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Singular Perturbations of Multibrot Set Polynomial[†]

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Abstract

We will give a complete description of the dynamics of the rational map $N_{F_{M_c}}(z) = \frac{3z^4-2z^3+c}{4z^3-3z^2+c}$ where c is a complex parameter. These are rational maps $N_{F_{M_c}}$ arising from Newton's method. The polynomial of Newton iteration function is obtained from singularly perturbed of the Multibrot set polynomial.

1. Introduction

Singular perturbations arise in all areas of dynamical systems from ODSs to discrete dynamical systems. There is a wide range of examples of singular perturbations in these areas [1–3]. For a simple example, suppose that we are applying Newton's method to find the roots of a complex polynomial equation $P(z) = z^2 - c$. The Newton iteration function is given by $N_P(z) = z - \frac{P(z)}{P'(z)} = \frac{z}{2} + \frac{c}{2z}$ when $c = 0$, the polynomial P has multiple roots at 0 and Newton iteration function is $N_P(z) = \frac{z}{2}$. In this case, of course, all orbits of $N_P(z) = \frac{z}{2}$ tend to be the unique root at 0. However, when $c \neq 0$, the degree of N_P jumps from 1 to 2, and dynamical behavior of N_P become excited. Moreover, instead of a fixed point at the origin, after perturbation, there is a pole at the origin, most orbits of N_P still do convergence to one of the two roots of P, that is $\pm\sqrt{c}$ but points on the straight line passing through the origin perpendicular to the line segment connecting $\pm\sqrt{c}$ have orbits that do not convergence to these roots. Rather all orbits on these lines behave chaotically, so the dynamical behavior is more complicated in this case.

In recent years, much attention has been paid to families of rational maps that arise as singular perturbation of polynomials. These are families of rational maps that depend on a parameter λ and have the property that, when $\lambda = 0$, the map involved is a polynomial of degree n, but for all other parameters, the maps are rational with a higher degree. When the parameter λ becomes non-zero, the dynamics of these maps are explored. Most of the studies of these singular perturbed rational maps have centered on families of the form $F_\lambda(z) = z^n + \frac{\lambda}{z^d}$ where $\lambda \in \mathbb{C}$, n, and d are positive integers [4]. A singular perturbation means that we have a complex analytic map which is the new map F_{M_c} obtained by multiplying Multibrot set polynomial $M_c(z) = z^n + c$ and a simple polynomial $P(z) = z - 1$ so that $F_{M_c}(z) = (z^n + c)(z-1)$ where c is a complex parameter and $n > 2$. In this study, specifically we consider the case when Newton's method is applied to the polynomial family $F_{M_c(z)} = (z^3 + c)(z-1)$. The dynamics of such a perturbation are very exciting for the following reasons:

1. they are non-polynomial examples,
2. their dynamical behavior is changed dramatically when the parameter c is non-zero quite small.

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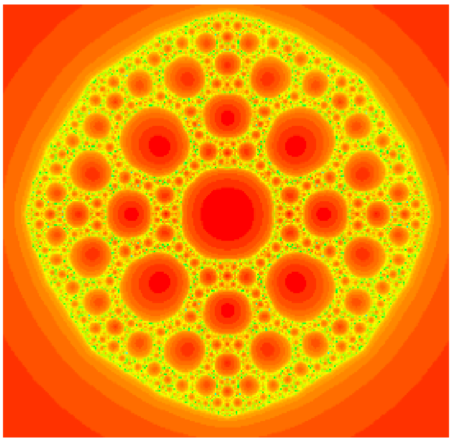
(a) $\lambda = 0$.



(b) $\lambda \neq 0$.

Figure 1.1: How the dynamics is explodes when the degree is changed. For more detail [4].

In complex dynamics, the most important object in the dynamical plane is the *Julia set* of F, which we denote by $J(F)$. From an analytic viewpoint, the Julia set is the set of points at which the family of iterates on the map fails to be a normal family in the sense of Montel. There are many other equivalent definitions of the Julia set such as the Julia set is the closure of the set of repelling periodic points of F. Equivalently, the Julia set is also the boundary of the set of points whose orbits escape to ∞. From a dynamic point of view, the Julia set is the set of points on which the map is chaotic. The complement of the Julia set is called the *Fatou set*. This is where the dynamical behavior is relatively tame [2, 3, 5, 6].

