Konuralp Journal of Mathematics, 10 (2) (2022) 276-281

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Some Differential Inequalities for Boundary Value Problems of Fractional Integro-Differential Equations

Hadi Kutlay[1] **and Ali Yakar**[1*]

[1]*Department of Mathematics, Tokat Gaziosmanpaşa University, Türkiye*
[*]*Corresponding author*

Abstract

We consider fractional integro-differential equations with boundary conditions and prove some differential inequalities related to given problem with the aid of technique of upper and lower solutions. We require these theorems because they serve as the basis for improvement of monotone iterative technique to such type of differential equations of boundary value problems.

Keywords: *Fractional differential equations, Differential inequalities, Upper and Lower solutions, Boundary Value Problem.*
2010 Mathematics Subject Classification: *26A33, 26D10, 34B05, 34B60.*

1. Introduction

The notions of the classical derivative and integral of integer order are generalized in the field of fractional calculus. The benefit and significance of using the fractional operators to simulate many physical processes across a range of disciplines have been revealed [9],[10],[6]. Theory of fractional differential equations has received a great deal of attention due to the fact that it is far more comprehensive than the theory of conventional ordinary differential equations. In recent years, there has been a substantial progress in the study of fractional differential equations as well as an increasing interest in this field. Following a survey of the literature, we identify several recent works on fundamental theoretical concepts including existence and uniqueness theorems for fractional differential equations, differential and integral inequalities and so on. See [1],[12],[4],[5],[3],[13],[14] and the references therein.
In this work, we improve some results in [14] related to fractional differential inequalities by means of the method of upper and lower solutions.

2. Mathematical Preliminaries

In this section, we recall some useful definitions and basic results to set up the main section.

Definition 2.1. *[9] **(Left and right Riemann-Liouville fractional integrals)**.*
Let $[a,b]\subset\mathbb{R}$, $Re(\alpha)>0$ *and* $f\in L_1[a,b]$ *. Then the left and right Riemann-Liouville fractional integrals* I_{a+}^{α} *and* I_{b-}^{α} *of order* α *are given by*

$$I_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}},\ x\in(a,b]$$

and

$$I_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}},\ x\in[a,b)$$

respectively.

Definition 2.2. *[9] **(Left and right Caputo fractional derivatives)** Let* $[a,b]\subset\mathbb{R}$, $Re(\alpha)\in(0,1)$ *and* $f\in L_1[a,b]$*. The left and right Caputo fractional derivatives of order* α *are*

$$\forall x\in(a,b],\ {}^{c}D_{a+}^{\alpha}f(x):=I_{a+}^{1-\alpha}Df(x)$$

Email addresses: hadi.kutlay@gop.edu.tr (Hadi Kutlay), ali.yakar@gop.edu.tr (Ali Yakar)

and

$$\forall x\in[a,b),{}^{c}D^{\alpha}_{b^-}f(x):=-I^{1-\alpha}_{b^-}Df(x)$$

respectively.

Let $F\in C[J\times\mathbb{R}\times\mathbb{R}_+,\mathbb{R}]$, $u\in C^1[J,\mathbb{R}]$, $J=[0,T]$. We consider the following fractional boundary value problem.

$$^{C}D^{q_1}u(t)=F(t,u(t),I^{q_2}u(t)),\ g(u(0),u(T))=0 \tag{2.1}$$

where $0<q_2\leq q_1<1$ and $g\in C[\mathbb{R}^2,\mathbb{R}]$. From now on, the fractional operator $^{C}D^{q}$ stands for the left Caputo fractional derivative as well as I^q represents the left Riemann Liouville fractional integral operator.

Definition 2.3. *[10] Let $\alpha,\beta\in C^1[J,\mathbb{R}]$. Then α and β are said to be*
(i) *natural lower and upper solutions of (2.1) respectively if*

$$\begin{aligned}{}^{C}D^{q_1}\alpha(t)&\leq F(t,\alpha,\alpha),\quad g(\alpha(0),\alpha(T))\leq 0\\ {}^{C}D^{q_1}\beta(t)&\geq F(t,\beta,\beta),\quad g(\beta(0),\beta(T))\geq 0\end{aligned}$$

(ii) *coupled lower and upper solutions of type I of (2.1) respectively if*

$$\begin{aligned}{}^{C}D^{q_1}\alpha(t)&\leq F(t,\alpha,\beta),\quad g(\alpha(0),\alpha(T))\leq 0\\ {}^{C}D^{q_1}\beta(t)&\geq F(t,\beta,\alpha),\quad g(\beta(0),\beta(T))\geq 0\end{aligned}$$

