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Research Article

Characterizations of L-concavities and L-convexities via derived relations

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Abstract

This paper is to characterize L-concavities and L-convexities via some derived forms of relations and operators. Specifically, notions of L-concave derived internal relation space and L-concave derived hull space are introduced. It is proved that the category of L-concave derived internal relation spaces and the category of L-concave derived hull spaces are isomorphic to the category of L-concave spaces. Also, notions of L-convex derived enclosed relation space and L-convex derived hull space are introduced. It is proved that the category of L-convex derived enclosed relation spaces and the category of L-convex derived hull spaces are isomorphic to the category of L-convex spaces.

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Keywords. L-concave derived internal relation space, L-convex derived enclosed relation space, L-concave derived hull space, L-convex derived hull space

1. Introduction

In an abstract convex space, a convex structure on a nonempty set is a family of subsets containing the empty set and the largest set, and is closed under arbitrary intersections and nested unions. Its theory is called the abstract convex theory which involves many mathematical structures such as lattice, graph, median algebra, metric space, poset and vector space [21].

Convex structure has been extended into fuzzy settings by many ways. Rosa introduced the notion of fuzzy convex structure [16] which was further extended by Maruyama who introduced L-convex structure [7]. In the framework of L-convex spaces, Pang and Shi as well as many other scholars investigated many properties of L-convex spaces [8, 11, 13, 14, 31, 32, 36, 38]. Later, Shi and Xiu introduced M-fuzzifying convex structures [19]. Many subsequent studies have been done [5, 9, 23, 30]. Further, Shi and Xiu introduced (L, M)-fuzzy convex structure which is a unified form of L-convex structure and M-fuzzifying convex structure [20]. Recently, Pang and Wu investigated many characterizations of (L, M)-fuzzy convex spaces [10, 24, 25]. Now, these fuzzy forms of convex structures have been being applied to many fuzzy mathematical structures such as fuzzy topology [6, 22, 25], fuzzy convergence [31, 37] and fuzzy matroid [23, 33].

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The notion of derived sets was originally defined by Georg Cantor in 1872 [4]. In a topological space, the derived set of a given set is composed by all of its adherent points. In addition, the closure of the given set is exact the union of itself and its derived set. In view of these properties, Shi presented an axiomatic concept of derived operators and studied its induced fuzzy derived operators [18]. In the framework of L-fuzzy setting, Wu et al introduced L-topologies derived internal relations, L-topological derived enclosed relations and L-topological derived neighborhood relations in L-topological spaces [26, 27]. Also, in the framework of M-fuzzifying settings, scholars introduced some derived operators in M-fuzzifying convex structures and M-fuzzifying matroids [3, 17, 29, 39]. These derived operators have a common feature which takes an axiom to show the relation between a set and its adherent points. In addition, they have being used to characterize their corresponding mathematical structures such as L-topology, M-fuzzifying convexity or M-fuzzifying matroid. From the perspective of the composition of derived sets, scholars extended adherent points and discussed Moore-Smith convergence theories in fuzzy topological spaces [1, 2, 15, 34, 35].

In view of the above statements, a natural question arises: can derived operator be applied to L-convex enclosed relation or L-concave internal relation? Specifically, is there any L-concave derived internal relation or L-convex derived enclosed relation such that the following diagrams communicate?

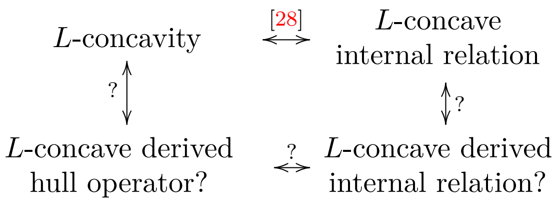

Figure 1. Problem 1.

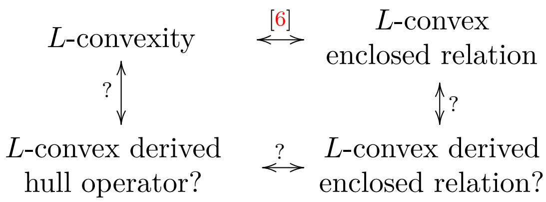

Figure 2. Problem 2.

Being motivated by the above problems, we present this paper. The arrangement of this paper is as follows. In Section 2, we recall some basic concepts, denotations and results of L-convex space and L-concave space. In Section 3, we introduce L-concave derived internal relation space and L-concave derived hull space by which we characterize L-concave space. In Section 4, we introduce L-convex derived enclosed relation space and L-convex derived hull space by which we characterize L-convex space.

2. Preliminaries

In this paper, X and Y are nonempty sets. The power set of X is denoted by 2^X. For any $A \in 2^X$, 2^A_{fin} is the set of all finite subsets of A. $(L, \vee, \wedge)$ is a completely distributive lattice with a partial order $\leq$ defined by $x \leq y$ iff $x \vee y = y$ (alternatively $x \wedge y = x$) for all $x, y \in L$. The smallest element and the largest element in L are respectively denoted by $\perp$ and $\top$. An element $a \in L$ is called a co-prime element, if for all $b, c \in L$, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all co-prime elements in $L \backslash \{\perp\}$ is denoted by $J(L)$. For any $a \in L$, there is $L_1 \subseteq J(L)$ such that $a = \bigvee_{b \in L_1} b$. A binary relation $\prec$ on L is defined by $a \prec b$ iff for any $L_1 \subseteq L$, $b \leq \bigvee L_1$ implies some $d \in L_1$ such that $a \leq d$. The mapping $\beta : L \to 2^L$, defined by $\beta(a) = \{b : b \prec a\}$, satisfies $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$ for any $\{a_i\}_{i \in I} \subseteq L$. For any $a \in L$, $\beta(a)$ and $\beta^*(a) = \beta(a) \cap J(L)$ satisfy $a = \bigvee \beta(a) = \bigvee \beta^*(a)$.

An L-fuzzy set on X is a mapping $A : X \to L$. The set of all L-fuzzy sets on X is denoted by L^X. The smallest element and the largest element in L^X are respectively denoted by $\underline{\perp}$ and $\underline{\top}$. A subset $\{A_i\}_{i \in I} \subseteq L^X$ is said to be up-directed (or down-directed), denoted by $\{A_i\}^{dir}_{i \in I} \subseteq L^X$ (or $\{A_i\}^{ddir}_{i \in I} \subseteq L^X$), if for all $i, j \in I$, there is $k \in I$ such that $A_i \vee A_j \leq A_k$ (or, $A_k \leq A_i \wedge A_j$). In this case, we denote $\bigvee_{i \in I} A_i$ (or $\bigwedge_{i \in I} A_i$) by $\bigvee^{dir}_{i \in I} A_i$ (or $\bigwedge^{ddir}_{i \in I} A_i$). For any $A \in L^X$, we denote $\beta^*(A) = \{x_\lambda \in L^X : \lambda \in \beta^*(A(x))\}$.

For a mapping $f : X \to Y$, the L-fuzzy mapping $f_L^{\rightarrow} : L^X \to L^Y$ is defined by $f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) : f(x) = y\}$ for $A \in L^X$ and $y \in Y$, and the mapping $f_L^{\leftarrow} : L^Y \to L^X$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$ for $B \in L^Y$ and $x \in X$.

Definition 2.1 ([7]). A subset $\mathcal{C} \subseteq L^X$ is called an L-convex structure on L^X and the pair $(X, \mathcal{C})$ is called an L-convex space if

(LC1) $\top, \bot \in \mathcal{C}$;
(LC2) $\bigwedge_{i\in I} A_i \in \mathcal{C}$ for any subset $\{A_i\}_{i\in I} \subseteq \mathcal{C}$;
(LC3) $\bigvee_{i\in I}^{dir} A_i \in \mathcal{C}$ for any $\{A_i\}_{i\in I}^{dir} \subseteq \mathcal{C}$.

Proposition 2.2 ([12]). *The L-convex hull operator $co_{\mathcal{C}} : L^X \to L^X$ of an L-convex space $(X, \mathcal{C})$ is defined by $co_{\mathcal{C}}(A) = \bigwedge\{B \in \mathcal{C} : A \leq B\}$ for any $A \in L^X$. It satisfies*

(LCO1) $co_{\mathcal{C}}(\bot) = \bot$;
(LCO2) $A \leq co_{\mathcal{C}}(A)$;
(LCO3) $co_{\mathcal{C}}(co_{\mathcal{C}}(A)) = co_{\mathcal{C}}(A)$;
(LCO4) $co_{\mathcal{C}}(\bigvee_{i\in I}^{dir} A_i) = \bigvee_{i\in I} co_{\mathcal{C}}(A_i)$ for any $\{A_i\}_{i\in I}^{dir} \subseteq L^X$.

Conversely, if an operator $co : L^X \to L^X$ satisfies (LCO1)–(LCO4), then the set $\mathcal{C}_{co} = \{A \in L^X : co(A) = A\}$ is an L-convex structure satisfying $co_{\mathcal{C}_{co}} = co$.

Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L-convex spaces. A mapping $f : X \to Y$ is called an L-convexity preserving mapping, if $f_L^{\leftarrow}(A) \in \mathcal{C}_X$ for any $A \in \mathcal{C}_Y$. It is proved that a mapping $f : X \to Y$ is an L-convexity preserving mapping if and only if $f_L^{\rightarrow}(co_{\mathcal{C}_X}(A)) \leq co_{\mathcal{C}_Y}(f_L^{\rightarrow}(A))$ for any $A \in L^X$. The category of L-convex spaces and L-convex preserving mappings is denoted by L-**CS** [12].

Definition 2.3 ([12]). A subset $\mathcal{A} \subseteq L^X$ is called an L-concave structure on L^X and the pair $(X, \mathcal{A})$ is called an L-concave space if

(LCA1) $\top, \bot \in \mathcal{A}$;
(LCA2) $\bigvee_{i\in I} A_i \in \mathcal{A}$ for any $\{A_i\}_{i\in I} \subseteq \mathcal{A}$;
(LCA3) $\bigwedge_{i\in I}^{ddir} A_i \in \mathcal{A}$ for any $\{A_i\}_{i\in I}^{ddir} \subseteq \mathcal{A}$.

Proposition 2.4 ([12]). *The L-concave hull operator $ca_{\mathcal{A}} : L^X \to L^X$ of an L-concave space $(X, \mathcal{A})$ is defined by $ca_{\mathcal{A}}(A) = \bigvee\{B \in \mathcal{A} : B \leq A\}$ for any $A \in L^X$. It satisfies*

(LCAH1) $ca_{\mathcal{A}}(\top) = \top$;
(LCAH2) $ca_{\mathcal{A}}(A) \leq A$;
(LCAH3) $ca_{\mathcal{A}}(ca_{\mathcal{A}}(A)) = ca_{\mathcal{A}}(A)$;
(LCAH4) $ca_{\mathcal{A}}(\bigwedge_{i\in I}^{ddir} A_i) = \bigwedge_{i\in I} ca_{\mathcal{A}}(A_i)$ for any $\{A_i\}_{i\in I}^{ddir} \subseteq L^X$.

Conversely, if an operator $ca : L^X \to L^X$ satisfies (LCAH1)–(LCAH4), then the set $\mathcal{A}_{ca} = \{A \in L^X : ca(A) = A\}$ is an L-concave structure satisfying $ca_{\mathcal{A}_{ca}} = ca$.

Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be L-concave spaces. A mapping $f : X \to Y$ is called an L-concavity preserving mapping, if $f_L^{\leftarrow}(A) \in \mathcal{A}_X$ for any $A \in \mathcal{A}_Y$. It is proved that a mapping $f : X \to Y$ is an L-concavity preserving mapping if and only if $f_L^{\leftarrow}(ca_{\mathcal{A}_Y}(B)) \leq ca_{\mathcal{A}_X}(f_L^{\leftarrow}(B))$ for any $B \in L^Y$. The category of L-concave spaces and L-concavity preserving mappings is denoted by L-**CAS** [12, 28].

Definition 2.5 ([6]). A binary relation $\lessdot$ on L^X is called an L-convex enclosed relation and the pair $(X, \lessdot)$ is called an L-convex enclosed relation space, if $\lessdot$ satisfies

(LCER1) $\bot \lessdot \bot$;
(LCER2) $A \lessdot B$ implies $A \leq B$;
(LCER3) $A \lessdot \bigwedge_{i\in I} B_i$ iff $A \lessdot B_i$ for all $i \in I$;
(LCER4) $A \lessdot B$ implies some $C \in L^X$ with $A \lessdot C \lessdot B$;
(LCER5) $\bigvee_{i\in I}^{dir} A_i \lessdot B$ iff $A_i \lessdot B$ for any $i \in I$.

Let $(X, \lessdot_X)$ and $(Y, \lessdot_Y)$ be L-convex enclosed relation spaces. A mapping $f : X \to Y$ is called an L-convex enclosed relation preserving mapping, if $f_L^{\leftarrow}(A) \lessdot_X f_L^{\leftarrow}(B)$ for all $A, B \in L^X$ with $A \lessdot_Y B$ [6].

The category of L-convex enclosed relation spaces and L-convex enclosed relation preserving mappings is denoted by L-**CERS** [6].