The aim of this paper is to investigate the dynamics and the Julia sets of Newton iteration function, $N_{F_{M_c}}(z)$, applied to the polynomial $F_{M_c}(z) = (z^n + c)(z-1)$. We shall pay attention to one special critical point and see how the orbit of this point affects the dynamics of the Newton iteration map.

Newton's method is the best known iterative method for finding roots(real or complex) of a function f. It is the iterating function $N_f(z) = z - \frac{f(z)}{f'(z)}$ by starting with some initial approximation z_0 and defining the $n+1$ approximation by $z_{n+1} = N_f(z_n)$. Whether the function $f(z)$ is a polynomial or a rational function, then the iteration function N_f will be a rational map of the form $N(z) = N_f(z) = \frac{p(z)}{q(z)}$ where p and q are polynomials. So the dynamics of Newton's method become more difficult even when applied to polynomials in one variable. Iteration of Newton's method function often allows one to find the roots of the corresponding polynomial, but this is not always the case. The orbit of a point z_0 is the set of iterates of the function f which gives the sequence $\{z_0, N_f(z_0), N_f^2(z_0), N_f^3(z_0), \ldots\}$. This sequence hopefully converges to a root, ζ, of f. That certainly happens most of the time but other things might happen. For instance, if a function is not differentiable at the root such as considering the function $f(x) = x^{1/3}$, this function is not differentiable at the root x=0 and $|N_f'(0)| > 1$, then all sequences tend to ∞. Thus we may have no convergence if there is no differentiability. In some cases, the convergence of Newton's method is guaranteed by Kantorovitch theorem [7].

We shall think of Newton's iterating function as being defined on the whole the Riemann sphere, i.e. the complex numbers with the point at infinity adjoined, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The orbit of a point ζ could converge to a cycle, or it could wander chaotically about Riemann sphere, or it could behave in other ways. A point $\zeta \in \mathbb{C}$ is called a *periodic point* of period n if $N_f^n(\zeta) = \zeta$ and $N_f^k(\zeta) \neq \zeta$ for all $k < n$, where $k, n \in \mathbb{N}$. The least such integer n is called the *period* and the orbit of ζ is then an n-cycle. If $n = 1$, we say that ζ is a fixed point of N_f and, as is well known, such points correspond to the roots of f. A point ζ is *eventually periodic* if $N^n(\zeta) = N^{n+k}(\zeta)$ for positive integers n and k. If ζ is a periodic point of period n, then the derivative $\lambda = (N_f^n)'(\zeta)$ is called the *eigenvalue* of the periodic point ζ. It follows from the chain rule that λ is the product of the derivatives of N_f at each point on the orbit of ζ. Hence λ is an invariant of the orbit. A periodic orbit is called *attracting* if $|\lambda| < 1$, *super-attracting* if $|\lambda| = 0$, *repelling* if $|\lambda| > 1$, and *neutral* if $|\lambda| = 1$. Using Taylor's series for $N_f(\zeta)$, it can be shown that $N_f(\zeta)$ will be linearly convergent at an attracting fixed point and at least quadratically convergent at a super-attracting fixed point. Recall that the sequence $\{\zeta_n\}$ *convergence linearly* to w if, for sufficiently large n, $|\zeta_{n+1} - w| < t|\zeta_n - w|$, where $0 < c < 1$, where $0 < t < 1$, and *convergence quadratically* if, for sufficiently large n, $|\zeta_{n+1} - w| < t|\zeta_n - w|^2$, for some constant t. The point at ∞ is always a repelling fixed point with derivative $d/(d-1)$, where d is the degree of f, so large values of ζ will tend to move away from infinity under iteration [3]. A point is a *critical point* if the derivative of the map vanishes at this point. Critical points of N_f are solutions of $N_f'(\zeta) = 0$, i.e., zeroes and inflection points of f. The critical point is non-degenerate if $N_f''(\zeta) \neq 0$ and it is degenerate if $N_f''(\zeta) = 0$. For example, $f(x) = x^n$ has a *degenerate critical point* at 0 when $n > 2$, but has a *non-degenerate* when $n = 2$. Note that degenerate critical points may be maxima, minima, or saddle points as in the case of $f(x) = x^3$ [4, 6].

Theorem 1.1 (Julia)**.** *For any holomorphic map of the extended complex plane to itself, an attracting periodic cycle must attract at least one critical point* [8].

Theorem 1.2 (P. Fatou)**.** *Every attracting cycle for a polynomial or a rational function attracts at least one critical point* [4].