(iii) *coupled lower and upper solutions of type II of (2.1) respectively if*

$$\begin{aligned}{}^{C}D^{q_1}\alpha(t)&\leq F(t,\beta,\alpha),\quad g(\alpha(0),\alpha(T))\leq 0\\ {}^{C}D^{q_1}\beta(t)&\geq F(t,\alpha,\beta),\quad g(\beta(0),\beta(T))\geq 0\end{aligned}$$

(iv) *coupled lower and upper solutions of type III of (2.1) respectively if*

$$\begin{aligned}{}^{C}D^{q_1}\alpha(t)&\leq F(t,\beta,\beta),\quad g(\alpha(0),\alpha(T))\leq 0\\ {}^{C}D^{q_1}\beta(t)&\geq F(t,\alpha,\alpha),\quad g(\beta(0),\beta(T))\geq 0\end{aligned}$$

Lemma 2.4. *[10] Let $m\in C^1[J,\mathbb{R}]$ and assume that $m(t_1)=0$ for $t_1\in(0,T]$ and $m(t)\leq 0$ for $0\leq t\leq t_1$. Then we have $^{C}D^{q}m(t_1)\geq 0$.*

As is well known, Laplace transform method is one of the useful technique for solving some initial value problems. Having utilized this technique, the given problem is converted to an algebraic equation. In this context, the next lemma is crucial and gives the inverse Laplace transform of given function.

Lemma 2.5. *[7] Let $\alpha\geq\beta>0$, $\alpha>\gamma$, $a,b\in\mathbb{R}$, $s^{\alpha-\beta}>|a|$ and $\left|s^{\alpha}+as^{\beta}\right|>|b|$. Then we get*

$$\mathscr{L}^{-1}\left\{\frac{s^{\gamma}}{(s^{\alpha}+as^{\beta}+b)}\right\}=t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}t^{k(\alpha-\beta)+n\alpha}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}$$

We must establish the following Lemma, which gives the solution of the given linear fractional initial value problem. As a new result, it admits the corresponding result in [8] as a special case.

Lemma 2.6. *Assume that $\lambda\in C^1[J,\mathbb{R}]$, $0<q_2\leq q_1<1$ and $L_1,M_1\in\mathbb{R}$. The explicit solution of the following linear fractional integro-differential equation,*

$$^{C}D^{q_1}\lambda(t)=L_1\lambda(t)+M_1I^{q_2}\lambda(t),\lambda(0)=\lambda_0 \tag{2.2}$$

is given by

$$\lambda(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n(L_1)^k\binom{n+k}{k}t^{q_1(n+k)+nq_2}}{\Gamma(q_1(n+k)+nq_2+1)}\lambda_0.$$

Proof. If we apply the Laplace transform on both side of the equation (2.2), we find the following relations

$$\begin{aligned}\mathscr{L}\left\{{}^{C}D^{q_1}\lambda(t)\right\}&=L_1\mathscr{L}\{\lambda(t)\}+M_1\mathscr{L}\{I^{q_2}\lambda(t)\}\\ \frac{s\lambda(s)-\lambda_0}{s^{1-q_1}}&=L_1\lambda(s)+M_1\frac{\lambda(s)}{s^{q_2}}\\ s^{q_1}\lambda(s)-s^{q_1-1}\lambda_0&=L_1\lambda(s)+M_1\lambda(s)s^{-q_2}\\ \lambda(s)\left(s^{q_1}-L_1-M_1s^{-q_2}\right)&=s^{q_1-1}\lambda_0\\ \lambda(s)&=\frac{s^{q_1-1}}{s^{q_1}-L_1-M_1s^{-q_2}}\lambda_0\\ \lambda(s)&=\frac{s^{q_1+q_2-1}}{s^{q_1+q_2}-L_1s^{q_2}-M_1}\lambda_0\end{aligned}$$