Proposition 2.6 ([6]). *(1) Let $(X, \lessdot)$ be an L-convex enclosed relation space. Define an operator $co_{\lessdot} : L^X \to L^X$ by*

$$\forall A \in L^X, \quad co_{\lessdot}(A) = \bigwedge\{B \in L^X : A \lessdot B\}.$$

Then $co_{\lessdot}$ is the L-convex hull operator of an L-convex structure $\mathcal{C}_{\lessdot}$.

(2) Let $(X, \mathcal{C})$ be an L-convex space. Define a binary relation $\lessdot_{\mathcal{C}}$ by

$$\forall A, B \in L^X, \quad A \lessdot_{\mathcal{C}} B \iff co_{\mathcal{C}}(A) \leq B.$$

Then $\lessdot_{\mathcal{C}}$ is an L-convex enclosed relation.

(3) $\lessdot_{\mathcal{C}_{\lessdot}} = \lessdot$ for any L-convex enclosed relation space $(X, \lessdot)$ and $\mathcal{C}_{\lessdot_{\mathcal{C}}} = \mathcal{C}$ for any L-convex space $(X, \mathcal{C})$.

Theorem 2.7 ([6]). *L-**CS** is isomorphic to L-**CERS**.*

Definition 2.8 ([28]). A binary relation $\leqslant$ on L^X is called an L-concave internal relation and the pair $(X, \leqslant)$ is called an L-concave internal relation space, if $\leqslant$ satisfies

(LCIR1) $\top \leqslant \top$;
(LCIR2) $A \leqslant B$ implies $A \leq B$;
(LCIR3) $\bigvee_{i \in I} A_i \leqslant B$ iff $A_i \leqslant B$ for all $i \in I$;
(LCIR4) $A \leqslant B$ implies $C \in L^X$ with $A \leqslant C \leqslant B$;
(LCIR5) $A \leqslant \bigwedge_{i \in I}^{ddir} B_i$ iff $A \leqslant B_i$ for any $i \in I$.

Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be L-concave internal relation spaces. A mapping $f : X \to Y$ is called an L-concave internal relation preserving mapping, if $f_L^{\leftarrow}(A) \leqslant_X f_L^{\leftarrow}(B)$ for all $A, B \in L^X$ with $A \leqslant_Y B$.

The category of L-concave internal relation spaces and L-concave internal relation preserving mappings is denoted by L-**CIRS** [28].

Proposition 2.9 ([28]). *(1) Let $(X, \mathcal{A})$ be an L-concave space. Define a binary relation $\leqslant_{\mathcal{A}}$ on L^X by*

$$\forall A, B \in L^X, \quad A \leqslant_{\mathcal{A}} B \iff A \leq ca_{\mathcal{A}}(B).$$

Then $\leqslant_{\mathcal{A}}$ is an L-concave internal relation.

(2) Let $(X, \leqslant)$ be an L-concave internal relation space. Define an operator $ca_{\leqslant} : L^X \to L^X$ by

$$\forall A \in L^X, \quad ca_{\leqslant}(A) = \bigvee\{B \in L^X : B \leqslant A\}.$$

Then $ca_{\leqslant}$ is an L-concave hull operator of an L-concave structure $\mathcal{A}_{\leqslant}$.

(3) $\leqslant_{\mathcal{A}_{\leqslant}} = \leqslant$ for any L-concave internal relation space $(X, \leqslant)$ and $\mathcal{A}_{\leqslant_{\mathcal{A}}} = \mathcal{A}$ for any L-concave space $(X, \mathcal{A})$.

Theorem 2.10 ([28]). *L-**CAS** is isomorphic to L-**CIRS**.*

3. L-concave derived internal relation spaces

In this section, we introduce the notion of L-concave derived internal relation space which can be used to characterize L-concave internal relation space and L-concave space. Also, we introduce the notion of L-concave derived hull operator by which we can obtain a simple characterization of L-concave derived internal relation space.

Definition 3.1. A binary relation $\preccurlyeq$ on L^X is called an L-concave derived internal relation and the pair $(X, \preccurlyeq)$ is called an L-concave derived internal relation space, if for all $A, B, C \in L^X$ and $x_\lambda \in \beta^*(\underline{\top})$,

(LCDIR1) $\underline{\top} \preccurlyeq \underline{\top}$;

(LCDIR2) $A \preccurlyeq B$ iff $x_\lambda \preccurlyeq B \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$;

(LCDIR3) $\bigvee_{i\in I} A_i \preccurlyeq B$ iff $A_i \preccurlyeq B$ for any $i \in I$;

(LCDIR4) $A \preccurlyeq B$ implies $C \in L^X$ such that $A \wedge B \preccurlyeq C \preccurlyeq B$ and $A \wedge B \leq C \leq B$;

(LCDIR5) $A \preccurlyeq \bigwedge^{ddir}_{i\in I} B_i$ iff $A \preccurlyeq B_i$ for any $i \in I$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave derived internal relation spaces. A mapping $f : X \to Y$ is called an L-concave derived internal relation preserving mapping, if for all $f^{\leftarrow}_L(A \wedge B) \preccurlyeq_X f^{\leftarrow}_L(B)$ for all $A, B \in L^Y$ with $A \preccurlyeq_Y B$.

The category of L-concave derived internal relation spaces and L-concave derived internal relation preserving mappings is denoted by L-**CDIRS**.

Now, we study relations between L-**CDIRS** and L-**CIRS**.

Proposition 3.2. *Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space. Define a binary relation $\leqslant_{\preccurlyeq}$ on L^X by for all $A, B \in L^X$,*

$$A \leqslant_{\preccurlyeq} B \iff \exists C \in L^X \text{ s.t. } C \preccurlyeq B \text{ and } A = B \wedge C.$$

Then $\leqslant_{\preccurlyeq}$ is an L-concave internal relation.

Proof. (LCIR1). $\underline{\top} \preccurlyeq \underline{\top}$ and $\underline{\top} \wedge \underline{\top} = \underline{\top}$ by (LCDIR1). Thus $\underline{\top} \leqslant_{\preccurlyeq} \underline{\top}$.

(LCIR2). It follows from the definition.

(LCIR3). Let $\bigvee_{i\in I} A_i \leqslant_{\preccurlyeq} B$. Then there is $C \in L^X$ such that $C \preccurlyeq B$ and $\bigvee_{i\in I} A_i = B \wedge C$. For any $i \in I$, $A_i \leq \bigvee_{i\in I} A_i \preccurlyeq B$. Thus $A_i \preccurlyeq B$ and $A_i = B \wedge A_i$. Hence $A_i \leqslant_{\preccurlyeq} B$.

Conversely, assume that $A_i \leqslant_{\preccurlyeq} B$ for any $i \in I$. For any $i \in I$, there is $C_i \in L^X$ such that $C_i \preccurlyeq B$ and $A_i = B \wedge C_i$. Thus $\bigvee_{i\in I} C_i \preccurlyeq B$ by (LCDIR3). Further,

$$\bigvee_{i\in I} A_i = \bigvee_{i\in I} (B \wedge C_i) = B \wedge \bigvee_{i\in I} C_i.$$

Hence $\bigvee_{i\in I} A_i \leqslant_{\preccurlyeq} B$.

(LCIR4). Let $A \leqslant_{\preccurlyeq} B$. There is $D \in L^X$ such that $D \preccurlyeq B$ and $A = B \wedge D$. By $D \preccurlyeq B$ and (LCDIR4), there is $C \in L^X$ such that $A = D \wedge B \preccurlyeq C \preccurlyeq B$ and $A \leq C \leq B$. Thus $A \leqslant_{\preccurlyeq} C \leqslant_{\preccurlyeq} B$.

(LCIR5). If $A \leqslant_{\preccurlyeq} \bigwedge^{ddir}_{i\in I} B_i$ then there is $D \in L^X$ such that $D \preccurlyeq \bigwedge^{ddir}_{i\in I} B_i$ and $A = \bigwedge^{ddir}_{i\in I} (B_i \wedge D)$. For any $j \in I$, $A \leq D \preccurlyeq \bigwedge^{ddir}_{i\in I} B_i \leq B_j$. Thus $A \preccurlyeq B_j$ and $A \leq B_j$. Hence $A \leqslant_{\preccurlyeq} B_j$.

Conversely, assume that $\{B_i\}^{ddir}_{i\in I} \subseteq L^X$ with $A \leqslant_{\preccurlyeq} B_i$ for any $i \in I$. Then there is $D_i \in L^X$ such that $D_i \preccurlyeq B_i$ and $A = D_i \wedge B_i$. Let $D = \bigwedge_{i\in I} D_i$. Then $D \leq D_i \preccurlyeq B_i$ for any $i \in I$. Thus $D \preccurlyeq B_i$ for any $i \in I$. Hence $D \preccurlyeq \bigwedge^{ddir}_{i\in I} B_i$ by (LCDIR5). Since

$$A = \bigwedge_{i\in I} (D_i \wedge B_i) = D \wedge \bigwedge^{ddir}_{i\in I} B_i,$$

we conclude that $A \leqslant_{\preccurlyeq} \bigwedge^{ddir}_{i\in I} B_i$. □

Proposition 3.3. *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave derived internal relation spaces. If $f : X \to Y$ is an L-concave derived internal relation preserving mapping, then $f : (X, \leqslant_{\preccurlyeq_X}) \to (Y, \leqslant_{\preccurlyeq_Y})$ is an L-concave internal relation preserving mapping.*

Proof. If $A \leqslant_{\preccurlyeq_Y} B$ then there is $C \in L^Y$ such that $C \preccurlyeq_Y B$ and $A = B \wedge C$. Thus $f^{\leftarrow}_L(C \wedge B) \preccurlyeq_X f^{\leftarrow}_L(B)$ and $f^{\leftarrow}_L(A) = f^{\leftarrow}_L(C \wedge B) \wedge f^{\leftarrow}_L(B)$. Hence $f^{\leftarrow}_L(A) \leqslant_{\preccurlyeq_X} f^{\leftarrow}_L(B)$. Therefore f is an L-concave internal relation preserving mapping. □

Proposition 3.4. *Let $(X, \leqslant)$ be an L-concave internal relation space. Define a binary relation $\preccurlyeq_{\leqslant}$ on L^X by for all $A, B \in L^X$,*

$$A \preccurlyeq_{\leqslant} B \iff \forall x_\lambda \in \beta^*(A),\ x_\lambda \leqslant B \vee x_\lambda.$$

Then $\preccurlyeq_{\leqslant}$ is an L-concave derived internal relation.

Proof. It is clear that $A \preccurlyeq_{\leqslant} B$ for any $A, B, C, D \in L^X$ with $A \leq C \preccurlyeq_{\leqslant} D \leq B$.

(LCDIR1). If $x_\lambda \in \beta^*(\underline{\top})$, then $x_\lambda \leq \underline{\top} \leqslant \underline{\top} = \underline{\top} \vee x_\lambda$. Thus $x_\lambda \leqslant \underline{\top}$. Hence $\underline{\top} \preccurlyeq_{\leqslant} \underline{\top}$.

(LCDIR2). Let $A \preccurlyeq_{\leqslant} B$ and let $x_\lambda \in \beta^*(A)$. To prove that $x_\lambda \preccurlyeq_{\leqslant} B \vee x_\lambda$, let $x_\eta \in \beta^*(x_\lambda)$. Then $x_\eta \in \beta^*(A)$. By $A \preccurlyeq_{\leqslant} B$, we have $x_\eta \leqslant B \vee x_\eta \leq (B \vee x_\lambda) \vee x_\eta$. Thus $x_\eta \leqslant (B \vee x_\lambda) \vee x_\eta$. Hence $x_\lambda \preccurlyeq_{\leqslant} B \vee x_\lambda$.

Conversely, assume that $x_\lambda \preccurlyeq_{\leqslant} B \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$. To prove that $A \preccurlyeq_{\leqslant} B$, let $x_\lambda \in \beta^*(A)$. By $x_\lambda \preccurlyeq_{\leqslant} B \vee x_\lambda$, we have $x_\eta \leqslant B \vee x_\lambda \vee x_\eta = B \vee x_\lambda$ for any $x_\eta \in \beta^*(x_\lambda)$. Thus $x_\lambda = \bigvee_{x_\eta \in \beta^*(x_\lambda)} \leqslant B \vee x_\lambda$ by (LCIR3). Hence $A \preccurlyeq_{\leqslant} B$.

(LCDIR3). If $\bigvee_{i \in I} A_i \preccurlyeq_{\leqslant} B$, then it is clear that $A_i \preccurlyeq_{\leqslant} B$ for any $i \in I$. Conversely, assume that $A_i \preccurlyeq_{\leqslant} B$ for any $i \in I$. To prove that $\bigvee_{i \in I} A_i \preccurlyeq_{\leqslant} B$, let $x_\lambda \in \beta^*(\bigvee_{i \in I} A_i)$. Then there is $i \in I$ such that $x_\lambda \in \beta^*(A_i)$. By $A_i \preccurlyeq_{\leqslant} B$, we have $x_\lambda \leqslant B \vee x_\lambda$. Therefore $\bigvee_{i \in I} A_i \preccurlyeq_{\leqslant} B$.

(LCDIR4). Let $A \preccurlyeq_{\leqslant} B$. Let

$$D = \bigvee \{F \in L^X : F \leq B,\ F \preccurlyeq_{\leqslant} B\}.$$

We have $A \wedge B \leq D \leq B$. In addition, $D \preccurlyeq_{\leqslant} B$ by (LCDIR3). To prove that $A \wedge B \preccurlyeq_{\leqslant} D$, we check that $y_\eta \leqslant D \vee y_\eta$ for any $y_\eta \in \beta^*(A \wedge B)$.