Theorem 1.3 (By The Riemann Hurtwitz Relation)**.** *A non-constant rational map with degree d has exactly $2d-2$ critical points in $\mathbb{C}_\infty$, counted with multiplicity* [8].

The critical points play a dominant role in determining the structure of the Julia set of rational iteration. In this paper, we will point out the case where the value of parameter c becomes non-zero, and when it happens, how the dynamical behavior changes strikingly.
We are interested in the dynamics of Newton's iteration map, N_f on the Riemann sphere. We can always conjugate N_f by an invertible linear(Möbius) transformation T, so the orbits of N_f will be essentially the same as the orbits of $T \circ N_f \circ T^{-1}$. On the Riemann sphere, the point at infinity is like any other point. In order to determine whether infinity is a fixed point of N_f and to find its eigenvalue there, we can conjugate N_f by the transformation $z \to \frac{1}{z}$ that interchanges 0 and ∞. Therefore the behavior of $N_f(z)$ at ∞ is the same as the behavior of $\frac{1}{N_f(1/z)}$ at 0.

The basin of attraction of a fixed point υ of the map N_f is the set $\left\{z | \lim_{n\to\infty} N_f^n(z) = \upsilon\right\}$, i.e., the set of all points whose orbits converge to υ under the iteration of N_f. This basin may have infinitely many components, and the immediate basin of attraction is the connected component containing the fixed point υ. The rational map N_f divides the Riemann sphere into two invariant sets, the Julia set, $J(N_f)$, and its complement. As mentioned earlier, the Julia set consists of points for which the dynamical behavior under iteration of N_f is complicated. Points in the complement of the Julia set will normally converge to a fixed point or an attracting cycle. This complement could also contain a Siegel disk or Herman ring in which the iterations are locally like an irrational rotation of a disk or an annulus.
A few basic facts about the Newton basin:

- The rational map N_f divides the Riemann sphere into two invariant sets, the *Julia set*, $J(N_f)$, and its complement.
- The points in the complement of the Julia set will normally converge to a fixed point, that could be infinity, or to an attractive cycle.
- $J(N_f)$ is the closure of the repelling periodic points.
- $J(N_f)$ is non-empty.
- $J(N_f)$ is completely invariant under N_f; i.e. $N_f(J(N_f)) = J(N_f) = N_f^{-1}(J(N_f))$.
- $J(N_f)$ is the boundary of the basin of attraction of each fixed point or attractive cycle: this guarantees that if there are more than two roots, $J(N_f)$ will be a fractal set.
- If $v \in J(N_f)$, then the closure of $\left\{z | N_f^n(z) = v \text{ for some non-negative integer} n\right\}$, the backward iterates of v, is the whole of $J(N_f)$.

It is well known that the Julia set is an unstable set. Iterates of points close to the Julia set will move away from that set. Hence Newton's method is very sensitive to initial conditions when the initial point is near the Julia set. Nearby points could converge to different roots or might not converge at all. If you start with a point actually on the Julia set, the iterates will also be on the Julia set because Julia set is a completely invariant set. As it is mentioned above, unfortunately, Newton's map does not converge to a root for every initial point. But the orbit could converge to an attractive cycle, rather than to a root.

2. The Dynamics of the Rational Map

In this section we consider the dynamics of the perturbed map which is a special class of rational functions, namely those obtained from Newton's method as applied to a polynomial of the form $F_{M_c(z)} = (z^3 + c)(z-1)$. We are interested in the collection of Newton iteration maps given by $N_{F_{M_c}}$ as their dynamical properties are related to the non-degenerate free critical point.

Proposition 2.1. *Infinity is a repelling fixed point for Newton's method applied to $F_{M_c}(z) = (z^3 + c)P(z)$ where $P(z) = z - 1$ and c is any constant.*

Proof. Newton's method function is the rational map:

$$N_{F_{M_c}}(z) = z - \frac{M_c(z)P(z)}{M_c^{'}(z)P(z) + M_c(z)P^{'}(z)} = \frac{3z^4 - 2z^3 + c}{4z^3 - 3z^2 + c},$$

∞ is a fixed point, since $\lim_{z\to\infty} N_{F_{M_c}}(z) = \infty$. To determine its nature, we map ∞ to 0 via $g(z) = \frac{1}{z} (= v)$: the conjugate function G is given by $g \circ N_{F_{M_c}} = G \circ g$ thus we obtain

