqslant tu(z) \leqslant \delta$ for all $z \in \overline{\Omega}$. Since $\tau < q \leqslant p$, we see that by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\widehat{\varphi}_+(tu) < 0,$$

so

$$\widehat{\varphi}_+(u_0) < 0 = \widehat{\varphi}_+(0)$$

(see (4.3)) and thus $u_0 \neq 0$..

From (4.3) we have

$$0 \leqslant \partial \widehat{\varphi}_+(u_0),$$

so

$$\langle V(u_0), h\rangle = \int_\Omega \widehat{g}_+^* h \, dz \quad \forall h \in W_0^{1,p}(\Omega), \tag{4.6}$$

here $\widehat{g}_+^*(z) \in \partial \widehat{j}_+(z, u_0(z))$ a.e. in Ω.

In (4.6) we use the test function $h = -u_0^- \in W_0^{1,p}(\Omega)$. We obtain

$$\frac{c_1}{p-1}\|Du_0^-\|_p^p \leqslant 0$$

(see Lemma 1.1 and (4.2)), so $u_0 \geqslant 0$, $u_0 \neq 0$.

Next in (4.6) we choose the test function $h = (u_0 - \vartheta_+)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} \langle V(u_0), (u_0 - \vartheta_+)^+\rangle &= \int_\Omega \widehat{g}_+^*(u_0 - \vartheta_+)^+ \, dz \\ &\leqslant -\int_\Omega \widehat{\eta}(u_0 - \vartheta_+)^+ \, dz \\ &< 0 = \langle V(\vartheta_+), (u_0 - \vartheta_+)^+\rangle, \end{aligned}$$

so $u_0 \leqslant \vartheta_+$. So, we have proved that

$$u_0 \in [0, \vartheta_+], \quad u_0 \neq 0. \tag{4.7}$$

We have

$$\begin{cases} -\operatorname{div} a(Du_0) = \widehat{g}_+^*(z) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

(see (4.6)). Since $u_0 \in L^\infty(\Omega)$ (see (4.7)), using the nonlinear regularity theory of Lieberman [15], we have that $u_0 \in C_+ \setminus \{0\}$.

Let $\widehat{\xi}_x > 0$ be as postulated by hypothesis $H_1'(v)$. Since by hypothesis $0 \in \partial j(z,0)$ for a.a. $z \in \Omega$, from hypothesis $H_1'(v)$, we have

$$u^* x + \widehat{\xi}_x x^p \geqslant \sigma x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \; u^* \in \partial j(z,x). \tag{4.8}$$

So, we have

$$-\operatorname{div} a(Du_0) + \widehat{\xi}_x u_0^{p-1} = \widehat{g}_+^*(z) + \widehat{\xi}_x u_0^{p-1} \geqslant 0 \quad \text{in } \Omega$$

(see (4.8)), so

$$\operatorname{div} a(Du_0) \leqslant \widehat{\xi}_x u_0^{p-1} \quad \text{in } \Omega,$$

thus $u_0 \in \operatorname{int} C_+$ (see Pucci-Serrin [27, p. 120]).

On account of hypothesis $H_1'(v)$, we have

$$w^* + \widehat{\xi}_x \vartheta_+^{p-1} \geqslant u^* + \widehat{\xi}_x x^{p-1} \tag{4.9}$$

for a.a. $z \in \Omega$, all $0 \leqslant x \leqslant \vartheta_+$, all $u^* \in \partial \widehat{j}(z,x)$, $w^* \in \partial j_+(z,\vartheta_+)$. Therefore

$$\begin{aligned} -\operatorname{div} a(Du_0) + \widehat{\xi}_x u_0^{p-1} &= \widehat{g}_+^*(z) + \widehat{\xi}_x u_0^{p-1} \\ &\leqslant g_{\vartheta_+}^*(z) + \widehat{\xi}_x \vartheta_+^{p-1} \\ &\leqslant -\operatorname{div} a(D\vartheta_+) + \widehat{\xi}_x \vartheta_+^{p-1} \quad \text{in } \Omega. \end{aligned} \tag{4.10}$$

with $g_{\vartheta_+}^*(z) \in \partial j(z,\vartheta_+)$ for a.a. $z \in \Omega$ (see (4.7), (4.9)).