Let $y_\eta \in \beta^*(A \wedge B)$. From $A \preccurlyeq_{\leqslant} B$, it is clear that $y_\eta \leqslant B \vee y_\eta = B$. By (LCIR4), there is $C \in L^X$ such that $y_\eta \leqslant C \leqslant B$. Thus $y_\eta \leq C \leq B$ by (LCIR2). For any $z_\theta \in \beta^*(C)$, $z_\theta \leq C \leqslant B \leq B \vee z_\theta$ which implies that $z_\theta \leqslant B \vee z_\theta$. Hence $C \preccurlyeq_{\leqslant} B$ and then $C \leq D$. Further, $y_\eta \leqslant D \vee y_\eta$ by $y_\eta \leqslant C \leq D \vee y_\eta$. Therefore $A \wedge B \preccurlyeq_{\leqslant} D \preccurlyeq_{\leqslant} B$ and $A \wedge B \leq D \leq B$ as desired.

(LCDIR5). If $A \preccurlyeq_{\leqslant} \bigwedge_{i \in I}^{ddir} B_i$, then $A \preccurlyeq_{\leqslant} B_i$ for any $i \in I$. Conversely, assume that $\{B_i\}_{i \in I}^{ddir} \subseteq L^X$ with $A \preccurlyeq_{\leqslant} B_i$ for any $i \in I$. Then $x_\lambda \leqslant B_i \vee x_\lambda$ for any $x_\lambda \in \beta^*(A)$ and any $i \in I$. Thus, by (LCIR5),

$$x_\lambda \leqslant \bigwedge_{i \in I}^{ddir} (B_i \vee x_\lambda) = (\bigwedge_{i \in I}^{ddir} B_i) \vee x_\lambda.$$

Therefore $A \preccurlyeq_{\leqslant} \bigwedge_{i \in I}^{ddir} B_i$. □

Proposition 3.5. *Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be L-concave internal relation spaces. If $f : X \to Y$ is an L-concave internal relation preserving mapping, then $f : (X, \preccurlyeq_{\leqslant_X}) \to (Y, \preccurlyeq_{\leqslant_Y})$ is an L-concave derived internal relation preserving mapping.*

Proof. Let $A \preccurlyeq_{\leqslant_Y} B$. If $f_L^{\leftarrow}(A \wedge B) = \bot$ then $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_{\leqslant_X} f_L^{\leftarrow}(B)$ is trivial. Let $f_L^{\leftarrow}(A \wedge B) \neq \bot$. If $x_\lambda \in \beta^*(f_L^{\leftarrow}(A \wedge B))$ then $f_L^{\rightarrow}(x_\lambda) \in \beta^*(A \wedge B)$. Thus $f_L^{\rightarrow}(x_\lambda) \leqslant_Y B \vee f_L^{\rightarrow}(x_\lambda)$ and

$$f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)) \leqslant_X f_L^{\leftarrow}(B) \vee f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda)) = f_L^{\leftarrow}(B).$$

Since $x_\lambda \leq f_L^{\leftarrow}(f_L^{\rightarrow}(x_\lambda))$ it follows that $x_\lambda \leqslant_X f_L^{\leftarrow}(B)$. Hence $x_\lambda \leqslant_X f_L^{\leftarrow}(B) \vee x_\lambda$ which implies that $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_{\leqslant_X} f_L^{\leftarrow}(B)$. Therefore f is an L-concave derived internal relation preserving mapping. □

Proposition 3.6. *If $(X, \leqslant)$ is an L-concave interval relation space then $\leqslant_{\preccurlyeq_{\leqslant}} = \leqslant$; if $(X, \preccurlyeq)$ is an L-concave derived internal relation space then $\preccurlyeq_{\leqslant_{\preccurlyeq}} = \preccurlyeq$.*

Proof. Let $(X, \leqslant)$ be an L-concave internal relation space. If $A \leqslant_{\preccurlyeq_{\leqslant}} B$, then $A \leq B$ by (LCIR2). In addition, there is $C \in L^X$ such that $C \preccurlyeq_{\leqslant} B$ and $A = B \wedge C$. Thus $A \preccurlyeq_{\leqslant} B$

which implies that $x_\lambda \leqslant B \vee x_\lambda = B$ for any $x_\lambda \in \beta^*(A)$. Hence $A = \bigvee_{x_\lambda \in \beta^*(A)} x_\lambda \leqslant B$ by (LCIR3).

Conversely, if $A \leqslant B$ then $A \leq B$ by (LCIR2). For any $x_\lambda \in \beta^*(A)$, $x_\lambda \leq A \leqslant B$ and so $x_\lambda \leqslant B = B \vee x_\lambda$. Hence $A \preccurlyeq_{\leqslant} B$. Since $A \wedge B = A$ it follows that $A \leqslant_{\preccurlyeq_{\leqslant}} B$.

In conclusion, for all $A, B \in L^X$, $A \leqslant_{\preccurlyeq_{\leqslant}} B$ iff $A \leqslant B$. That is, $\leqslant_{\preccurlyeq_{\leqslant}} = \leqslant$.

Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space.

If $A \preccurlyeq_{\leqslant_{\preccurlyeq}} B$ and $x_\lambda \in \beta^*(A)$, then $x_\lambda \leqslant_{\preccurlyeq} B \vee x_\lambda$. Thus there is $C \in L^X$ such that $C \preccurlyeq B$ and $x_\lambda = (B \vee x_\lambda) \wedge C$. Hence $x_\lambda \leq C \preccurlyeq B \vee x_\lambda$ and so $x_\lambda \preccurlyeq B \vee x_\lambda$. Therefore $A \preccurlyeq B$ by (LCDIR2).

Conversely, assume that $A \preccurlyeq B$. If $x_\lambda \in \beta^*(A)$ then $x_\lambda \leq A \preccurlyeq B \leq B \vee x_\lambda$. Thus $x_\lambda \preccurlyeq B \vee x_\lambda$. By this result and $x_\lambda = x_\lambda \wedge (B \vee x_\lambda)$, we have $x_\lambda \leqslant_{\preccurlyeq} B \vee x_\lambda$. Hence $A \preccurlyeq_{\leqslant_{\preccurlyeq}} B$.

In conclusion, for all $A, B \in L^X$, we have $A \preccurlyeq_{\leqslant_{\preccurlyeq}} B$ iff $A \preccurlyeq B$. That is, $\preccurlyeq_{\leqslant_{\preccurlyeq}} = \preccurlyeq$. □

Based on Propositions 3.2 and 3.3, we define a functor $\mathbb{U} : L$-**CDIRS** $\to L$-**CIRS** by

$$\mathbb{U}((X, \preccurlyeq)) = (X, \leqslant_{\preccurlyeq}), \quad \mathbb{U}(f) = f.$$

Based on Propositions 3.2–3.6, $\mathbb{U}$ is an isomorphic functor.

Theorem 3.7. *L-**CDIRS** is isomorphic to L-**CIRS**.*

Based on Propositions 2.9, 3.2–3.6 and Theorem 3.7, relationships between L-concave space and L-concave derived internal relation space can be presented as follows.

Corollary 3.8. *(1) Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space. Define an operator $ca_{\preccurlyeq} : L^X \to L^X$ by*

$$\forall A \in L^X, \quad ca_{\preccurlyeq}(A) = A \wedge \bigvee\{B \in L^X : B \preccurlyeq A\}.$$

Then $ca_{\preccurlyeq}$ is an L-concave hull operator which induces an L-concave structure denoted by $\mathcal{A}_{\preccurlyeq}$.

(2) Let $(X, \mathcal{A})$ be an L-concave space. Define a binary relation $\preccurlyeq_{\mathcal{A}}$ on L^X by

$$\forall A, B \in L^X, \quad A \preccurlyeq_{\mathcal{A}} B \iff \forall x_\lambda \in \beta^*(A),\ x_\lambda \leq ca_{\mathcal{A}}(B \vee x_\lambda).$$

Then $\preccurlyeq_{\mathcal{A}}$ is an L-concave derived internal relation.

(3) If $(X, \preccurlyeq)$ is an L-concave derived internal relation space then $\preccurlyeq_{\mathcal{A}_{\preccurlyeq}} = \preccurlyeq$. If $(X, \mathcal{A})$ is an L-concave space then $\mathcal{A}_{\preccurlyeq_{\mathcal{A}}} = \mathcal{A}$.

Proof. (1). Let $A, B \in L^X$. Then

$$\begin{aligned} B \leqslant_{\preccurlyeq} A &\iff \exists C \in L^X,\ B = A \wedge C \leq C \preccurlyeq A \\ &\iff A \wedge B = B \preccurlyeq A. \end{aligned}$$

Thus, by Proposition 2.9,

$$\begin{aligned} ca_{\preccurlyeq}(A) &= A \wedge \bigvee\{B \in L^X : B \preccurlyeq A\} \\ &= \bigvee\{A \wedge B : \exists B \in L^X,\ B \preccurlyeq A\} \\ &= \bigvee\{B \in L^X : B \leqslant_{\preccurlyeq} A\} \\ &= ca_{\leqslant_{\preccurlyeq}}(A). \end{aligned}$$

Hence $ca_{\preccurlyeq} = ca_{\leqslant_{\preccurlyeq}}$. Therefore $ca_{\preccurlyeq}$ is an L-concave hull operator.

(2). Let $A, B \in L^X$. By Proposition 2.9(1), we have

$$\begin{aligned} A \preccurlyeq_{\leqslant_{\mathcal{A}}} B &\iff \forall x_\lambda \in \beta^*(A),\ x_\lambda \leqslant_{\mathcal{A}} B \vee x_\lambda \\ &\iff \forall x_\lambda \in \beta^*(A),\ x_\lambda \leq ca_{\mathcal{A}}(B \vee x_\lambda) \\ &\iff A \preccurlyeq_{\mathcal{A}} B. \end{aligned}$$

Thus $\preccurlyeq_{\mathcal{A}}=\preccurlyeq_{\leqslant_{\mathcal{A}}}$. Hence, by Proposition 3.4, $\preccurlyeq_{\mathcal{A}}$ is an L-concave derived internal relation.

(3). Let $(X,\preccurlyeq)$ be an L-concave derived internal relation space. Let $A, B \in L^X$. By Propositions 2.9 and 3.6,

$$A \preccurlyeq_{\mathcal{A}_{\preccurlyeq}} B \iff A \preccurlyeq_{\leqslant_{\mathcal{A}_{\leqslant_{\preccurlyeq}}}} B \iff A \preccurlyeq_{\leqslant_{\preccurlyeq}} B \iff A \preccurlyeq B.$$

Thus $\preccurlyeq_{\mathcal{A}_{\preccurlyeq}}=\preccurlyeq$.

Let $(X,\mathcal{A})$ be an L-concave space. Let $A \in L^X$. By Propositions 2.9 and 3.6,

$$ca_{\preccurlyeq_{\mathcal{A}}}(A) = ca_{\leqslant_{\preccurlyeq_{\leqslant_{\mathcal{A}}}}}(A) = ca_{\leqslant_{\mathcal{A}}}(A) = ca_{\mathcal{A}}(A).$$

Therefore $\mathcal{A}_{\preccurlyeq_{\mathcal{A}}} = \mathcal{A}_{ca_{\preccurlyeq_{\mathcal{A}}}} = \mathcal{A}_{ca_{\mathcal{A}}} = \mathcal{A}$. □

Theorem 3.9. *L-**CDIRS** is isomorphic to L-**CAS**.*

To simply characterize L-**CDIRS**, we introduce L-concave derived hull space as follows.

Definition 3.10. An operator $\mathcal{I} : L^X \to L^X$ is called an L-concave derived hull operator on L^X and the pair $(X,\mathcal{I})$ is called an L-concave derived hull space if for all $A, B \in L^X$,

(LCADH1) $\mathcal{I}(\underline{\top}) = \underline{\top}$;

(LCADH2) $A \le \mathcal{I}(B)$ iff $x_\lambda \le \mathcal{I}(B \vee x_\lambda)$ for any $x_\lambda \in \beta^*(A)$;

(LCADH3) $A \wedge \mathcal{I}(A) \le \mathcal{I}(A \wedge \mathcal{I}(A))$;

(LCADH4) $\mathcal{I}(\bigwedge^{ddir}_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{I}(A_i)$.

Let $(X,\mathcal{I}_X)$ and $(Y,\mathcal{I}_Y)$ be L-concave derived hull spaces. A mapping $f : X \to Y$ is called an L-concave derived hull preserving mapping, if $f_L^{\leftarrow}(\mathcal{I}_Y(B) \wedge B) \le \mathcal{I}_X(f_L^{\leftarrow}(B))$ for all $B \in L^Y$. The category of L-concave derived hull spaces and L-concave derived hull preserving mappings is denoted by L-**CADHS**.

To characterize L-**CADHS**, we first show that an L-concave derived hull operator induces an L-concave derived internal relation.

Proposition 3.11. *Let $(X,\mathcal{I})$ be an L-concave derived hull space. Define a binary relation $\preccurlyeq_{\mathcal{I}}$ on L^X by*

$$\forall A, B \in L^X, \quad A \preccurlyeq_{\mathcal{I}} B \iff A \le \mathcal{I}(B).$$

Then $(X,\preccurlyeq_{\mathcal{I}})$ is an L-concave derived internal relation space.