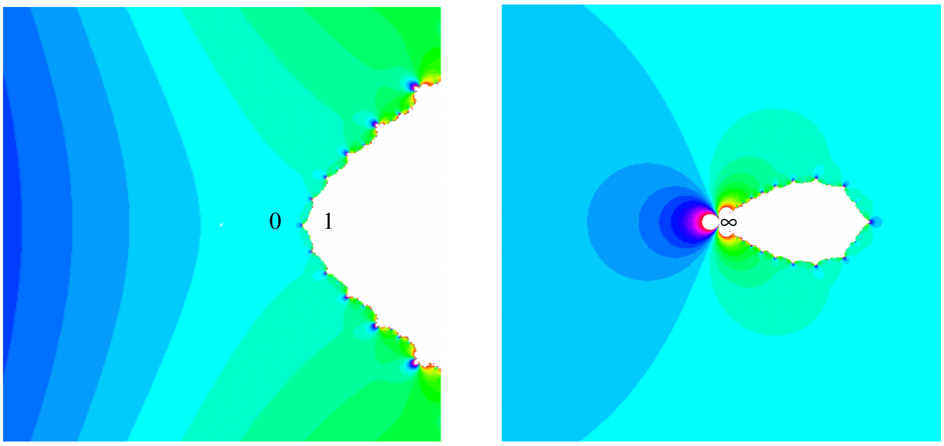
$$G(v) = g\left(N_{F_{M_c}}\left(\frac{1}{v}\right)\right) = \frac{1}{N_{F_{M_c}}\left(\frac{1}{v}\right)} = \frac{4v - 3v^2 + 4v^4}{3 - 2v + cv^4}.$$

∞ is a repelling fixed point, since $G(0) = 0$ and $|G^{'}(0)| > 1$. ☐

Before examining the dynamics of F_{M_c} when c is small, we will consider the dynamics of the case $c = 0$.

2.1. The dynamics of $F_{M_0} = z^3(z-1)$

The Newton iterating function of F_{M_0} is a rational map of the form $N_{F_{M_0}}(z) = \frac{3z^4 - 2z^3}{4z^3 - 3z^2}$. The finite fixed points of $N_{F_{M_0}}(z)$ are 0 and 1 which are an attracting fixed point and a super-attracting fixed point, respectively. In addition, ∞ is a repelling fixed point. In Figures 2.1a and 2.1b, the computer graphics pictures illustrate $N_{F_{M_0}}(z)$ on the dynamical plane. Each color in the picture belongs to a finite root of $N_{F_{M_0}}(z)$. In Figure 2.1a, the area from blue to turquoise is the basin of attraction for the attracting fixed point 0 and the white area is the attracting basin for the super-attracting fixed point 1 of $N_{F_{M_0}}(z)$. In Figure 2.1b, the same basins are shown when viewed from infinity. It is the simple case $c = 0$ for Newton iteration that has decorations on the Julia set on the boundary of the basin; rather this boundary is a simple closed curve passing through ∞.



(a) **(b)**

Figure 2.1

One of the most important goals of Newton's method is to approximate the roots of a function - for which the convergence of the initial values is an important matter in dynamical systems. In Figure 2.1a and 2.1b, the speed of convergence for Newton's map of the function $z^3(z-1)$ is clearly observed. The critical orbits play a dominant role in determining the structure of the Julia sets in dynamical systems. Points $0,1$ and $1/2$ are critical points of $N_{F_{M_0}}$.
The aim of this paper is to draw attention to the case where the value of the parameter c becomes non-zero but quite small. When this happens, the dynamical behavior changes dramatically. We will now describe those changes.

2.2. The dynamics of $F_{M_c}(z) = (z^3+c)(z-1)$ for $c \neq 0$

We will deal with the value of c being different from zero but rather small. When we applied Newton's method to the polynomial $F_{M_c}(z) = (z^3+0.001)(z-1)$ obtained the rational map,

$$N(z) = N_{F_{M_{0.001}}}(z) = \frac{3z^4-2z^3+0.001}{4z^3-3z^2+0.001}.$$

∞ is a repelling fixed point and the real roots -0.1 and 1 are super-attracting fixed points of N. In addition to this, the complex roots are $0.05 \pm 0.0866025i$ for the rational maps N with the parameter $c = 0.001$. The points $0.05 \pm 0.0866025i, -0.1, 0, 1$ and $1/2$ are critical points for N. Critical points $0, 1/2$ and 1 are common critical points for the maps $N_{F_{M_{0.001}}}$ and $N_{F_{M_0}}$ with different critical values and also they are non-degenerate critical points. In addition to this, the common critical point 1 is a super-attracting fixed point for the maps $N_{F_{M_{0.001}}}$ and $N_{F_{M_0}}$.