Since $g_{\vartheta_+}^*(z) \leqslant -\widehat{\eta} < 0$ for a.a. $z \in \Omega$, from (4.10) and Proposition 2.4 of Papageorgiou-Winkert [26], we have

$$u_0(z) < \vartheta_+ \quad \forall z \in \overline{\Omega},$$

so

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0,\vartheta_+]. \tag{4.11}$$

From (4.11), (4.2) and (4.6) it follows that $u_0 \in \operatorname{int} C_+$ is a positive solution of (1.1).

Note that

$$\varphi|_{[0,\varphi_+]} = \widehat{\varphi}_+|_{[0,\vartheta_+]}$$

(see (4.1)). From (4.11) and (4.3) it follows that

$$u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi,$$

so

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi$$

(see Gasiński-Papageorgiou [10]). Similarly working with the locally Lipschitz integrand

$$\widehat{j}_-(z,x) = \begin{cases} j(z,\vartheta_-) & \text{if} \quad x < \vartheta_-, \\ j(z,-x^-) & \text{if} \quad \vartheta_- \leqslant x \end{cases}$$

and the corresponding locally Lipschitz functional $\widehat{\varphi}_- \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_-(u) = \int_\Omega G(Du)\,dz - \int_\Omega j_-(z,-u^-)\,dz \quad \forall u \in W_0^{1,p}(\Omega),$$

we produce a negative solution

$$v_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\vartheta_-,0]$$

for problem (1.1), which is a local minimizer of φ. □

Using the two solutions from Proposition 3.8 and the nonsmooth mountain pass theorem (see Theorem 2.9), we can have two more solutions of constant sign, for a total of four solutions of constant sign.

Proposition 4.4. *If hypotheses H_0', H_1' hold, then problem* (1.1) *has two more constant sign solutions $\widehat{u} \in \operatorname{int} C_+$, $\widehat{v} \in -\operatorname{int} C_+$, which are distinct from u_0 and v_0.*

Proof. Recall that we have assumed that

$$K_{\varphi_+} \text{ and } K_{\varphi_-} \text{ are finite.} \tag{4.12}$$

From Proposition 4.3 we already have two constant sign $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$. From the proof of Proposition 4.3, we know that

$$u_0 \text{ is a minimizer of } \widehat{\varphi}_+.$$

But $\widehat{\varphi}_+|_{[0,\vartheta_+]} = \varphi_+|_{[0,\vartheta_+]}$ (see (4.1)). So, from (4.11), we see that

$$u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_+$$

so

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_+ \tag{4.13}$$

(see Gasiński-Papageorgiou [10]).

Then (4.12) and (4.13) and Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [23, p. 449] (the result is also valid in the present nonsmooth setting with the same proof), imply that we can find $\varrho \in (0,1)$ small such that

$$\varphi_+(u_0) < \inf\{\varphi_+(u) :\ \|u-u_0\| = \varrho\} = m_+. \tag{4.14}$$

On account of hypothesis $H_1'(ii)$ if $u \in \mathrm{int}C_+$, then

$$\varphi_+(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{4.15}$$

Moreover, from Proposition 3.1 we know that

$$\varphi_+ \text{ satisfies the Cerami condition.} \tag{4.16}$$

Then (4.14), (4.15) and (4.16), permit the use of Theorem 2.9. We can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\varphi_+} \quad \text{and} \quad m_+ \leqslant \varphi_+(\widehat{u}). \tag{4.17}$$