At this point, by using the inverse Laplace transform in Lemma 2.5, we arrive at

$$\begin{aligned}
\mathscr{L}^{-1}\{\lambda(s)\} &= \mathscr{L}^{-1}\left\{\frac{s^{q_1+q_2-1}}{s^{q_1+q_2}-L_1 s^{q_2}-M_1}\right\} \\
\lambda(t) &= t^{q_1+q_2-(q_1+q_2-1)-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n (L_1)^k \binom{n+k}{k} t^{k(q_1+q_2-q_2)+n(q_1+q_2)}}{\Gamma(k(q_1+q_2-q_2)+(n+1)(q_1+q_2)-(q_1+q_2-1))}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n (L_1)^k \binom{n+k}{k} t^{kq_1+n(q_1+q_2)}}{\Gamma(kq_1+(n+1)(q_1+q_2)-(q_1+q_2-1))}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n (L_1)^k \binom{n+k}{k} t^{kq_1+n(q_1+q_2)}}{\Gamma(kq_1+n(q_1+q_2)+1)}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n (L_1)^k \binom{n+k}{k} t^{q_1(n+k)+nq_2}}{\Gamma(q_1(n+k)+nq_2+1)}\lambda_0
\end{aligned}$$

provided that $s^{q_1} > |L_1|$ and $\left|s^{q_1+q_2}-L_1 s^{q_2}\right| > |M_1|$ □

3. Main Results

We are able to provide some differential inequalities via upper and lower solutions of (2.1)

Theorem 3.1. *Let α and β be natural lower and upper solutions of (2.1). Moreover following condition holds*

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \le L(u_1-u_2)+M(v_1-v_2) \tag{3.1}$$

$L,M \ge 0$, whenever $u_1 \ge u_2, v_1 \ge v_2$.
Then we have $\alpha(t) \le \beta(t)$ provided $\alpha(0) \le \beta(0)$.

Proof. We first set $\beta_\varepsilon(t) = \beta(t) + \varepsilon\lambda(t)$ for $\varepsilon > 0$, where

$$\lambda(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M)^n 2^k (L)^k \binom{n+k}{k} t^{q_1(n+k)+nq_2}}{\Gamma(q_1(n+k)+nq_2+1)}.$$

is taken as a unique solution of the equation

$${}^{C}D^{q_1}\lambda(t) = 2L\lambda(t) + MI^{q_2}\lambda(t), \quad \lambda(0)=1 \tag{3.2}$$

Notice that $\beta_\varepsilon(0) = \beta(0)+\varepsilon\lambda(0) > \beta(0)$ and $\beta_\varepsilon(t) > \beta(t)$ for $0 \le t \le T$. If we differentiate $\beta_\varepsilon(t)$ in terms of Caputo's sense , we get

$$\begin{aligned}
{}^{C}D^{q_1}\beta_\varepsilon(t) &= {}^{C}D^{q_1}\beta(t) + {}^{C}D^{q_1}\varepsilon\lambda(t) \\
&\ge F(t,\beta(t),I^{q_2}\beta(t)) + 2L\varepsilon\lambda(t) + M\varepsilon I^{q_2}\lambda(t)
\end{aligned} \tag{3.3}$$

Since $\beta_\varepsilon(t) > \beta(t)$ on J, we can use the Liphitzlike inequality in (3.1) such that

$$F(t,\beta_\varepsilon(t),I^{q_2}\beta_\varepsilon(t)) - F(t,\beta(t),I^{q_2}\beta(t)) \le L(\beta_\varepsilon-\beta) + MI^{q_2}(\beta_\varepsilon-\beta)$$

which leads to

$$F(t,\beta(t),I^{q_2}\beta(t)) \ge F(t,\beta_\varepsilon(t),I^{q_2}\beta_\varepsilon(t)) - L\varepsilon\lambda(t) - M\varepsilon I^{q_2}\lambda(t) \tag{3.4}$$

If we substitute (3.4) into (3.3), we have

$$\begin{aligned}
{}^{C}D^{q_1}\beta_\varepsilon(t) &\ge F(t,\beta(t),I^{q_2}\beta(t)) + 2L\varepsilon\lambda(t) + M\varepsilon I^{q_2}\lambda(t) \\
&\ge F(t,\beta_\varepsilon(t),I^{q_2}\beta_\varepsilon(t)) - L\varepsilon\lambda(t) - M\varepsilon I^{q_2}\lambda(t) + 2L\varepsilon\lambda(t) + M\varepsilon I^{q_2}\lambda(t) \\
&= F(t,\beta_\varepsilon(t),I^{q_2}\beta_\varepsilon(t)) + L\varepsilon\lambda(t) \\
&> F(t,\beta_\varepsilon(t),I^{q_2}\beta_\varepsilon(t))
\end{aligned} \tag{3.5}$$