Proof. We check that $\preccurlyeq_{\mathcal{I}}$ satisfies (LCDIR1)–(LCDIR5).

(LCDIR1). $\mathcal{I}(\underline{\top}) = \underline{\top}$ by (LCADH1). Thus $\underline{\top} \preccurlyeq_{\mathcal{I}} \underline{\top}$.

(LCDIR2). It follows from (LCADH2).

(LCDIR3). If $\bigvee_{i\in I} A_i \preccurlyeq_{\mathcal{I}} B$ then $A_j \le \bigvee_{i\in I} A_i \le \mathcal{I}(B)$ for any $j \in I$. Thus $A_j \preccurlyeq_{\mathcal{I}} B$ for any $j \in I$. Conversely, assume that $A_i \preccurlyeq_{\mathcal{I}} B$ for any $i \in I$. Then $A_i \le \mathcal{I}(B)$ for any $i \in I$. Hence $\bigvee_{i\in I} A_i \le \mathcal{I}(B)$. Therefore $\bigvee_{i\in I} A_i \preccurlyeq_{\mathcal{I}} B$.

(LCDIR4). Let $A \preccurlyeq_{\mathcal{I}} B$ and let $E = B \wedge \mathcal{I}(B)$. By $A \preccurlyeq_{\mathcal{I}} B$, $A \le \mathcal{I}(B)$. Thus $A \wedge B \le E \le B$. Since $E \le \mathcal{I}(B)$, $E \preccurlyeq_{\mathcal{I}} B$. In addition, by (LCADH3),

$$A \wedge B \le E = B \wedge \mathcal{I}(B) \le \mathcal{I}(B \wedge \mathcal{I}(B)) = \mathcal{I}(E).$$

Thus $A \wedge B \preccurlyeq_{\mathcal{I}} E$. Therefore $A \wedge B \preccurlyeq_{\mathcal{I}} E \preccurlyeq_{\mathcal{I}} B$ and $A \wedge B \le E \le B$ as desired.

(LCDIR5). By (LCADH4), $\mathcal{I}$ is monotonic. So the desired result is direct. □

Proposition 3.12. *Let $(X,\mathcal{I}_X)$ and $(Y,\mathcal{I}_Y)$ be L-concave derived hull spaces. If $f : X \to Y$ is an L-concave derived hull preserving mapping, then $f : (X,\preccurlyeq_{\mathcal{I}_X}) \to (Y,\preccurlyeq_{\mathcal{I}_Y})$ is an L-concave derived internal relation preserving mapping.*

Proof. If $A \preccurlyeq_{\mathcal{I}_Y} B$ then $A \le \mathcal{I}_Y(B)$. Thus $f_L^{\leftarrow}(A) \le f_L^{\leftarrow}(\mathcal{I}_Y(B))$ and

$$f_L^{\leftarrow}(A \wedge B) \le f_L^{\leftarrow}(\mathcal{I}_Y(B)) \wedge f_L^{\leftarrow}(B) \le \mathcal{I}_X(f_L^{\leftarrow}(B)).$$

Hence $f_L^{\leftarrow}(A \wedge B) \preccurlyeq_{\mathcal{I}_X} f_L^{\leftarrow}(B)$. Therefore f is an L-concave derived internal relation preserving mapping. □

Also, we can obtain an L-concave derived hull operator from an L-concave internal relation.

Proposition 3.13. *Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space. Define an operator $\mathcal{I}_{\preccurlyeq} : L^X \to L^X$ by*

$$\forall A \in L^X, \quad \mathcal{I}_{\preccurlyeq}(A) = \bigvee\{B \in L^X : B \preccurlyeq A\}.$$

Then $(X, \mathcal{I}_{\preccurlyeq})$ is an L-concave derived hull space.

Proof. (LCADH1). $\underline{\top} \leq \mathcal{I}_{\preccurlyeq}(\underline{\top})$ by (LCDIR1). Thus $\mathcal{I}_{\preccurlyeq}(\underline{\top}) = \underline{\top}$.

(LCADH2). Let $A \leq \mathcal{I}_{\preccurlyeq}(B)$. If $x_\lambda \in \beta^*(A)$, then $x_\lambda \prec \mathcal{I}_{\preccurlyeq}(B)$. Thus there is $D \in L^X$ such that $x_\lambda \leq D$ and $D \preccurlyeq B$. Hence $x_\lambda \leq D \preccurlyeq B \leq B \vee x_\lambda$ and then $x_\lambda \preccurlyeq B \vee x_\lambda$. Therefore $x_\lambda \leq \mathcal{I}_{\preccurlyeq}(B \vee x_\lambda)$.

Conversely, assume that $x_\lambda \leq \mathcal{I}_{\preccurlyeq}(B \vee x_\lambda)$ for any $x_\lambda \in \beta^*(A)$. To prove that $A \leq \mathcal{I}_{\preccurlyeq}(B)$, let $x_\lambda \in \beta^*(A)$. Then $x_\lambda \leq \mathcal{I}_{\preccurlyeq}(B \vee x_\lambda)$. For any $x_\eta \in \beta^*(x_\lambda)$, $x_\eta \prec \mathcal{I}_{\preccurlyeq}(B \vee x_\lambda)$. Thus there is a $D \in L^X$ such that $x_\eta \leq D \preccurlyeq B \vee x_\lambda$. Hence $x_\eta \preccurlyeq B \vee x_\lambda$. So $x_\lambda \preccurlyeq B \vee x_\lambda$ by (LCDIR3). Thus $A \preccurlyeq B$ by (LCDIR2). Therefore $A \leq \mathcal{I}_{\preccurlyeq}(B)$.

(LCADH3). By (LCDIR3), $\mathcal{I}_{\preccurlyeq}(A) \preccurlyeq A$. For any $x_\lambda \in \beta^*(A \wedge \mathcal{I}_{\preccurlyeq}(A))$, there is $D \in L^X$ such that $x_\lambda \leq D \preccurlyeq A$. By $D \preccurlyeq A$ and (LCDIR4), there is $C \in L^X$ such that $D \wedge A \preccurlyeq C \preccurlyeq A$ and $x_\lambda \leq D \wedge A \leq C \leq A$. Thus $C \leq \mathcal{I}_{\preccurlyeq}(A) \wedge A$. Hence $D \wedge A \preccurlyeq \mathcal{I}_{\preccurlyeq}(A) \wedge A$ which implies that

$$x_\lambda \leq D \wedge A \leq \mathcal{I}_{\preccurlyeq}(\mathcal{I}_{\preccurlyeq}(A) \wedge A).$$

Therefore $A \wedge \mathcal{I}_{\preccurlyeq}(A) \leq \mathcal{I}_{\preccurlyeq}(\mathcal{I}_{\preccurlyeq}(A) \wedge A)$.

(LCADH4). If $\{A_i\}_{i \in I}^{ddir} \subseteq L^X$, then it is cleat that $\mathcal{I}_{\preccurlyeq}(\bigwedge_{i \in I}^{ddir} A_i) \leq \bigwedge_{i \in I} \mathcal{I}_{\preccurlyeq}(A_i)$. Conversely, let $x_\lambda \in \beta^*(\bigwedge_{i \in I} \mathcal{I}_{\preccurlyeq}(A_i))$. Since $x_\lambda \in \beta^*(\mathcal{I}_{\preccurlyeq}(A_i))$ for any $i \in I$, there is $C_i \in L^X$ such that $x_\lambda \leq C_i \preccurlyeq A_i$. Let $C = \bigwedge_{i \in I} C_i$. Then $x_\lambda \leq C \preccurlyeq A_i$ for any $i \in I$. Thus $x_\lambda \leq C \preccurlyeq \bigwedge_{i \in I}^{ddir} A_i$ which implies that $x_\lambda \leq \mathcal{I}_{\preccurlyeq}(\bigwedge_{i \in I}^{ddir} A_i)$. Therefore $\bigwedge_{i \in I} \mathcal{I}_{\preccurlyeq}(A_i) \leq \mathcal{I}_{\preccurlyeq}(\bigwedge_{i \in I}^{ddir} A_i)$. □

Proposition 3.14. *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-concave derived internal relation spaces. If $f : X \to Y$ is an L-concave derived internal relation preserving mapping, then $f : (X, \mathcal{I}_{\preccurlyeq_X}) \to (Y, \mathcal{I}_{\preccurlyeq_Y})$ is an L-concave derived hull preserving mapping.*

Proof. Let $B \in L^Y$. In order to prove that $f_L^{\leftarrow}(\mathcal{I}_{\preccurlyeq_Y}(B) \wedge B) \leq \mathcal{I}_{\preccurlyeq_X}(f_L^{\leftarrow}(B))$, let $x_\lambda \in \beta^*(f_L^{\leftarrow}(\mathcal{I}_{\preccurlyeq_Y}(B) \wedge B))$. Then $f_L^{\rightarrow}(x_\lambda) \prec \mathcal{I}_{\preccurlyeq_Y}(B) \wedge B$. By $f_L^{\rightarrow}(x_\lambda) \prec \mathcal{I}_{\preccurlyeq_Y}(B)$, there is $D \in L^Y$ such that $f_L^{\rightarrow}(x_\lambda) \leq D \preccurlyeq_Y B$. Thus

$$x_\lambda \leq f_L^{\leftarrow}(D) \wedge f_L^{\leftarrow}(B) = f_L^{\leftarrow}(D \wedge B) \preccurlyeq_X f_L^{\leftarrow}(B).$$

Hence $x_\lambda \leq \mathcal{I}_{\preccurlyeq_X}(f_L^{\leftarrow}(B))$ and then $f_L^{\leftarrow}(\mathcal{I}_{\preccurlyeq_Y}(B) \wedge B) \leq \mathcal{I}_{\preccurlyeq_X}(f_L^{\leftarrow}(B))$. Therefore f is an L-concave derived hull preserving mapping. □

Proposition 3.15. *If $(X, \mathcal{I})$ is an L-concave derived hull space, then $\mathcal{I}_{\preccurlyeq_{\mathcal{I}}} = \mathcal{I}$; if $(X, \preccurlyeq)$ is an L-concave derived internal relation space, then $\preccurlyeq_{\mathcal{I}_{\preccurlyeq}} = \preccurlyeq$.*

Proof. Let $(X, \mathcal{I})$ be an L-concave derived hull space and $A \in L^X$. It is clear that

$$\mathcal{I}_{\preccurlyeq_{\mathcal{I}}}(A) = \bigvee\{D \in L^X : D \preccurlyeq_{\mathcal{I}} A\} \leq \mathcal{I}(A).$$

Conversely, $\mathcal{I}(A) \preccurlyeq_{\mathcal{I}} A$ by $\mathcal{I}(A) \leq \mathcal{I}(A)$. Thus $\mathcal{I}(A) \leq \mathcal{I}_{\preccurlyeq_{\mathcal{I}}}(A)$. Therefore $\mathcal{I}_{\preccurlyeq_{\mathcal{I}}} = \mathcal{I}$.

Let $(X, \preccurlyeq)$ be an L-concave derived internal relation space. If $A \preccurlyeq B$ then

$$A \leq \bigvee\{E \in L^X : E \preccurlyeq B\} = \mathcal{I}_{\preccurlyeq}(B).$$

Thus $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$. Conversely, if $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$, then $A \leq \mathcal{I}_{\preccurlyeq}(B)$. For any $x_\lambda \in \beta^*(A)$, $x_\lambda \prec \mathcal{I}_{\preccurlyeq}(B)$. Thus there is a set $E \in L^X$ such that $x_\lambda \leq E \preccurlyeq B$. Hence $x_\lambda \preccurlyeq B \vee x_\lambda$. By (LCDIR2), it follows that $A \preccurlyeq B$.

In conclusion, for any $A, B \in L^X$, $A \preccurlyeq B$ iff $A \preccurlyeq_{\mathcal{I}_{\preccurlyeq}} B$. That is, $\preccurlyeq_{\mathcal{I}_{\preccurlyeq}} = \preccurlyeq$. □

Based on Propositions 3.13 and 3.14, we get a functor $\mathbb{W}: L$-**CDIRS**$\to L$-**CADHS** by

$$\mathbb{W}((X,\preccurlyeq)) = (X,\mathcal{I}_{\preccurlyeq}), \quad \mathbb{W}(f) = f.$$

Based on Propositions 3.11–3.15, $\mathbb{W}$ is isomorphic. So we obtain the following result.

Theorem 3.16. *L-**CDIRS** is isomorphic to L-**CADHS**.*

Based on Corollary 3.8 and Propositions 3.11–3.15, relationships between L-concave derived hull spaces and L-concave spaces can be presented as follows.