From (4.17), (4.14) and Proposition 3.3, we have

$$\widehat{u} \neq u_0, \quad \widehat{u} \in \mathrm{int}C_+ \cup \{0\}.$$

It remains to show that $\widehat{u}$ is nontrivial. On account of hypothesis $H_1'(iv)$, given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\left| j(z,x) - \frac{\widehat{\eta}}{\tau}|x|^\tau \right| \leqslant \varepsilon |x|^\tau \text{for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta. \tag{4.18}$$

Consider the C^1-functional $\psi_+ : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi_+(u) = \int_\Omega G(Du)\,dz - \frac{1}{\tau}\|u^+\|_\tau^\tau \quad \forall u \in W_0^{1,p}(\Omega).$$

Let $\widetilde{\varphi}_+ = \varphi_+|_{C_0^1(\overline{\Omega})}$ and $\widetilde{\psi}_+ = \psi_+|_{C_0^1(\overline{\Omega})}$. From Palais [19, Theorem 16], we have

$$C_k(\widetilde{\varphi}_+,0) = C_k(\varphi_+,0) \quad \text{and} \quad C_k(\widetilde{\psi}_+,0) = C_k(\psi_+,0) \quad \forall k \in \mathbb{N}_0. \tag{4.19}$$

Then using (4.18) and Theorem 5.1 of Corvellec-Hantoute [8] (with $U = B_\delta^{C_0^1(\overline{\Omega})} = \{C_0^1(\overline{\Omega}) : \|u\|_{C_0^1(\overline{\Omega})} < \delta\}$ and choosing $\varepsilon > 0$ small), we obtain

$$C_k(\widetilde{\varphi}_+,0) = C_k(\widetilde{\psi}_+,0) \quad \forall k \in \mathbb{N}_0,$$

so

$$C_k(\varphi_+,0) = C_k(\psi_+,0) \quad \forall k \in \mathbb{N}_0 \tag{4.20}$$

(see (4.20)).

As in the Proposition 3.7 of Papageorgiou-Rădulescu [21], the presence of the concave term near zero (see hypothesis $H_1'(iv)$) implies that

$$C_k(\psi_+,0) = 0 \quad \forall k \in \mathbb{N}_0,$$

so

$$C_k(\varphi_+,0) = 0 \quad \forall k \in \mathbb{N}_0. \tag{4.21}$$

On the other hand $\widehat{u} \in K_{\varphi_+}$ is of mountain pass type and so Theorem 4.7 of Corvellec [6] implies that

$$C_1(\varphi_+,\widehat{u}) \neq 0. \tag{4.22}$$

From (4.21) and (4.22) we infer that $\widehat{u} \neq 0$. So, $\widehat{u} \in \mathrm{int} C_+$ is the second positive solution of 1.1.

Similarly, working with φ_- and $v_0 \in -\mathrm{int} C_+$, we produce a second negative solution $\widehat{v} \neq v_0$, $\widehat{v} \in -\mathrm{int} C_+$. $\square$

In fact we can produce extremal constant sign solution, that is, a smallest positive solution and a biggest negative solution (barrier solutions). We will use these solutions to produce a nodal solution.

On account of hypotheses $H_1'(i)$, (iv), for a given $\varepsilon > 0$, we can find $\eta_1 \in (0, \eta_0)$ and $c_{25} > 0$ such that

$$u^* x \geqslant \eta_1 |x|^\tau - c_{25} |x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \ u^* \in \partial j(z,x). \tag{4.23}$$

Motivated by this unilateral growth condition for the subdifferential $\partial j(z, \cdot)$, we consider the following auxiliary Dirichlet problem

$$\begin{cases} -\mathrm{div}\, a(Du) = \eta_1 |u|^{\tau-2} u - c_{25} |u|^{r-2} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4.24}$$

From this problem, we have the following result (see Papageorgiou-Rădulescu [21, Proposition 3.5]).

Proposition 4.5. *If hypotheses H_0', H_1' hold, then problem* (4.24) *has a unique positive solution $\overline{u} \in \mathrm{int} C_+$ and since the equation is odd, $\overline{v} = -\overline{u} \in -\mathrm{int} C_+$ is the unique negative solution of* (4.24).