We aim to prove $\alpha(t) < \beta_\varepsilon(t)$ on $t \in [0,J]$, which completes the proof by letting $\varepsilon \longrightarrow 0$. Suppose that $\alpha(t) < \beta_\varepsilon(t)$ on $t \in [0,J]$ is not true. Then there would exists a $t_1 \in J$ such that $\alpha(t) < \beta_\varepsilon(t)$ for $0 \le t < t_1$ and $\alpha(t_1) = \beta_\varepsilon(t_1)$.
By composing $m(t) = \alpha(t) - \beta_\varepsilon(t)$, it is found that $m(t) \le 0$ for $0 \le t < t_1$ and $m(t_1) = 0$. This results in ${}^{C}D^{q_1}m(t_1) \ge 0$, on account of Lemma 2.4. It then follows that

$$F(t_1,\alpha(t_1),I^{q_2}\alpha(t_1)) \ge {}^{C}D^{q_1}\alpha(t_1) \ge {}^{C}D^{q_1}\beta_\varepsilon(t_1) > F(t_1,\beta_\varepsilon(t_1),I^{q_2}\beta_\varepsilon(t_1))$$

giving rise to a contradiction because of the fact that $\alpha(t_1) = \beta_\varepsilon(t_1)$. Then the inequality

$$\alpha(t) < \beta_\varepsilon(t), \forall t \in J$$

holds, which means $\alpha(t) \le \beta(t)$ on J.

□

Theorem 3.2. *Let α and β be coupled lower and upper solutions of type I of (2.1). Assume further that following inequalities are satisfied:*

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \leq L(u_1-u_2) \tag{3.6}$$

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \geq -M(v_1-v_2) \tag{3.7}$$

$L,M \geq 0$, whenever $u_1 \geq u_2$ and $v_1 \geq v_2$.
Then $\alpha(0) \leq \beta(0)$ implies that $\alpha(t) \leq \beta(t)$ on J.

Proof. We begin by constructing $\beta_\varepsilon(t) = \beta(t) + \varepsilon\lambda(t)$ and $\alpha_\varepsilon(t) = \alpha(t) - \varepsilon\lambda(t)$ for $\varepsilon > 0$. The function $\lambda(t)$ is also supposed to be unique solution of (3.2). It is clear that $\beta_\varepsilon(0) = \beta(0) + \varepsilon\lambda(0) > \beta(0)$ and $\alpha_\varepsilon(0) = \alpha(0) - \varepsilon\lambda(0) < \alpha(0)$ imply $\alpha_\varepsilon(0) < \beta_\varepsilon(0)$. In addition to that for $0 < t \leq T$ we get $\beta_\varepsilon(t) > \beta(t)$ and $\alpha_\varepsilon(t) < \alpha(t)$.
Differentiating both sides of $\beta_\varepsilon(t) = \beta(t) + \varepsilon\lambda(t)$ leads to

$$\begin{aligned} {}^CD^{q_1}\beta_\varepsilon(t) &= {}^CD^{q_1}\beta(t) + {}^CD^{q_1}\varepsilon\lambda(t) \\ &\geq F(t,\beta(t),I^{q_2}\alpha(t)) + 2L\varepsilon\lambda(t) + M\varepsilon I^{q_2}\lambda(t) \end{aligned} \tag{3.8}$$

Since $\beta_\varepsilon(t) > \beta(t)$ for $0 \leq t \leq T$, we can employ the inequality (3.6) then it yields

$$\begin{aligned} F(t,\beta_\varepsilon(t),I^{q_2}\alpha(t)) - F(t,\beta(t),I^{q_2}\alpha(t)) &\leq L(\beta_\varepsilon - \beta) \\ F(t,\beta(t),I^{q_2}\alpha(t)) &\geq F(t,\beta_\varepsilon(t),I^{q_2}\alpha(t)) - L\varepsilon\lambda(t) \end{aligned}$$

Inserting foregoing inequality into (3.8) gives

$$\begin{aligned} {}^CD^{q_1}\beta_\varepsilon(t) &\geq F(t,\beta_\varepsilon(t),I^{q_2}\alpha(t)) - L\varepsilon\lambda(t) + 2L\varepsilon\lambda(t) + M\varepsilon I^{q_2}\lambda(t) \\ &= F(t,\beta_\varepsilon(t),I^{q_2}\alpha(t)) + L\varepsilon\lambda(t) + M\varepsilon I^{q_2}\lambda(t) \end{aligned} \tag{3.9}$$