Corollary 3.17. *(1) Let $(X,\mathcal{I})$ be an L-concave derived hull space. Define an operator $ca_{\mathcal{I}}: L^X \to L^X$ by*

$$\forall A \in L^X, \quad ca_{\mathcal{I}}(A) = A \wedge \mathcal{I}(A).$$

Then $ca_{\mathcal{I}}$ is the L-concave hull operator of an L-concave space $(X,\mathcal{A}_{\mathcal{I}})$;

(2) Let $(X,\mathcal{A})$ be an L-concave space. Define an operator $\mathcal{I}_{\mathcal{A}}: L^X \to L^X$ by

$$\forall A \in L^X, \quad \mathcal{I}_{\mathcal{A}}(A) = \bigvee\{B \in L^X : \forall x_\lambda \in \beta^*(B), x_\lambda \leq ca_{\mathcal{A}}(A \vee x_\lambda)\}.$$

Then $\mathcal{I}_{\mathcal{A}}$ is an L-concave derived hull operator;

(3) $\mathcal{I}_{\mathcal{A}_{\mathcal{I}}} = \mathcal{I}$ for any L-concave derived hull space $(X,\mathcal{I})$ and $\mathcal{A}_{\mathcal{I}_{\mathcal{A}}} = \mathcal{A}$ for any L-concave space $(X,\mathcal{A})$.

Proof. (1). Let $A \in L^X$. By Corollary 3.8(2),

$$\begin{aligned}\mathcal{I}_{\preccurlyeq_{\mathcal{A}}}(A) &= \bigvee\{B \in L^X : B \preccurlyeq_{\mathcal{A}} A\} \\ &= \bigvee\{B \in L^X : \forall x_\lambda \in \beta^*(B),\ x_\lambda \leq ca_{\mathcal{A}}(A \vee x_\lambda)\} \\ &= \mathcal{I}_{\mathcal{A}}(A).\end{aligned}$$

Thus $\mathcal{I}_{\mathcal{A}} = \mathcal{I}_{\preccurlyeq_{\mathcal{A}}}$. Hence, by Proposition 3.13, $\mathcal{I}_{\mathcal{A}}$ is an L-concave derived hull operator.

(2). Let $A \in L^X$. By Corollary 3.8(1) and Proposition 3.11,

$$\begin{aligned}ca_{\preccurlyeq_{\mathcal{I}}}(A) &= A \wedge \bigvee\{B \in L^X : B \preccurlyeq_{\mathcal{I}} A\} \\ &= A \wedge \bigvee\{B \in L^X : B \leq \mathcal{I}(A)\} \\ &= A \wedge \mathcal{I}(A) \\ &= ca_{\mathcal{I}}(A).\end{aligned}$$

Thus $ca_{\mathcal{I}} = ca_{\preccurlyeq_{\mathcal{I}}}$. Hence $ca_{\preccurlyeq_{\mathcal{I}}}$ is an L-concave derived interior operator. In addition, $\mathcal{A}_{\mathcal{I}} = \mathcal{A}_{\preccurlyeq_{\mathcal{I}}}$. Therefore $\mathcal{A}_{\mathcal{I}}$ is an L-concave structure.

(3). Let $(X,\mathcal{I})$ be an L-concave derived hull space. Let $A \in L^X$. By Corollary 3.8(3) and Proposition 3.15,

$$\mathcal{I}_{\mathcal{A}_{\mathcal{I}}}(A) = \mathcal{I}_{\preccurlyeq_{\mathcal{A}_{\preccurlyeq_{\mathcal{I}}}}}(A) = \mathcal{I}_{\preccurlyeq_{\mathcal{I}}}(A) = \mathcal{I}(A).$$

Thus $\mathcal{I}_{\mathcal{A}_{\mathcal{I}}} = \mathcal{I}$.

Let $(X,\mathcal{A})$ be an L-concave space. By Corollary 3.8(3) and Proposition 3.15,

$$ca_{\mathcal{I}_{\mathcal{A}}}(A) = ca_{\preccurlyeq_{\mathcal{I}_{\preccurlyeq_{\mathcal{A}}}}}(A) = ca_{\preccurlyeq_{\mathcal{A}}}(A) = ca_{\mathcal{A}}(A)$$

for any $A \in L^X$. Thus $\mathcal{A}_{\mathcal{I}_{\mathcal{A}}} = \mathcal{A}_{ca_{\mathcal{I}_{\mathcal{A}}}} = \mathcal{A}_{ca_{\mathcal{A}}} = \mathcal{A}$. □

Theorem 3.18. *L-**CADHS** is isomorphic to L-**CAS**.*

Relations among categories mentioned in this section is showed by the following diagram.

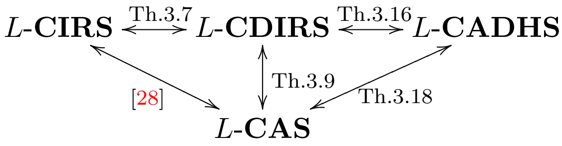

Figure 3. Relations 1.

4. L-convex derived enclosed relation spaces

In this section, we introduce the notion of L-convex derived enclosed relations by which we characterize L-convex enclosed relation spaces and L-convex spaces. Further, we introduce the notion of L-convex derived hull space and get a brief characterization of L-convex derived enclosed relation space. For these purposes, we introduce the following notions.

For $A \in L^X$ and $x_\lambda \in \beta^*(\top)$, we denote $\beta^*_\lambda(L) = \{\mu \in \beta^*(\top) : \lambda \in \beta^*(\mu)\}$ and

$$A_{x_\lambda} = \bigvee \{y_\mu \in \beta^*(A) : x_\lambda \not\leq y_\mu\}.$$

For convenience, we also denote $y_\eta \not\leq^* A$ for $y_\eta \in \beta^*(\top)$ and $y_\eta \not\leq A$.

Proposition 4.1 ([27])**.** *For all $x_\lambda, y_\eta \in \beta^*(\top)$, $A \in L^X$ and $\{A_i\}_{i\in I} \subseteq L^X$,*

(1) $x_\lambda \not\leq^* A$ implies $A_{x_\lambda} = A$;
(2) $A \leq B$ implies $A_{x_\lambda} \leq B_{x_\lambda}$;
(3) $(A_{x_\lambda})_{x_\lambda} = A_{x_\lambda}$;
(4) $\mu \in \beta^*_\lambda(L)$ implies $A_{x_\lambda} \leq A_{x_\mu}$ and $(A_{x_\mu})_{x_\lambda} = (A_{x_\lambda})_{x_\mu} = A_{x_\lambda}$;
(5) $y_\eta \not\leq \top_{x_\lambda}$ iff $x = y$ and $\eta \in \beta^*_\lambda(L)$;
(6) $A = \bigwedge_{x_\lambda \not\leq^* A} \top_{x_\lambda}$;
(7) $(\bigvee_{i\in I} A_i)_{x_\lambda} = \bigvee_{i\in I} (A_i)_{x_\lambda}$.

Definition 4.2. A binary relation $\preccurlyeq$ on L^X is called an L-convex derived enclosed relation and the pair $(X, \preccurlyeq)$ is called an L-convex derived enclosed relation space, if for all $A, B, C \in L^X$ and $x_\lambda \in \beta^*(\top)$,

(LCDER1) $\bot \preccurlyeq \bot$;
(LCDER2) $A \preccurlyeq B$ iff $A_{x_\mu} \preccurlyeq \top_{x_\lambda}$ and $A_{x_\mu} \leq \top_{x_\lambda}$ for any $x_\lambda \not\leq^* B$ and any $\mu \in \beta^*_\lambda(L)$;
(LCDER3) $A \preccurlyeq \bigwedge_{i\in I} B_i$ iff $A \preccurlyeq B_i$ for any $i \in I$;
(LCDER4) $A \preccurlyeq B$ implies $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq A \vee B$ and $A \leq C \leq A \vee B$;
(LCDER5) $\bigvee^{dir}_{i\in I} A_i \preccurlyeq B$ iff $A_i \preccurlyeq B$ for any $i \in I$.

It directly follows from (LCDER3) and (LCDER5) that $C \preccurlyeq D$ for all $A, B, C, D \in L^X$ with $C \leq A \preccurlyeq B \leq D$.

Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-convex derived enclosed relation spaces. A mapping $f : X \to Y$ is called an L-convex derived enclosed relation preserving mapping if $f^{\leftarrow}_L(A) \preccurlyeq_X f^{\leftarrow}_L(B) \vee f^{\leftarrow}_L(A)$ for all $A, B \in L^Y$ with $A \preccurlyeq_Y B$.

The category of L-convex derived enclosed relation spaces and L-convex derived enclosed relation preserving mappings is denoted by L-**CDERS**.

Proposition 4.3. *Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. Define a binary relation $\leqslant_{\preccurlyeq}$ on L^X by for all $A, B \in L^X$,*

$$A \leqslant_{\preccurlyeq} B \iff \exists C \in L^X \text{ s.t. } A \preccurlyeq C \text{ and } A \vee C = B.$$

Then $\leqslant_{\preccurlyeq}$ is an L-convex enclosed relation space.

Proof. (LCER1). We have $\bot \preccurlyeq \bot$ by (LCDER1). In addition, it follows from $\bot = \bot \vee \bot$ that $\bot \leqslant_{\preccurlyeq} \bot$.

(LCER2). It directly follows from the definition.

(LCER3). If $A \leqslant_{\preccurlyeq} \bigwedge_{i\in I} B_i$, then there is $C \in L^X$ such that $A \preccurlyeq C$ and $A \vee C = \bigwedge_{i\in I} B_i$. Then $A \vee C \leq B_i$ for any $i \in I$. Thus $A \preccurlyeq B_i$ and $A \vee B_i = B_i$. That is, $A \leqslant_{\preccurlyeq} B_i$ for any

$i \in I$. Conversely, assume that $A \leqslant_{\preccurlyeq} B_i$ for any $i \in I$. Then there is $C_i \in L^X$ such that $A \preccurlyeq C_i$ and $A \vee C_i = B_i$ for any $i \in I$. Thus $A \preccurlyeq \bigwedge_{i\in I} B_i$ by (LCDER3) and

$$A \vee \bigwedge_{i\in I} C_i = \bigwedge_{i\in I} (A \vee C_i) = \bigwedge_{i\in I} B_i.$$

Thus $A \leqslant_{\preccurlyeq} \bigwedge_{i\in I} B_i$.

(LCER4). Let $A \leqslant_{\preccurlyeq} B$. Then there is $D \in L^X$ such that $A \preccurlyeq D$ and $A \vee D = B$. By (LCDER4), there is $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq B$ and $A \leq C \leq B$. Hence $A \leqslant_{\preccurlyeq} C \leqslant_{\preccurlyeq} B$. Therefore C satisfies the requirement.

(LCER5). Let $\bigvee^{dir}_{i\in I} A_i \leqslant_{\preccurlyeq} B$. Then there is $D \in L^X$ such that $\bigvee^{dir}_{i\in I} A_i \preccurlyeq D$ and $\bigvee^{dir}_{i\in I} A_i \vee D = B$. Thus $A_i \preccurlyeq B$ and $A_i \vee B = B$ for any $i \in I$. That is $A_i \leqslant_{\preccurlyeq} B$ for any $i \in I$. Conversely, assume that $A_i \leqslant_{\preccurlyeq} B$ for any $i \in I$. Then there is $D_i \in L^X$ such that $A_i \preccurlyeq D_i$ and $A_i \vee D_i = B$ for any $i \in I$. Let $D = \bigvee_{i\in I} D_i$. Then $A_i \preccurlyeq D$ for any $i \in I$. Thus $\bigvee^{dir}_{i\in I} A_i \preccurlyeq D$ by (LCDER5). Since $\bigvee^{dir}_{i\in I} A_i \vee D = B$, it follows that $\bigvee^{dir}_{i\in I} A_i \leqslant_{\preccurlyeq} B$. □

Proposition 4.4. *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-convex derived enclosed relation spaces. If $f : X \to Y$ is an L-convex derived enclosed relation preserving mapping, then $f : (X, \leqslant_{\preccurlyeq_X}) \to (Y, \leqslant_{\preccurlyeq_Y})$ is an L-convex enclosed relation preserving mapping.*

Proof. If $A \leqslant_{\preccurlyeq_Y} B$ then there is $C \in L^Y$ such that $A \preccurlyeq_Y C$ and $A \vee C = B$. Thus $f^{\leftarrow}_L(A) \preccurlyeq_X f^{\leftarrow}_L(A \vee C)$ and

$$f^{\leftarrow}_L(A) \vee f^{\leftarrow}_L(A \vee C) = f^{\leftarrow}_L(A \vee C) = f^{\leftarrow}_L(B).$$

Hence $f^{\leftarrow}_L(A) \leqslant_{\preccurlyeq_X} f^{\leftarrow}_L(B)$. So f is an L-convex enclosed relation preserving mapping. □

Proposition 4.5. *Let $(X, \leqslant)$ be an L-convex enclosed relation space. Define a binary relation $\preccurlyeq_{\leqslant}$ on L^X by*

$$\forall A, B \in L^X, \quad A \preccurlyeq_{\leqslant} B \iff \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L),\ A_{x_\mu} \leqslant \top_{x_\lambda}.$$

Then $(X, \preccurlyeq_{\leqslant})$ is an L-convex derived enclosed relation space.

Proof. Clearly, $A \preccurlyeq_{\leqslant} B$ for any $A, B, C, D \in L^X$ with $A \leq C \preccurlyeq_{\leqslant} D \leq B$.

(LCDER1). It directly follows from (LCER1) of $\leqslant$.

(LCDER2). Let $A \preccurlyeq_{\leqslant} B$. Let $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. Then $B \leq \top_{x_\lambda}$ by $x_\lambda \not\leq B$. Thus $A_{x_\mu} \leq A \preccurlyeq_{\leqslant} B \leq \top_{x_\lambda}$ which implies that $A_{x_\mu} \preccurlyeq_{\leqslant} \top_{x_\lambda}$. Further, by $A \preccurlyeq_{\leqslant} B$, $A_{x_\mu} \leqslant \top_{x_\lambda}$. Hence $A_{x_\mu} \leq \top_{x_\lambda}$ by (LCER2).