Let S_+ (respectively S_-) be the set of positive (respectively negative) solution of (1.1). We know that

$$\emptyset \neq S_+ \subseteq \mathrm{int} C_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq -\mathrm{int} C_+$$

(see Proposition 3.8). The solutions from Proposition 4.5 provide bounds for the elements of S_+ and of S_- respectively.

Proposition 4.6. *If hypotheses H_0' and H_1' hold, then $\overline{u} \leqslant u$ for all $u \in S_+$ and $v \leqslant \overline{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+ \subseteq \mathrm{int} C_+$ and introduce the Carathéodory function $\gamma_+(z,x)$ defined by

$$\gamma_+(z,x) = \begin{cases} \eta_1 (x^+)^{\tau-1} - c_{25} (x^+)^{r-1} & \text{if } \ x \leqslant u(z), \\ \eta_1 u(z)^{\tau-1} - c_{25} u(z)^{r-1} & \text{if } \ u(z) < x. \end{cases} \tag{4.25}$$

We set

$$\Gamma_+(z,x) = \int_0^x \gamma_+(z,s)$$

and consider the C_1-functional $k_+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$k_+(u) = \int_\Omega G(Du)\, dz - \int_\Omega \Gamma_+(z,u)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Corollary 2.3 and (4.25) imply that k_+ is coercive. Also via the Sobolev embedding theorem, we see that k_+ is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$k_+(\widehat{u}) = \inf\{k_+(u) : \ u \in W_0^{1,p}(\Omega)\}. \tag{4.26}$$

Since $\tau < q < p < r$, we see that

$$k_+(\widehat{u}) < 0 = k_+(0),$$

so $\widehat{u} \neq 0$.

From (4.26), we have

$$k_+'(\widetilde{u}) = 0,$$

so
$$\langle V(\widehat{u}), h\rangle = \int_\Omega k_+(z,\widetilde{u})h\,dz \quad \forall h \in W_0^{1,p}(\Omega). \tag{4.27}$$
In (4.27) we choose the test function $h = -\widetilde{u}^- \in W_0^{1,p}(\Omega)$. Using Lemma 2.2, we obtain
$$\frac{c_1}{p-1}\|D\widetilde{u}^-\|_p^p \leqslant 0,$$
so
$$\widetilde{u} \geqslant 0, \quad \widetilde{u} \neq 0.$$
In (4.27) we use the test function $h = (\widetilde{u}-u)^+ \in W_0^{1,p}(\Omega)$. We have
$$\begin{aligned}
&\langle V(\widetilde{u}), (\widetilde{u}-u)^+\rangle \\
= &\int_\Omega (\eta_1 u^{\tau-1} - c_{25}u^{r-1})(\widetilde{u}-u)^+\,dz \\
\leqslant &\int_\Omega g_+^*(\widetilde{u}-u)^+\,dz \\
= &\langle V(u), (\widetilde{u}-u)^+
\end{aligned}$$
(since $u \in S_+$) with $g_+^* \in S^{r'}_{\partial j(\cdot,u(\cdot))}$ corresponding to $u \in S_+$ (see (4.25) and (4.23)), so
$$\widetilde{u} \leqslant u.$$
So, we have proved that
$$\widetilde{u} \in [0,u],\ \widetilde{u} \neq 0. \tag{4.28}$$
From (4.28), (4.25), (4.27) and Proposition 4.5, it follows that $\widetilde{u} = \overline{u}$, so
$$\overline{u} \leqslant u \quad \forall u \in S_+$$
(see (4.28)).

Similarly we show that $v \leqslant \overline{v}$ for all $v \in S_-$. □

Now that we have these bounds, we can generate the extremal constant sign solutions of (1.1).