On the other side, It is convenient to write

$$\begin{aligned} F(t,\beta_\varepsilon(t),I^{q_2}\alpha(t)) - F(t,\beta_\varepsilon(t),I^{q_2}\alpha_\varepsilon(t)) &\geq -MI^{q_2}(\alpha - \alpha_\varepsilon) \\ F(t,\beta_\varepsilon(t),I^{q_2}\alpha(t)) &\geq F(t,\beta_\varepsilon(t),I^{q_2}\alpha_\varepsilon(t)) - M\varepsilon I^{q_2}\lambda(t) \end{aligned}$$

where since $\alpha_\varepsilon(t) < \alpha(t)$ on $[0,T]$, we have used (3.7). Now, substituting that inequality in the right hand side of (3.9)

$$\begin{aligned} {}^CD^{q_1}\beta_\varepsilon(t) &\geq F(t,\beta_\varepsilon(t),I^{q_2}\alpha_\varepsilon(t)) - M\varepsilon I^{q_2}\lambda(t) + L\varepsilon\lambda(t) + M\varepsilon I^{q_2}\lambda(t) \\ &= F(t,\beta_\varepsilon(t),I^{q_2}\alpha_\varepsilon(t)) + L\varepsilon\lambda(t) \\ &> F(t,\beta_\varepsilon(t),I^{q_2}\alpha_\varepsilon(t)) \end{aligned} \tag{3.10}$$

follows immediately.
A similar procedure can be applied to $\alpha_\varepsilon(t) = \alpha(t) - \varepsilon\lambda(t)$ to achieve the following results

$${}^CD^{q_1}\alpha_\varepsilon(t) < F(t,\alpha_\varepsilon(t),I^{q_2}\beta_\varepsilon(t)) \tag{3.11}$$

on $[0,T]$
We next prove that $\alpha_\varepsilon(t) < \beta_\varepsilon(t)$ on $[0,T]$. Contrary to this claim, we presume for a moment that the inequality is not true and, setting $m(t) = \alpha_\varepsilon(t) - \beta_\varepsilon(t)$ there would exist a point t_1 such that $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t < t_1$. We get at once ${}^CD^{q_1}m(t_1) \geq 0$ by Lemma 2.4. Obviously, it causes a contradiction. Then, it should has been

$$\alpha_\varepsilon(t) < \beta_\varepsilon(t)$$

on J. Finally, letting $\varepsilon \longrightarrow 0$, we reach at

$$\begin{aligned} \lim_{\varepsilon\longrightarrow 0}(\alpha(t) - \varepsilon\lambda(t)) &\leq \lim_{\varepsilon\longrightarrow 0}(\beta(t) + \varepsilon\lambda(t)) \\ \alpha(t) &\leq \beta(t), \end{aligned}$$

for $t \in J$, ending the proof. □

Theorem 3.3. *Let α and β be coupled lower and upper solutions of type II of (2.1). Additionally, following inequalities*

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \geq -L(u_1-u_2) \tag{3.12}$$

$$F(t,u_1(t),v_1(t)) - F(t,u_2(t),v_2(t)) \leq M(v_1-v_2) \tag{3.13}$$

hold whenever $u_1 \geq u_2$ and $v_1 \geq v_2$ and $L,M \geq 0$ on J.
Then $\alpha(0) \leq \beta(0)$ implies that $\alpha(t) \leq \beta(t)$.

Proof. In that case, for some $\varepsilon > 0$, we compose $\beta_{\varepsilon}(t) = \beta(t) + \varepsilon\lambda(t)$ and $\alpha_{\varepsilon}(t) = \alpha(t) - \varepsilon\lambda(t)$ where the function $\lambda(t)$ is taken as the unique solution of the following linear equation

$$^{C}D^{q_1}\lambda(t) = 2L\lambda(t) + MI^{q_2}\lambda(t), \quad \lambda(0) = 1$$

which is given by

$$\lambda(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(M)^n 2^k (L)^k \binom{n+k}{k} t^{q_1(n+k)+nq_2}}{\Gamma(q_1(n+k)+nq_2+1)}.$$