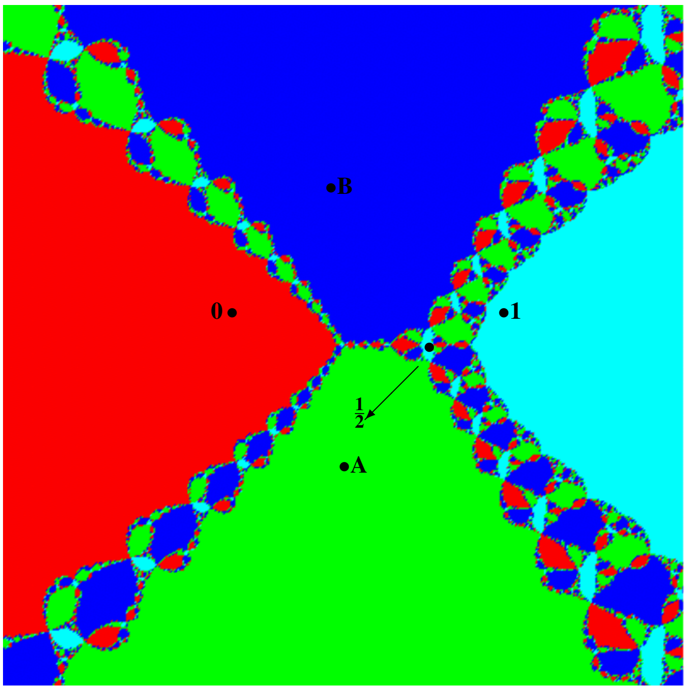
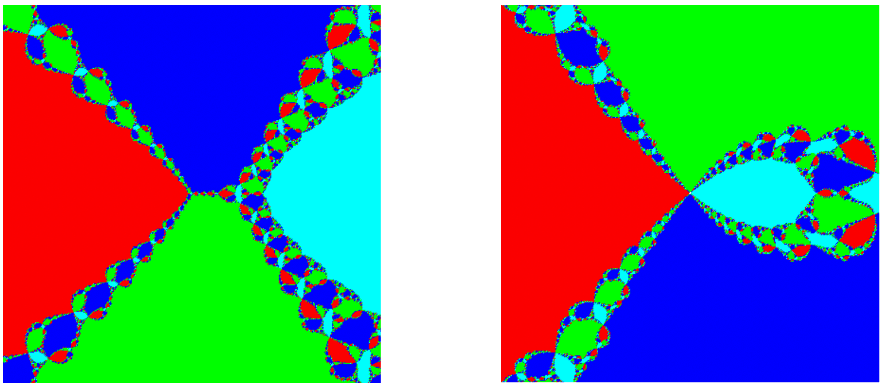


Figure 2.2: $A = 0.05 - 0.0866025i$ and $B = 0.05 + 0.0866025i$.

In Figure 2.2, the computer graphics picture illustrates how points behave under iteration of $N(z)$ in a dynamical plane. First of all, we will make clear the fact that we are considering the complex plane, the x-axis is the real direction and the y-axis is the imaginary direction.

Newton's map, N, for the polynomial $F_{M_{0.001}}(z) = z^4 - z^3 + (0.001)z - 0.001$ has degree 4. Since the function has four roots, the graph of the complex plane is divided into four parts, each of which is a basin of attraction for a root. Colors indicate to which of the four roots a given starting point converges to the finite roots of Newton's map which are contained in the Fatou set. The turquoise area is the basin of super-attracting fixed point for the map N, $\mathscr{A}_N(1) = \{z \in \mathbb{C} : N^n(z) \to 1, n \to \infty\}$. The boundary of the Newton basin including decorations is the Julia set, $\mathscr{A}_N(1) = \mathscr{J}(N_F)$, on which the dynamics of Newton iteration map are chaotic. The free critical point $1/2$ lies in the real axis and in a pre-image of the immediate basin of 1. Every root can be connected ∞ within its basin of attraction. Note importantly that there are no black regions in the basins, so Newton's map does not fail anywhere on that basin. The decorations on the boundary of the four immediate basins correspond to their pre-images. In addition, the immediate basin of attraction is a connected component containing the fixed points of N. It is no longer just a simple closed curve as in the case $c = 0$.



(a) (b)

Figure 2.3: The parameter plane pictures for the view from 0 and ∞ for the parameter $c = 0.001$.