Proposition 4.7. *If hypotheses H_0', H_1', then problem* (1.1) *has a smallest positive solution $u_* \in S_+ \subseteq \mathrm{int}C_+$ (that is, $u_* \leqslant u$ for all $u \in S_+$) and a biggest negative solution $v_* \in -\mathrm{int}C_+$ (that is, $v \leqslant v_+$ for all $v \in S_-$).*

Proof. From Filippakis-Papageorgiou [9], we know that S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then there exists $u \in S_+$ such that $u \leqslant u_1$, $u \leqslant u_2$). So, we may focus only on $S_+ \cap [0,\vartheta_+] \neq \emptyset$ (see (4.7)). Then using Lemma 3.10 of Hu-Papageorgiou [14, p.178], we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq S_+ \cap [0,\vartheta_+]$ such that
$$\inf S_+ = \inf_{n\in\mathbb{N}} u_n.$$
Evidently $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded and so, we may assume that
$$u_n \xrightarrow{w} u_* \quad \text{in } W_0^{1,p}(\Omega), \quad u_n \longrightarrow u_* \quad \text{in } L^r(\Omega). \tag{4.29}$$
We have
$$V(u_n) = g_n^* \quad \text{in } W^{-1,p'}(\Omega) \quad \forall n \in \mathbb{N}, \tag{4.30}$$
with $g_n^* \in S^{r'}_{\partial j(\cdot,u_n(\cdot))}$. On (4.30) we act with $u_n - u_* \in W_0^{1,p}(\Omega)$. Then
$$\langle V(u_n), u_n - u_*\rangle = \int_\Omega g_n^*(u_n - u_*)\,dz,$$
so
$$\lim_{n\to+\infty} \langle V(u_n), u_n - u\rangle = 0$$
(see (4.29) and hypothesis $H_1'(i)$), thus
$$u_n \longrightarrow u_* \quad \text{in } W_0^{1,p}(\Omega) \tag{4.31}$$

(see Proposition 2.6) with $\overline{u} \leqslant u_*$ (see Proposition 4.6).

On account of hypothesis $H_1'(i)$, we may assume that

$$g_n^* \xrightarrow{w} g^* \quad \text{in } L^{r'}(\Omega). \tag{4.32}$$

Moreover, we may also assume that

$$u_n(z) \longrightarrow u(z) \quad \text{for a.a. } z \in \Omega \tag{4.33}$$

(see (4.33)). Since $g_n^*(z) \in \partial j(z, u_n(z))$ for a.a. $z \in \Omega$, all $n \in \mathbb{N}$, from (4.32), (4.33), Proposition 3.9 of Hu-Papageorgiou [14, p. 694] and recalling that the multifunction $x \mapsto \partial j(z,x)$ is upper semicontinuous, we have

$$g^*(z) \in \overline{\text{conv}} \limsup_{n\to+\infty} \partial j(z, u_n(a)) \subseteq \partial j(z, u(z)) \quad \text{for a.a. } z \in \Omega. \tag{4.34}$$

If in (4.30) we pass to the limit as $n \to +\infty$ and use (4.31), (4.32), we obtain

$$V(u_*) = g^* \quad \text{in } W^{-1,p'}(\Omega),$$

with $g^* \in S^{r'}_{\partial j(\cdot, u_*(\cdot))}$, so $u_* \in S_+ \subseteq \text{int} C_+$ (see (4.31)), with $u_* = \inf S_+$.

Working similarly we produce $v_* \in S_-$ such that $v \leqslant v_*$ for all $v \in S_-$. We mention that S_- is upward directed (that is, if $v_1, v_2 \in S_-$, then there exists $v \in S_-$ such that $v_1 \leqslant v$, $v_2 \leqslant v$; see Filippakis-Papageorgiou [9]). □

Now we are ready to produce a nodal solution. The idea is the following. Using suitable truncation, we focus on the order interval $[v_*, u_*]$. Any nontrivial solution of (1.1) in this interval district from u_* and v_*, will be nodal. So, the goal is to find such a solution in $[v_*, u_*]$.