Conversely, assume that $A_{x_\mu} \preccurlyeq_{\leqslant} \top_{x_\lambda}$ and $A_{x_\mu} \leq \top_{x_\lambda}$ for any $x_\lambda \not\leq^* B$ and any $\mu \in \beta^*_\lambda(L)$. Suppose that $A \not\preccurlyeq_{\leqslant} B$. Then there are $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$ such that $A_{x_\mu} \not\leqslant \top_{x_\lambda}$. Since $\mu \in \beta^*_\lambda(L)$, $x_\mu \not\leq \top_{x_\lambda}$. Further, by $A_{x_\mu} \preccurlyeq_{\leqslant} \top_{x_\lambda}$,

$$A_{x_\mu} = (A_{x_\mu})_{x_\mu} \leqslant (\top_{x_\lambda})_{x_\mu} = \top_{x_\lambda}.$$

It is a contradiction. Therefore $A \preccurlyeq_{\leqslant} B$.

(LCDER3). If $A \preccurlyeq_{\leqslant} \bigwedge_{i\in I} B_i$, then it is clear that $A \preccurlyeq_{\leqslant} B_i$ for any $i \in I$. Conversely, assume that $A \preccurlyeq_{\leqslant} B_i$ for any $i \in I$. For any $x_\lambda \not\leq^* \bigwedge_{i\in I} B_i$, there is $i \in I$ such that $x_\lambda \not\leq^* B_i$. By $A \preccurlyeq_{\leqslant} B_i$, it follows that $A_{x_\mu} \leqslant \top_{x_\lambda}$ for any $\mu \in \beta^*_\lambda(L)$. Therefore $A \preccurlyeq_{\leqslant} \bigwedge_{i\in I} B_i$.

(LCDER4). Let $A \preccurlyeq_{\leqslant} B$ and let

$$D = \bigwedge \{F \in L^X : A \preccurlyeq_{\leqslant} F\}.$$

Then $A \leq A \vee D \leq A \vee B$. Next, we verify that $E = A \vee D$ satisfies that $A \preccurlyeq_{\leqslant} E \preccurlyeq_{\leqslant} A \vee B$.

Indeed, we have $A \preccurlyeq_{\leqslant} D$ by (LCDER3). Thus $A \preccurlyeq_{\leqslant} E$. To prove that $E \preccurlyeq_{\leqslant} A \vee B$, let $x_\lambda \not\leq^* A \vee B$. We prove that $E_{x_\mu} \leqslant \top_{x_\lambda}$ for any $\mu \in \beta^*_\lambda(L)$.

By $A \preccurlyeq_{\leqslant} B$, $A = A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$. By (LCER4), there is $C \in L^X$ such that $A \leqslant C \leqslant \underline{\top}_{x_\lambda}$. Thus $A \leq C \leq \underline{\top}_{x_\lambda}$ by (LCER2). For any $z_\eta \not\leq^* C$ and any $\theta \in \beta^*_\eta(L)$,

$$A_{z_\theta} = A \leqslant C = C_{z_\eta} \leq \underline{\top}_{z_\eta}.$$

Hence $A \preccurlyeq_{\leqslant} C$ which implies that $D \leq C \leqslant \underline{\top}_{x_\lambda}$. So $E \leq C \leqslant \underline{\top}_{x_\lambda}$ and then $E_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$. Therefore $E \preccurlyeq_{\leqslant} A \vee B$. In conclusion, E satisfies the requirement.

(LCDER5). If $\bigvee^{dir}_{i\in I} A_i \preccurlyeq_{\leqslant} B$, then $A_i \preccurlyeq_{\leqslant} B$ for any $i \in I$. Conversely, let $\{A_i\}_{i\in I} \overset{dir}{\subseteq} L^X$ with $A_i \preccurlyeq_{\leqslant} B$ for any $i \in I$. Let $x_\lambda \not\leq^* B$ and $\mu \in \beta^*(L)$. Then $(A_i)_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ and $(A_i)_{x_\mu} \leq \underline{\top}_{x_\lambda}$ for any $i \in I$. Thus $(\bigvee^{dir}_{i\in I} A_i)_{x_\mu} = \bigvee^{dir}_{i\in I}(A_i)_{x_\mu}$ by (7) of Proposition 4.1. Hence $(\bigvee^{dir}_{i\in I} A_i)_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ by (LCER5). Therefore $\bigvee^{dir}_{i\in I} A_i \preccurlyeq_{\leqslant} B$. □

Proposition 4.6. *Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be L-convex enclosed relation spaces. If $f : X \to Y$ is an L-convex enclosed relation preserving mapping, then $f : (X, \preccurlyeq_{\leqslant_X}) \to (Y, \preccurlyeq_{\leqslant_Y})$ is an L-convex derived enclosed relation preserving mapping.*

Proof. Let $A \preccurlyeq_{\leqslant_Y} B$. To prove that $f^{\leftarrow}_L(A) \preccurlyeq_{\leqslant_X} f^{\leftarrow}_L(A \vee B)$, let $x_\lambda \not\leq^* f^{\leftarrow}_L(A \vee B)$ and $\mu \in \beta^*_\lambda(L)$. Then $f(x)_\mu \not\leq^* A \vee B$. By $A \preccurlyeq_{\leqslant_Y} B$, $A = A_{f(x)_\mu} \leqslant_Y \underline{\top}_{f(x)_\lambda}$ and $A \leq \underline{\top}_{f(x)_\lambda}$. Thus

$$f^{\leftarrow}_L(A)_{x_\mu} = f^{\leftarrow}_L(A) \leqslant_X f^{\leftarrow}_L(\underline{\top}_{f(x)_\lambda}) \leq \underline{\top}_{x_\lambda}.$$

Hence $f^{\leftarrow}_L(A) \preccurlyeq_{\leqslant_X} f^{\leftarrow}_L(A \vee B)$. Therefore f is an L-convex derived enclosed relation preserving mapping. □

Proposition 4.7. *If $(X, \preccurlyeq)$ is an L-convex derived enclosed relation space then $\preccurlyeq_{\leqslant_{\preccurlyeq}} = \preccurlyeq$; if $(X, \leqslant)$ is an L-convex enclosed relation space then $\leqslant_{\preccurlyeq_{\leqslant}} = \leqslant$.*

Proof. Let $(X, \leqslant)$ be an L-convex enclosed relation space. If $A \leqslant_{\preccurlyeq_{\leqslant}} B$, then there is $C \in L^X$ such that $A \preccurlyeq_{\leqslant} C$ and $A \vee C = B$. Thus $A \preccurlyeq_{\leqslant} B$ and $A \leq B$. By $A \preccurlyeq_{\leqslant} B$, $A = A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$ for any $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. Hence $A \leqslant \bigwedge_{x_\lambda \not\leq^* B} \underline{\top}_{x_\lambda} = B$ by (LCER3). That is, $A \leqslant B$.

Conversely, if $A \leqslant B$ then $A \leq B$ by (LCER2). For any $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$, $A_{x_\mu} = A \leqslant B \leq \underline{\top}_{x_\lambda}$. Thus $A_{x_\mu} \leqslant \underline{\top}_{x_\lambda}$. Hence $A \preccurlyeq_{\leqslant} B$. Further, by $A \preccurlyeq_{\leqslant} B$ and $A \vee B = B$, it follows that $A \leqslant_{\preccurlyeq_{\leqslant}} B$.

In conclusion, $A \leqslant_{\preccurlyeq_{\leqslant}} B$ iff $A \leqslant B$ for all $A, B \in L^X$. That is, $\leqslant_{\preccurlyeq_{\leqslant}} = \leqslant$.

Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. Let $A \preccurlyeq_{\leqslant_{\preccurlyeq}} B$. If $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$, we have $A_{x_\mu} \leqslant_{\preccurlyeq} \underline{\top}_{x_\lambda}$. Thus there is $C \in L^X$ such that $A_{x_\mu} \preccurlyeq C$ and $A_{x_\mu} \vee C = \underline{\top}_{x_\lambda}$. Hence $A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$. Therefore $A \preccurlyeq B$ by (LCDER2).

Conversely, let $A \preccurlyeq B$. To prove that $A \preccurlyeq_{\leqslant_{\preccurlyeq}} B$, let $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. We need to prove that $A_{x_\mu} \leqslant_{\preccurlyeq} \underline{\top}_{x_\lambda}$.

Actually, by $A \preccurlyeq B \leq \underline{\top}_{x_\lambda}$, it is clear that $A_{x_\mu} \preccurlyeq \underline{\top}_{x_\lambda}$ and $A_{x_\mu} \leq \underline{\top}_{x_\lambda}$ by (LCDER2). In addition, by $A_{x_\mu} \vee \underline{\top}_{x_\lambda} = \underline{\top}_{x_\lambda}$, $A_{x_\mu} \leqslant_{\preccurlyeq} \underline{\top}_{x_\lambda}$. Therefore $A \preccurlyeq_{\leqslant_{\preccurlyeq}} B$

In conclusion, for all $A, B \in L^X$, $A \preccurlyeq_{\leqslant_{\preccurlyeq}} B$ iff $A \preccurlyeq B$. That is, $\preccurlyeq_{\leqslant_{\preccurlyeq}} = \preccurlyeq$. □

Based on Propositions 4.3 and 4.4, we obtain a functor $\mathbb{F} : L$-**CDERS**$\to L$-**CERS** defined by

$$\mathbb{F}((X, \preccurlyeq)) = (X, \leqslant_{\preccurlyeq}), \qquad \mathbb{F}(f) = f.$$

Based on Propositions 4.3–4.7, $\mathbb{F}$ is an isomorphic functor. So we get the following result.

Theorem 4.8. *L-**CDERS** is isomorphic to L-**CERS**.*

Based on Propositions 2.6, 4.3–4.7 and Theorem 4.8, relationships between L-convex derived enclosed relation spaces and L-convex spaces are presented as follows.

Corollary 4.9. *(1) Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. Define an operator $co_{\preccurlyeq} : L^X \to L^X$ by*

$$\forall A \in L^X, \quad co_{\preccurlyeq}(A) = A \vee \bigwedge\{B \in L^X : A \preccurlyeq B\}.$$

Then $co_{\preccurlyeq}$ is an L-convex hull operator which induces an L-convex structure denoted by $\mathcal{C}_{\preccurlyeq}$.

(2) Let $(X, \mathcal{C})$ be an L-convex space. Define a binary operator $\preccurlyeq_{\mathcal{C}}$ on L^X by

$$\forall A, B \in L^X, \quad A \preccurlyeq_{\mathcal{C}} B Ł \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L),\ co_{\mathcal{C}}(A_{x_\mu}) \leq \top_{x_\lambda}.$$

Then $(X, \preccurlyeq_{\mathcal{C}})$ is an L-convex derived enclosed relation space.

(3) $\preccurlyeq_{\mathcal{C}_{\preccurlyeq}} = \preccurlyeq$ for any L-convex derived enclosed relation space $(X, \preccurlyeq)$ and $\mathcal{C}_{\preccurlyeq_{\mathcal{C}}} = \mathcal{C}$ for any L-convex space $(X, \mathcal{C})$.

Proof. (1). Let $A, B \in L^X$. Then

$$\begin{aligned} A \leqslant_{\preccurlyeq} B &\iff \exists C \in L^X,\ A \preccurlyeq C \leq A \vee C = B \\ &\iff A \preccurlyeq A \vee B = B. \end{aligned}$$

Thus, by Proposition 2.6(2),

$$\begin{aligned} co_{\preccurlyeq}(A) &= A \vee \bigwedge\{B \in L^X : A \preccurlyeq B\} \\ &= \bigwedge\{A \vee B : \exists B \in L^X,\ A \preccurlyeq B\} \\ &= \bigwedge\{B \in L^X : A \leqslant_{\preccurlyeq} B\} \\ &= co_{\leqslant_{\preccurlyeq}}(A). \end{aligned}$$

Hence $co_{\preccurlyeq} = co_{\leqslant_{\preccurlyeq}}$. Therefore $co_{\preccurlyeq}$ is an L-convex hull operator.

(2). Let $A, B \in L^X$. By Proposition 2.6(1), it follows that

$$\begin{aligned} A \preccurlyeq_{\leqslant_{\mathcal{C}}} B &\iff \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L),\ A_{x_\mu} \leqslant_{\mathcal{C}} \top_{x_\lambda} \\ &\iff \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L), co_{\mathcal{C}}(A_{x_\mu}) \leq \top_{x_\lambda} \\ &\iff A \preccurlyeq_{\mathcal{C}} B. \end{aligned}$$

Thus $\preccurlyeq_{\mathcal{C}} = \preccurlyeq_{\leqslant_{\mathcal{C}}}$. Hence, by Proposition 4.5, $\preccurlyeq_{\mathcal{C}}$ is an L-convex derived enclosed relation.

(3). Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. Let $A, B \in L^X$. By Propositions 2.6 and 4.7,

$$A \preccurlyeq_{\mathcal{C}_{\preccurlyeq}} B \iff A \preccurlyeq_{\leqslant_{\mathcal{C}_{\leqslant_{\preccurlyeq}}}} B \iff A \preccurlyeq_{\leqslant_{\preccurlyeq}} B \iff A \preccurlyeq B.$$

Thus $\preccurlyeq_{\mathcal{C}_{\preccurlyeq}} = \preccurlyeq$.