Theorem 2.2 (P. Fatou)**.** *The immediate basin of an attracting fixed point or cycle of N contains at least one critical point of N.*

In this paper, Newton's iteration map has free critical points that determine the fate of orbit in the complex dynamical behavior of N. The vital effect in the formation of this situation is in parameter c. When the parameter c takes a non-zero value, which is quite small, a dramatical change in the dynamics of the iteration map is observed. The importance of the periodic point in this change is seen in Figure 2.4. The parameter value of c after changing the parameter from 0 to any constant on a circle in a complex plane we see the periodic channels leading to ∞. In order to explain this situation, we change the parameter c from real to complex. For instance, in Figures 2.4-2.5, the value of parameter $c = 0.001 + 0.001i$. In Figure 2.4, the four roots of the function $F_{M_{0.001+0.001i}} : z \to (z^3 + 0.001 + 0.001i)(z-1)$ are $-0.100008 - 0.00303118i$, $0.0471496 - 0.0845998i$, $0.0528568 + 0.08863i$, $1. - 0.000998994i$. These are finite fixed points of Newton's iteration which are contained in the Fatou set. Since the function has four roots, the graph of the complex plane is divided into four parts, each of which is a basin for a root. The boundary of the basin is the chaotic part of Newton's fractal which is the Julia set. By the definition of Julia set, Newton's method does not converge on the boundary points, but it is chaotic. The Newton iteration functions for the values c have critical points 1 and $1/2$. In Figure 2.4, the green area goes to infinity and contains the free critical point. In Figure 2.5 the same area view from the point ∞.

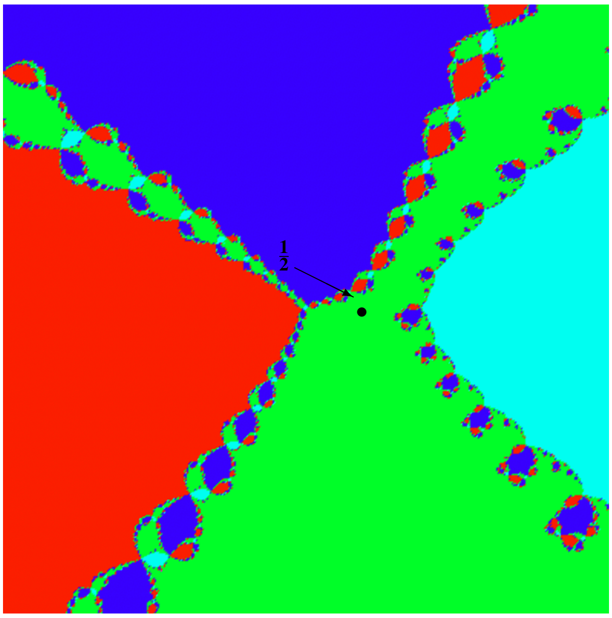


Figure 2.4: Dynamical plane view from 0.

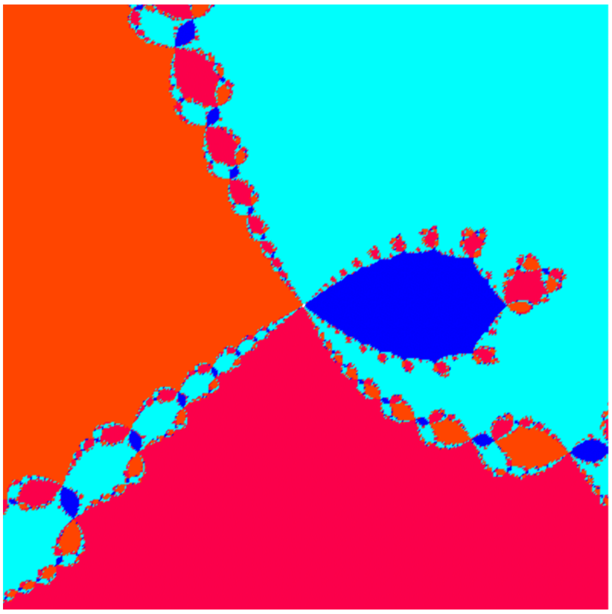


Figure 2.5: Dynamical plane view from ∞.

Corollary 2.3. *The non-degenerate free critical point plays a vital role in determining the dynamics of the rational map which arising in complex Newton's method is applied to polynomial family* $F_{M_c}(z) = (z^3 + c)(z-1)$, *where c is a complex (or non-complex) parameter.*

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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