Implementing this strategy, first we truncate $j(z,\cdot)$ at $u_*(z)$ and at $v_*(z)$. So, we introduce the function

$$i(z,x) = \begin{cases} j(z, v_*(z)) & \text{if} \quad x < v_*(z), \\ j(z,x) & \text{if} \quad v_*(z) \leqslant x \leqslant u_*(z), \\ j(z, u_*(z)) & \text{if} \quad u_*(z) < x. \end{cases} \tag{4.35}$$

Evidently $z \mapsto i(z,x)$ is measurable and $x \mapsto i(z,x)$ is locally Lipschitz. We have

$$\partial i(z,x) = \begin{cases} = \{0\} & \text{if} \quad x < v_*(z), \\ \subseteq \text{conv}\{s\partial j(z,0):\ 0 \leqslant s \leqslant 1\} & \text{if} \quad x = v_*(z), \\ = \partial j(z,x) & \text{if} \quad v_*(z) < x < u_*(z), \\ \subseteq \text{conv}\{t\partial j(z,0):\ 0 \leqslant t \leqslant 1\} & \text{if} \quad x = u_*(z), \\ \{0\} & \text{if} \quad u_*(z) < x. \end{cases} \tag{4.36}$$

Also, we introduce the positive and negative truncation of $i(z,\cdot)$, namely the locally Lipschitz integrands

$$i_\pm(z,x) = i(z, \pm x^\pm). \tag{4.37}$$

We have

$$\partial i_+(z,x) \quad = \quad \begin{cases} = \{0\} & \text{if} \quad x < 0, \\ \subseteq \text{conv}\{s\partial i(z,0):\ 0 \leqslant s \leqslant 1\} & \text{if} \quad x = 0, \\ = \partial i(z,x) & \text{if} \quad 0 < x < u_*(z), \\ \subseteq \text{conv}\{t\partial i(z,0):\ 0 \leqslant t \leqslant 1\} & \text{if} \quad x = u_*(z), \\ \{0\} & \text{if} \quad u_*(z) < x. \end{cases} \tag{4.38}$$

$$\partial i_-(z,x) \quad = \quad \begin{cases} = \{0\} & \text{if} \quad x < v_*(z), \\ \subseteq \text{conv}\{s\partial i(z,0):\ 0 \leqslant s \leqslant 1\} & \text{if} \quad x = v_*(z), \\ = \partial i(z,x) & \text{if} \quad v_*(z) < x < 0, \\ \subseteq \text{conv}\{t\partial i(z,0):\ 0 \leqslant t \leqslant 1\} & \text{if} \quad x = 0, \\ \{0\} & \text{if} \quad 0 < x. \end{cases} \tag{4.39}$$

We consider the following locally Lipschitz functionals defined on $W_0^{1,p}(\Omega)$:

$$\begin{aligned} \sigma(u) &= \int_\Omega G(Du)\,dz - \int_\Omega i(z,u)\,dz, \\ \sigma_\pm(u) &= \int_\Omega G(Du)\,dz - \int_\Omega i_\pm(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega). \end{aligned}$$

Proposition 4.8. *If hypotheses H_0', H_1' hold, then $u_* \in \mathrm{int}C_+$ and $v_* \in -\mathrm{int}C_+$ are local minimizers of σ.*

Proof. From Corollary 2.3 and (4.35), (4.37), we see that the functional σ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\widetilde{u}_*) = \inf_{u \in W_0^{1,p}(\Omega)} \sigma_+(u) < 0 = \sigma_+(0) \tag{4.40}$$

(using hypothesis $H_1'(iv)$ and since $\tau < q < p < r$), so $\widetilde{u}_* \neq 0$.

Using (4.38) and the nonlinear regularity theory of Lieberman [15], we show that

$$K_{\sigma_+} \subseteq [0, u_*] \cap C_+,$$

so

$$K_{\sigma_+} = \{0, u_*\}$$

(since u_* is extremal), thus

$$\widetilde{u}_* = u_* \in \mathrm{int}C_+ \tag{4.41}$$

(see (4.40)).