Let $(X, \mathcal{C})$ be an L-convex space. Let $A \in L^X$. By Propositions 2.6 and 4.7,

$$co_{\preccurlyeq_{\mathcal{C}}}(A) = co_{\leqslant_{\preccurlyeq_{\leqslant_{\mathcal{C}}}}}(A) = co_{\leqslant_{\mathcal{C}}}(A) = co_{\mathcal{C}}(A).$$

Therefore $\mathcal{C}_{\preccurlyeq_{\mathcal{C}}} = \mathcal{C}_{co_{\preccurlyeq_{\mathcal{C}}}} = \mathcal{C}_{co_{\mathcal{C}}} = \mathcal{C}$. □

Theorem 4.10. *L-**CDERS** is isomorphic to L-**CS**.*

To simply characterize L-**CDERS**, we introduce L-convex derived hull operator.

Definition 4.11. An operator $\mathcal{D} : L^X \to L^X$ is called an L-convex derived hull operator on L^X and the pair $(X, \mathcal{D})$ is called an L-convex derived hull space if for all $A, B \in L^X$ and any $x_\lambda \in \beta^*(\top)$,

(LCDH1) $\mathcal{D}(\perp) = \perp$;

(LCDH2) $\mathcal{D}(A) \leq B$ iff $\bigvee_{\mu \in \beta^*_\lambda(L)} \mathcal{D}(A_{x_\mu}) \vee A_{x_\mu} \leq \top_{x_\lambda}$ for any $x_\lambda \not\leq^* B$;

(LCDH3) $\mathcal{D}(\mathcal{D}(A) \vee A) \leq \mathcal{D}(A) \vee A$;

(LCDH4) $\mathcal{D}(\bigvee^{dir}_{i \in I} A_i) = \bigvee_{i \in I} \mathcal{D}(A_i)$.

Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be L-convex derived hull spaces. A mapping $f : X \to Y$ is called an L-convex derived hull preserving mapping, if for any $A \in L^X$,

$$f_L^{\rightarrow}(\mathcal{D}_X(A)) \leq \mathcal{D}_Y(f_L^{\rightarrow}(A)) \vee f_L^{\rightarrow}(A).$$

The category of L-convex derived hull spaces and L-convex derived hull preserving mappings is denoted by L-**CDHS**.

Proposition 4.12. *Let $(X, \mathcal{D})$ be an L-convex derived hull space. Define a binary relation $\preccurlyeq_{\mathcal{D}}$ on L^X by*

$$\forall A, B \in L^X, \quad A \preccurlyeq_{\mathcal{D}} B \iff \mathcal{D}(A) \leq B.$$

Then $\preccurlyeq_{\mathcal{D}}$ is an L-convex derived enclosed relation space.

Proof. (LCDER1). $\mathcal{D}(\perp) = \perp$ by (LCDH1). Thus $\perp \preccurlyeq_{\mathcal{D}} \perp$.

(LCDER2). It follows from (LCDH2).

(LCDER3). If $A \preccurlyeq_{\mathcal{D}} \bigwedge_{i\in I} B_i$ then $\mathcal{D}(A) \leq \bigwedge_{i\in I} B_i \leq B_j$ for any $j \in I$. Thus $A \preccurlyeq_{\mathcal{D}} B_j$ for any $j \in I$. Conversely, if $A \preccurlyeq_{\mathcal{D}} B_i$ for any $i \in I$, then $\mathcal{D}(A) \leq B_i$. Thus $\mathcal{D}(A) \leq \bigwedge_{i\in I} B_i$ which implies that $A \preccurlyeq_{\mathcal{D}} \bigwedge_{i\in I} B_i$.

(LCDER4). Let $A \preccurlyeq_{\mathcal{D}} B$ and let $E = \mathcal{D}(A) \vee A$. Then $E \preccurlyeq_{\mathcal{D}} E \leq B$ by (LCDH3). Thus $E \preccurlyeq_{\mathcal{D}} B$. In addition, $A \preccurlyeq_{\mathcal{D}} E$ by $\mathcal{D}(A) \leq E$. Therefore $A \preccurlyeq_{\mathcal{D}} E \preccurlyeq_{\mathcal{D}} A \vee B$ and $A \leq E \leq A \vee B$ as desired.

(LCDER5). It directly follows from (LCDH4). □

Proposition 4.13. *Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be L-convex derived hull spaces. If $f : X \to Y$ is an L-convex derived hull preserving mapping, then $f : (X, \preccurlyeq_{\mathcal{D}_X}) \to (Y, \preccurlyeq_{\mathcal{D}_Y})$ is an L-convex derived enclosed relation preserving mapping.*

Proof. If $A \preccurlyeq_{\mathcal{D}_Y} B$ then $\mathcal{D}_Y(A) \leq B$. Thus

$$\begin{aligned} f_L^{\rightarrow}(\mathcal{D}_X(f_L^{\leftarrow}(A))) &\leq \mathcal{D}_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(A))) \vee f_L^{\rightarrow}(f_L^{\leftarrow}(A)) \\ &\leq A \vee \mathcal{D}_Y(A) \\ &\leq A \vee B. \end{aligned}$$

Hence $f_L^{\leftarrow}(A) \preccurlyeq_{\mathcal{D}_X} f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(B)$ and

$$\mathcal{D}_X(f_L^{\leftarrow}(A)) \leq f_L^{\leftarrow}(A \vee B) = f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(B).$$

Therefore f is an L-convex derived enclosed relation preserving mapping. □

Proposition 4.14. *Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. Define an operator $\mathcal{D}_{\preccurlyeq} : L^X \to L^X$ by*

$$\forall A \in L^X, \quad \mathcal{D}_{\preccurlyeq}(A) = \bigwedge\{B \in L^X : A \preccurlyeq B\}.$$

Then $(X, \mathcal{D}_{\preccurlyeq})$ is an L-convex derived hull space.

Proof. (LCDH1). We have $\mathcal{D}_{\preccurlyeq}(\perp) \leq \perp$ by (LCDER1). Thus $\mathcal{D}_{\preccurlyeq}(\perp) = \perp$.

(LCDH2). If $\mathcal{D}_{\preccurlyeq}(A) \leq B$ then $A \preccurlyeq \mathcal{D}_{\preccurlyeq}(A) \leq B$ which implies $A \preccurlyeq B$. By (LCDER2), it is clear that $A_{x_\mu} \leq \top_{x_\lambda}$ and $A_{x_\mu} \preccurlyeq \top_{x_\lambda}$ for all $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. Thus $\bigvee_{\mu\in\beta^*(L)}(\mathcal{D}_{\preccurlyeq}(A_{x_\mu}) \vee A_{x_\mu}) \leq \top_{x_\lambda}$.

Conversely, assume that $\bigvee_{\mu\in\beta^*_\lambda(L)}(\mathcal{D}_{\preccurlyeq}(A_{x_\mu}) \vee A_{x_\mu}) \leq \top_{x_\lambda}$ for any $x_\lambda \not\leq^* B$. By (LCDER3), it follows that $A_{x_\mu} \preccurlyeq \mathcal{D}_{\preccurlyeq}(A_{x_\mu})$ for all $x_\lambda \not\leq^* B$ and $\mu \in \beta^*_\lambda(L)$. Thus $A_{x_\mu} \preccurlyeq \top_{x_\lambda}$ and $A_{x_\mu} \leq \top_{x_\lambda}$. Hence $A \preccurlyeq \top_{x_\lambda}$ by (LCDER2). Therefore $\mathcal{D}_{\preccurlyeq}(A) \leq \bigwedge_{x_\lambda \not\leq^* B} \top_{x_\lambda} = B$ by Proposition 4.1(6).

(LCDH3). Let $x_\lambda \in \beta^*(\top)$ with $x_\lambda \not\leq \mathcal{D}_{\preccurlyeq}(A) \vee A$. Then $x_\lambda \not\leq A$ and $x_\lambda \not\leq \mathcal{D}_{\preccurlyeq}(A)$. By $x_\lambda \not\leq \mathcal{D}_{\preccurlyeq}(A)$, there is $B \in L^X$ such that $x_\lambda \not\leq B$ and $A \preccurlyeq B$. By (LCDER4), there is $E \in L^X$ such that $A \preccurlyeq E \preccurlyeq B \vee A$ and $A \leq E \leq B \vee A$. By $A \preccurlyeq E$, it is clear that $\mathcal{D}_{\preccurlyeq}(A) \vee A \leq E \preccurlyeq A \vee B$. Thus

$$\mathcal{D}_{\preccurlyeq}(\mathcal{D}_{\preccurlyeq}(A) \vee A) \leq (A \vee B) \ngeq x_\lambda.$$

Hence $x_\lambda \nleq \mathcal{D}_{\preccurlyeq}(\mathcal{D}_{\preccurlyeq}(A) \vee A)$. Therefore we conclude that $\mathcal{D}_{\preccurlyeq}(\mathcal{D}_{\preccurlyeq}(A) \vee A) \leq \mathcal{D}_{\preccurlyeq}(A) \vee A$.

(LCDH4). Clearly, we have $\bigvee_{i\in I} \mathcal{D}_{\preccurlyeq}(A_i) \leq \mathcal{D}_{\preccurlyeq}(\bigvee^{dir}_{i\in I} A_i)$. Conversely, let $x_\lambda \in J(L^X)$ with $x_\lambda \nleq \bigvee_{i\in I} \mathcal{D}_{\preccurlyeq}(A_i)$. Thus there is $\mu \in \beta^*(\lambda)$ such that $x_\mu \nleq \bigvee_{i\in I} \mathcal{D}_{\preccurlyeq}(A_i)$. For any $i \in I$, by $x_\mu \nleq \mathcal{D}_{\preccurlyeq}(A_i)$, there is $C_i \in L^X$ such that $x_\mu \nleq C_i$ and $A_i \preccurlyeq C_i$. Let $C = \bigvee_{i\in I} C_i$. Then $A_i \preccurlyeq C$ for any $i \in I$. So $\bigvee^{dir}_{i\in I} A_i \preccurlyeq C \ngeq x_\lambda$. Hence $x_\lambda \nleq \mathcal{D}_{\preccurlyeq}(\bigvee^{dir}_{i\in I} A_i)$. Therefore $\mathcal{D}_{\preccurlyeq}(\bigvee^{dir}_{i\in I} A_i) \leq \bigvee_{i\in I} \mathcal{D}_{\preccurlyeq}(A_i)$. □

Proposition 4.15. *Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be L-convex derived enclosed relation spaces. If $f : X \to Y$ is an L-convex derived enclosed relation preserving mapping, then $f : (X, \mathcal{D}_{\preccurlyeq_X}) \to (Y, \mathcal{D}_{\preccurlyeq_Y})$ is an L-convex derived hull preserving mapping.*

Proof. Let $A \in L^X$ and $x_\lambda \in J(L^X)$ with $x_\lambda \nleq f_L^{\leftarrow}(\mathcal{D}_{\preccurlyeq_Y}(f_L^{\rightarrow}(A))) \vee f_L^{\leftarrow}(f_L^{\rightarrow}(A))$. Then

$$f_L^{\rightarrow}(x_\lambda) \nleq \mathcal{D}_{\preccurlyeq_Y}(f_L^{\rightarrow}(A)) \vee f_L^{\rightarrow}(A).$$

By $f_L^{\rightarrow}(x_\lambda) \nleq \mathcal{D}_{\preccurlyeq_Y}(f_L^{\rightarrow}(A))$, there is $B \in L^X$ such that $f_L^{\rightarrow}(x_\lambda) \nleq B$ and $f_L^{\rightarrow}(A) \preccurlyeq_Y B$. Thus $x_\lambda \nleq f_L^{\leftarrow}(B)$ and $A \leq f_L^{\leftarrow}(f_L^{\rightarrow}(A)) \preccurlyeq_X f_L^{\leftarrow}(B)$. Hence $A \preccurlyeq_X f_L^{\leftarrow}(B)$ and $x_\lambda \nleq \mathcal{D}_{\preccurlyeq_X}(A)$. So

$$\begin{aligned} f_L^{\rightarrow}(\mathcal{D}_{\preccurlyeq_X}(A)) &\leq f_L^{\rightarrow}(f_L^{\leftarrow}(\mathcal{D}_{\preccurlyeq_Y}(f_L^{\rightarrow}(A))) \vee f_L^{\leftarrow}(f_L^{\rightarrow}(A))) \\ &\leq \mathcal{D}_{\preccurlyeq_Y}(f_L^{\rightarrow}(A)) \vee f_L^{\rightarrow}(A). \end{aligned}$$

Therefore f is an L-convex derived hull preserving mapping. □

Proposition 4.16. *If $(X, \preccurlyeq)$ is an L-convex derived enclosed relation space then $\preccurlyeq_{\mathcal{D}_{\preccurlyeq}} = \preccurlyeq$; if $(X, \mathcal{D})$ is an L-convex derived hull space then $\mathcal{D}_{\preccurlyeq_{\mathcal{D}}} = \mathcal{D}$.*

Proof. Let $(X, \mathcal{D})$ be an L-convex derived hull space and $A \in L^X$. We have

$$\mathcal{D}(A) \leq \bigwedge\{B \in L^X : A \preccurlyeq_{\mathcal{D}} B\} = \mathcal{D}_{\preccurlyeq_{\mathcal{D}}}(A).$$

Conversely, $\mathcal{D}(A) \leq \top_{x_\lambda}$ for any $x_\lambda \nleq^* \mathcal{D}(A)$. Thus $A \preccurlyeq_{\mathcal{D}} \top_{x_\lambda}$ and $\mathcal{D}_{\preccurlyeq_{\mathcal{D}}}(A) \leq \top_{x_\lambda}$. So

$$\mathcal{D}_{\preccurlyeq_{\mathcal{D}}}(A) \leq \bigwedge_{x_\lambda \nleq \mathcal{D}(A)} \top_{x_\lambda} = \mathcal{D}(A).$$

Therefore $\mathcal{D}_{\preccurlyeq_{\mathcal{D}}}(A) = \mathcal{D}(A)$. So $\mathcal{D}_{\preccurlyeq_{\mathcal{D}}} = \mathcal{D}$.