Taking derivatives in Caputo sense on both sides of constructed functions and using (3.12) and (3.13), we have the following strict inequalities

$$^{C}D^{q_1}\beta_{\varepsilon}(t) \quad > \quad F(t, \alpha_{\varepsilon}(t), I^{q_2}\beta_{\varepsilon}(t))$$

and

$$^{C}D^{q_1}\alpha_{\varepsilon}(t) \quad < \quad F(t, \beta_{\varepsilon}(t), I^{q_2}\alpha_{\varepsilon}(t))$$

with $\alpha_{\varepsilon}(t) < \beta_{\varepsilon}(t)$. At this stage we apply proof by contradiction with the help of Lemma 2.4 to show $\alpha_{\varepsilon}(t) < \beta_{\varepsilon}(t)$ on J. As a final step, performing $\varepsilon \longrightarrow 0$, we get the desired result

$$\alpha(t) \quad \leq \quad \beta(t),$$

for $t \in J$, which completes the proof.
To complete the proof, we can utilize the same procedure discussed before. So, we skip over the details of the proof. □

Theorem 3.4. *Let α and β be coupled lower and upper solutions of type III of (2.1). We also assume that*

$$F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) \geq -L(u_1 - u_2) - M(v_1 - v_2) \tag{3.14}$$

$L, M \geq 0$, whenever $u_1 \geq u_2, v_1 \geq v_2$.
Then we have $\alpha(t) \leq \beta(t)$ provided $\alpha(0) \leq \beta(0)$.

Proof. We can use the same technique as before to finish the proof. As a result, we will pass through all of the proof's details. □

4. Conclusions

This paper deals with the extension of some differential inequalities in the literature to fractional integro-differential equations with boundary conditions. Our main tool to derive the results is the concept of upper and lower solutions, as well as the theory of strict and nonstrict fractional differential inequalities.These theorems are necessary because they provide the framework for adapting the monotone iterative technique to such a class of fractional boundary value problem (2.1).

Acknowledgements

The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

There are no competing interests.

Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

References

[1] Agarwal, R., Benchohra, M. & Hamani, S. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Applicandae Mathematicae*. **109**, 973-1033 (2010)
[2] Yakar, A. & Koksal, M. Existence results for solutions of nonlinear fractional differential equations. *Abstract And Applied Analysis*. **2012** (2012)
[3] Refice, A., Souid, M. & Yakar, A. Some qualitative properties of nonlinear fractional integro-differential equations of variable order. *An International Journal Of Optimization And Control: Theories & Applications (IJOCTA)*. **11**, 68-78 (2021)
[4] Guezane, L. & Ashyralyev, A. Existence of solutions for weighted p (t)-Laplacian mixed Caputo fractional differential equations at resonance. *Filomat*. **36**, 231-241 (2022)
[5] Gambo, Y., Ameen, R., Jarad, F. & Abdeljawad, T. Existence and uniqueness of solutions to fractional differential equations in the frame of generalized Caputo fractional derivatives. *Advances In Difference Equations*. **2018**, 1-13 (2018)
[6] Singh, H., Kumar, D. & Baleanu, D. Methods of mathematical modelling: fractional differential equations. (CRC Press,2019)
[7] Kazem, S. Exact solution of some linear fractional differential equations by Laplace transform. *International Journal Of Nonlinear Science*. **16**, 3-11 (2013)
[8] Devi, J. & Sreedhar, C. Generalized Monotone Iterative Method for Caputo Fractional Integro-differential Equation. *European Journal Of Pure And Applied Mathematics*. **9**, 346-359 (2016)
[9] Kilbas, A., Srivastava, H. & Trujillo, J. Theory and applications of fractional differential equations. (elsevier,2006)
[10] Lakshmikantham, V., Leela, S. & Devi, J. Theory of fractional dynamic systems. (Cambridge Academic ,2009)
[11] Lakshmikantham, V. & Vatsala, A. Theory of fractional differential inequalities and applications. *Communications In Applied Analysis*. **11**, 395-402 (2007)
[12] Al-Refai, M. & Luchko, Y. Comparison principles for solutions to the fractional differential inequalities with the general fractional derivatives and their applications. *Journal Of Differential Equations*. **319** pp. 312-324 (2022)
[13] Yakar, A. & Kutlay, H. A note on comparison results for fractional differential equations. *AIP Conference Proceedings*. **1676**, 020064 (2015)
[14] Yakar, A. Some generalizations of comparison results for fractional differential equations. *Computers & Mathematics With Applications*. **62**, 3215-3220 (2011)