From (4.35) and (4.37), we see that

$$\sigma|_{C_+} = \sigma_+|_{C_+}.$$

Then (4.40) and (4.41) imply that

$$u_* \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \sigma$$

so

$$u_* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \sigma$$

(see Gasiński-Papageorgiou [10]). Similarly for $v_* \in -\mathrm{int}C_+$, using this time the functional σ_-. □

We are ready to have the nodal solution.

Proposition 4.9. *If hypotheses H_0', H_1' hold, then problem* (1.1) *has a nodal solution*

$$y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega}).$$

Proof. From Proposition 4.8 we know that

$$u_* \in \mathrm{int}C_+ \text{ and } v_* \in -\mathrm{int}C_+ \text{ are local minimizers of } \sigma. \tag{4.42}$$

We may assume that

$$\sigma(v_*) \leqslant \sigma(u_*).$$

The reasoning is similar if the opposite inequality holds.

Using (4.36) and the nonlinear regularity theory, we show that

$$K_\sigma \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}). \tag{4.43}$$

Again we assume that K_σ is finite (otherwise on account of (4.43) we already have an infinity of smooth nodal solutions and so we are done). We can find $\varrho \in (0,1)$ small such that

$$\begin{cases} \tau(v_*) \leqslant \sigma(u_*) < \inf\{\sigma(u) : \ \|u - u_*\| = \varrho\} = \widehat{m}, \\ \|v_* - u_*\| > \varrho \end{cases} \tag{4.44}$$

(see Papageorgiou-Rădulescu-Repovš [23, p. 449]).

Since σ is coercive (see (4.35)), we have that

$$\sigma \text{ satisfies the nonsmooth Cerami condition.} \tag{4.45}$$

Then (4.44) and (4.45) permit the use of Theorem 2.9. So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_\sigma, \quad \widehat{m} \leqslant \sigma(y_0), \quad C_1(\sigma, y_0) \neq 0 \tag{4.46}$$

(see Corvellec [6]). From (4.46) and (4.44), we see that

$$y_0 \notin \{u_*, v_*\}.$$

Note that

$$\sigma|_{[v_*,u_*]} = \varphi|_{[v_*,u_*]}$$

(see (4.35)).

Since $v_* \in -\mathrm{int}C_+$, $u_* \in \mathrm{int}C_+$ via a homotopy invariance argument as in the proof of Theorem 3.6 and using Theorem 5.2 of Corvellec-Hantoute [8], we have

$$C_k(\sigma, 0) = C_k(\varphi, 0) \quad \forall k \in \mathbb{N}_0,$$

so

$$C_k(\sigma, 0) = 0 \quad \forall k \in \mathbb{N}_0 \tag{4.47}$$

(see the proof of Proposition 3.9).

Comparing (4.47) and (4.46), we infer that $y_0 \neq 0$. The nonlinear regularity theory implies that $y_0 \in [v_*, u_*] \cap (C_0^1(\overline{\Omega}) \setminus \{0\})$, hence y_0 is a nodal solution of (1.1). □

Summarizing, we can state the following multiplicity theorem for problem (1.1).

Theorem 4.10. *If hypotheses H_0' and H_1' hold, then problem* (1.1) *has at least five nontrivial solutions*

$$u_*, \widehat{u} \in \mathrm{int}C_+, \quad u_* \leqslant \widehat{u}, \quad u_* \neq \widehat{u},$$
$$v_*, \widehat{v} \in -\mathrm{int}C_+, \quad \widehat{v} \leqslant v_*, \quad v_* \neq \widehat{v},$$
$$y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega}) \text{ nodal.}$$

Remark 4.11. We stress that in the above multiplicity theorem, we provide sign information for all the solutions produced and the solutions are ordered $\widehat{v} \leqslant v_* \leqslant y_0 \leqslant u_* \leqslant \widehat{u}$.

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