Let $(X, \preccurlyeq)$ be an L-convex derived enclosed relation space. If $A \preccurlyeq B$ then $\mathcal{D}_{\preccurlyeq}(A) \leq B$ and so $A \preccurlyeq_{\mathcal{D}_{\preccurlyeq}} B$. Conversely, if $A \preccurlyeq_{\mathcal{D}_{\preccurlyeq}} B$, then $\mathcal{D}_{\preccurlyeq}(A) \leq B$ and $A \preccurlyeq \mathcal{D}_{\preccurlyeq}(A)$ by (LCDER3). Thus $A \preccurlyeq B$. In conclusion, $A \preccurlyeq B$ iff $A \preccurlyeq_{\mathcal{D}_{\preccurlyeq}} B$. That is, $\preccurlyeq_{\mathcal{D}_{\preccurlyeq}} = \preccurlyeq$. □

Based on Propositions 4.14 and 4.15, we get a functor $\mathbb{G} : L$-**CDERS**$\to L$-**CDHS** by

$$\mathbb{G}((X, \preccurlyeq)) = (X, \mathcal{D}_{\preccurlyeq}), \quad \mathbb{G}(f) = f.$$

Based on Propositions 4.12–4.16, $\mathbb{G}$ is isomorphic. So we obtain the following result.

Theorem 4.17. *L-**CDERS** is isomorphic to L-**CDHS**.*

Based on Corollary 4.9 and Propositions 4.12–4.16, and Theorem 4.17, relationships between L-convex derived hull relation spaces and L-convex spaces are presented as follows.

Corollary 4.18. *(1) Let $(X, \mathcal{D})$ be an L-convex derived hull space. Define an operator $co_{\mathcal{D}} : L^X \to L^X$ by*

$$\forall A \in L^X, \quad co_{\mathcal{D}}(A) = \mathcal{D}(A) \vee A.$$

Then $co_{\mathcal{D}}$ is the L-convex hull operator of an L-convex structure denoted by $\mathcal{C}_{\mathcal{D}}$.

(2) Let $(X,\mathcal{C})$ *be an L-convex space. Define an operator* $\mathcal{D}_{\mathcal{C}}: L^X \to L^X$ *by*

$$\forall A \in L^X, \quad \mathcal{D}_{\mathcal{C}}(A) \;=\; \bigwedge\{B \in L^X : \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L), co_{\mathcal{C}}(A_{x_\mu}) \leq \top_{x_\lambda}\}.$$

Then $\mathcal{D}_{\mathcal{C}}$ *is an L-convex derived hull operator.*

(3) $\mathcal{D}_{\mathcal{C}_{\mathcal{D}}} = \mathcal{D}$ *for any L-convex derived hull space* $(X,\mathcal{D})$ *and* $\mathcal{C}_{\mathcal{D}_{\mathcal{C}}} = \mathcal{C}$ *for any L-convex space* $(X,\mathcal{C})$.

Proof. (1). Let $A \in L^X$. By Corollary 4.9(2),

$$\begin{aligned}\mathcal{D}_{\preccurlyeq_{\mathcal{C}}}(A) &= \bigwedge\{B \in L^X : A \preccurlyeq_{\mathcal{C}} B\}\\ &= \bigwedge\{B \in L^X : \forall x_\lambda \not\leq^* B, \forall \mu \in \beta^*_\lambda(L), co_{\mathcal{C}}(A_{x_\mu}) \leq \top_{x_\lambda}\}\\ &= \mathcal{D}_{\mathcal{C}}(A).\end{aligned}$$

Thus $\mathcal{D}_{\mathcal{C}} = \mathcal{D}_{\preccurlyeq_{\mathcal{C}}}$. Hence, by Proposition 4.14, $\mathcal{D}_{\mathcal{C}}$ is an L-convex derived hull operator.

(2). Let $A \in L^X$. By Corollary 4.9(1) and Proposition 4.12, we have

$$\begin{aligned}co_{\preccurlyeq_{\mathcal{D}}}(A) &= A \vee \bigwedge\{B \in L^X : A \preccurlyeq_{\mathcal{D}} B\}\\ &= A \vee \bigwedge\{B \in L^X : \mathcal{D}(A) \leq B\}\\ &= A \vee \mathcal{D}(A)\\ &= co_{\mathcal{D}}(A).\end{aligned}$$

Thus $co_{\mathcal{D}} = co_{\preccurlyeq_{\mathcal{D}}}$. Hence $co_{\preccurlyeq_{\mathcal{D}}}$ is an L-convex derived hull operator. In addition, $\mathcal{C}_{\mathcal{D}} = \mathcal{C}_{\preccurlyeq_{\mathcal{D}}}$. Therefore $\mathcal{C}_{\mathcal{D}}$ is an L-convex structure.

(3). Let $(X,\mathcal{D})$ be an L-convex derived hull space. For any $A \in L^X$. it follows from Corollary 4.9(3) and Proposition 4.16 that

$$\mathcal{D}_{\mathcal{C}_{\mathcal{D}}}(A) = \mathcal{D}_{\preccurlyeq_{\mathcal{C}_{\preccurlyeq_{\mathcal{D}}}}}(A) = \mathcal{D}_{\preccurlyeq_{\mathcal{D}}}(A) = \mathcal{D}(A).$$

Thus $\mathcal{D}_{\mathcal{C}_{\mathcal{D}}} = \mathcal{D}$.

Let $(X,\mathcal{C})$ be an L-convex space. For any $A \in L^X$, it follows from Corollary 4.9(3) and Proposition 4.16 that

$$co_{\mathcal{D}_{\mathcal{C}}}(A) = co_{\preccurlyeq_{\mathcal{D}_{\preccurlyeq_{\mathcal{C}}}}}(A) = co_{\preccurlyeq_{\mathcal{C}}}(A) = co_{\mathcal{C}}(A).$$

Thus $\mathcal{C}_{\mathcal{D}_{\mathcal{C}}} = \mathcal{C}_{co_{\mathcal{D}_{\mathcal{C}}}} = \mathcal{C}_{co_{\mathcal{C}}} = \mathcal{C}$. □

Theorem 4.19. *L-**CDHS** is isomorphic to L-**CS**.*

The following diagram shows the relations among categories mentioned in this section.

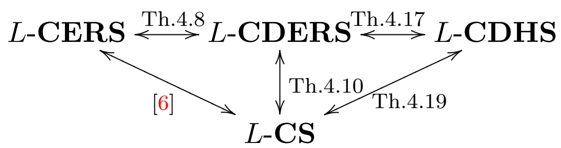

Figure 4. Relations 2.

5. Conclusions

(1) In this paper, we introduce notions of L-concave derived internal relation space and L-concave derived internal space by which we characterize L-concave internal relation space and L-concave space. Also, we introduce L-convex derived enclosed relation space and L-convex derived hull space by which we characterize L-convex enclosed relation space and L-convex space.

(2) Solutions of Problems 1 and 2 in the introduction section are presented as follows.

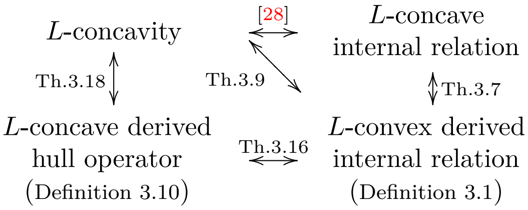

Figure 5. Solution 1.

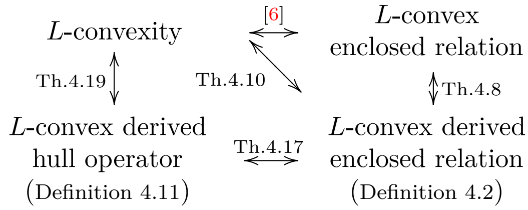

Figure 6. Solution 2.

(3) If L has an inverse involution $'$ then a subset $\mathcal{A} \subseteq L^X$ is an L-concave structure iff the set $\mathcal{A}' = \{A' : A \in \mathcal{A}\}$ is an L-convex structure. Similarly, a binary relation $\leqslant$ on L^X is an L-concave internal relation iff the binary relation $\leqslant'$ is an L-convex enclosed relation, where $\leqslant'$ is defined by $A \leqslant' B$ iff $B' \leqslant A'$ for all $A, B \in L^X$. By this consideration, we usually say that L-concave space and L-convex space are dual concepts. Similarly, L-concave internal relation and L-convex enclosed relation are dual concepts too. However, L-concave derived internal relation and L-convex derived enclosed relation can not to be regarded as dual concepts in this sense. We have the following example.

Let $X = \{x\}$ and $L = \{\bot, c, a, b, d, \top\}$ be a lattice defined by Figure 7 as follows. The inverse involution $'$ is defined by $\bot' = \top$, $a' = b$ and $c' = d$. Let $\mathcal{C} = \{x_\bot, x_c, x_a, x_\top\}$. Then the L-convex derived enclosed relation $\preccurlyeq_{\mathcal{C}}$ is presented by Table 1.

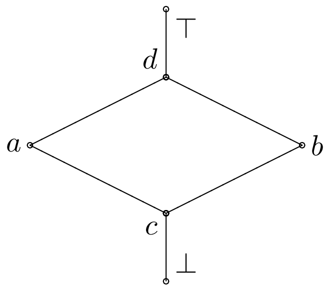

Figure 7

	$x_\bot$	x_c	x_a	x_b	x_d	$x_\top$
$x_\bot$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$
x_c	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$
x_a	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$
x_b	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$
x_d			$\preccurlyeq_{\mathcal{C}}$		$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$
$x_\top$			$\preccurlyeq_{\mathcal{C}}$		$\preccurlyeq_{\mathcal{C}}$	$\preccurlyeq_{\mathcal{C}}$

Table 1

Let $\preccurlyeq'_{\mathcal{C}}$ be defined by $A \preccurlyeq'_{\mathcal{C}} B$ iff $B' \preccurlyeq_{\mathcal{C}} A'$ for all $A, B \in L^X$. Clearly, $\beta^*(x_d) = \{x_c, x_a, x_b\}$. Suppose that $\preccurlyeq'_{\mathcal{C}}$ is an L-concave derived internal relation. We have $x_c \preccurlyeq'_{\mathcal{C}} x_c \vee x_c$, $x_a \preccurlyeq'_{\mathcal{C}} x_c \vee x_a$ and $x_b \preccurlyeq'_{\mathcal{C}} x_c \vee x_b$ since $x_d \preccurlyeq_{\mathcal{C}} x_d$, $x_b \preccurlyeq_{\mathcal{C}} x_b$ and $x_a \preccurlyeq_{\mathcal{C}} x_a$. Thus $x_d \preccurlyeq'_{\mathcal{C}} x_c$ by (LCDIR2). So $x_d \preccurlyeq_{\mathcal{C}} x_c$. But this is a contradiction. Therefore $\preccurlyeq'_{\mathcal{C}}$ is not an L-concave derived internal relation.

Actually, $\mathcal{A} = \mathcal{C}' = \{x_\bot, x_b, x_d, x_\top\}$ is an L-concave space. From Table 2 and Table 3, we find that $\preccurlyeq_{\mathcal{A}}$ is quite different from $\preccurlyeq'_{\mathcal{C}}$.

	$x_\bot$	x_c	x_a	x_b	x_d	$x_\top$
$x_\bot$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
x_c				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
x_a				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
x_b				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
x_d				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$
$x_\top$				$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$	$\preccurlyeq_{\mathcal{A}}$

Table 2

	$x_\bot$	x_c	x_a	x_b	x_d	$x_\top$
$x_\bot$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$
x_c	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$
x_a			$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$
x_b	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$
x_d			$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$
$x_\top$			$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$	$\preccurlyeq'_{\mathcal{C}}$

Table 3

(4) In abstract convex structures, algebraic property of convex hulls is an essential feature of convex structure which is different from many other mathematic structures such as topological structures, convergence structure and matroid. With the development

of fuzzy extensions of convex theory, such property has been accordingly extended by many means [3,12,17,19,24]. Thus it could be worth to discuss presentations of algebraic property in L-convex enclosed relation space, L-convex derived enclosed relation space and L-convex derived hull space.

(5) The notions L-concave derived internal relation, L-concave derived hull operator, L-convex derived enclosed relation and L-convex derived hull operator may provide some alternative ways in discussing separation axioms of L-convex spaces and relations among L-convex spaces, L-matroids and L-convergence spaces.

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