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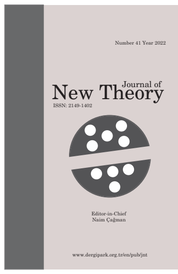

On Quasi Quadratic Modules of Lie Algebras

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Abstract — This study introduces the category of quasi-quadratic modules of Lie algebras and discusses the functorial relations between quasi-quadratic modules and quadratic modules of Lie algebras.

Keywords — *Crossed modules, quasi quadratic modules, quadratic modules*

Mathematics Subject Classification (2020) — 18A40, 18G45

1. Introduction

The concept of crossed modules first appears in Whitehead's work on groups, [1, 2], related to the homotopy type of 3-dimensional complexes. Its Lie algebra version has been introduced by Kassel and Loday in [3]. Since then, many algebraic contents have been created based on this concept. Some of them are considered two-dimensional analogous to crossed modules. One of them is the notion of 2-crossed modules for groups introduced by Conduché, [4], with the objective of generalizing the well known result by which the category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length 2. The Lie algebra version of this notion belongs to Ellis, [5]. Another notion is quadratic modules of groups defined by Baues, [6], which is a special case of it. Ulualan and Uslu have adapted this algebraic 3-type model to the Lie algebras, [7] as well as Arvasi and Ulualan have worked on that of commutative algebra case, [8], [9]. Many studies have been conducted in these contexts for various algebraic cases, including the homotopy theory and some categorical results, [10–14]. Carrasco and Porter have initially introduced 2-quasi-crossed modules of groups in [15] as an auxiliary tool that they are in between 2-pre-crossed modules and 2-crossed modules. It is well known that we can get a crossed module associated to a pre-crossed module. It is seen in [15] that it is possible to construct the 2-crossed module associated to a 2-quasi-crossed module of groups by similar ideas. Thus, we can obtain some functorial relations and some categorical results. In [16], the second author and Kaplan have also given the concept of quasi 2-crossed modules of Lie algebras and some functorial results. In this work we focus on the concept of the quadratic module which is another two-dimensional analogous of crossed modules. We will introduce the definition of a quasi-quadratic module as an auxiliary concept in the construction of some categorical content and also examine how this relates to quadratic modules as functorially.

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2. Preliminaries

A pre-crossed module of Lie algebras (Y, Z, ∂) is given by a Lie homomorphism $\partial : Y \to Z$, together with a left Lie algebra action "$*_1$" of Z on Y such that the condition

XMod$_L$1: $\partial(z *_1 y) = [z, \partial(y)]$

is satisfied for each $z \in Z$ and $y \in Y$.

A crossed module of Lie algebras (Y, Z, ∂) is a pre-crossed module satisfying, in addition condition:

XMod$_L$2: $\partial(y) *_1 y' = [y, y']$

for all $y_1, y_2 \in Y$.

A crossed module morphism $f : (Y, Z, \partial) \to (Y', Z', \partial')$ consists of Lie algebra morphisms f_1 and f_0 such that the following diagram is commutative and preserves the action of Z on Y:

$$\begin{tikzcd} Y \arrow[r, "\partial"] \arrow[d, "f_1"'] & Z \arrow[d, "f_0"] \\ Y' \arrow[r, "\partial'"'] & Z' \end{tikzcd}$$

Therefore we can define the category of crossed modules of Lie algebras denoting it as $\mathbf{XMod_L}$. If we fix the bottom part, the Z Lie algebra, then $\mathbf{XMod_L/Z}$ will be the category of crossed Z-modules.

2.1. Quadratic modules of Lie algebras

We recall the definition of quadratic modules of Lie algebras given [7].

A quadratic module $(\omega, \delta, \partial)$ of Lie algebras consists of Lie algebra homomorphisms as illustrated in the below diagram, satisfying following conditions:

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "\Phi"] & \\ X \arrow[r, "\delta"'] & Y \arrow[r, "\partial"'] & Z \end{tikzcd}$$

QM$_\mathbf{L}$1 : The homomorphism $\partial : Y \to Z$ is a $nil(2)$-module and $Y \twoheadrightarrow C = Y^{cr}/[Y^{cr}, Y^{cr}]$ is defined by $y \mapsto [y]$ and Φ is defined by

$$\Phi([y] \otimes [y']) = \partial(y) *_1 y' - [y, y']$$

for $y, y' \in Y$.

QM$_\mathbf{L}$2 : The boundary Lie homomorphisms composition of ∂ and δ is zero map and $\delta\omega = \Phi$.

QM$_\mathbf{L}$3 : X is a Lie Z-algebra, all of the homomorphisms in the diagram are Z-equivariant, and the left Lie algebra action "$*_3$" of Z on X also holds the following equality

$$\partial(y) *_3 x = \omega([\delta(x)] \otimes [y] + [y] \otimes [\delta(x)])$$

for $x \in X$ and $y \in Y$.

QM$_\mathbf{L}$4 :

$$\omega([\delta(x)] \otimes [\delta(x')]) = [x', x]$$

for $x, x' \in X$.

Remark 2.1. It should be noted that $X \xrightarrow{\delta} Y$ is a crossed module, with a left Lie algebra action "$*_2$"

$$y *_2 x = \omega([\delta(x)] \otimes [y])$$

for each $y \in Y$ and $x \in X$. On the other hand, generally, $Y \xrightarrow{\partial} Z$ is only a $nil(2)$ module.

Remark 2.2. By $\mathbf{QM}_L\mathbf{3}$, we have:

$$\partial(y) *_3 x - y *_2 x = \omega([y] \otimes [\delta(x)])$$

Lemma 2.3. Let $\mathcal{L} = \left(X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-])\right)$ be a quadratic module and consider the "$*_2$" and "$*_3$" Lie algebra actions. Then for all $z \in Z$ and $y_1, y_2, y_3 \in Y$, we have:

$$z *_3 \omega([y_1] \otimes [y_2]) = \omega([z *_1 y_1] \otimes [y_2]) + \omega([y_1] \otimes [z *_1 y_2]) \tag{1}$$

$$\begin{aligned} \omega([[y_1, y_2]] \otimes [y_3]) &= \partial(y_1) *_3 \omega([y_2] \otimes [y_3]) + \omega([y_1] \otimes [[y_2, y_3]]) \\ &\quad - \partial(y_2) *_3 \omega([y_1] \otimes [y_3]) - \omega([y_2] \otimes [[y_1, y_3]]) \end{aligned} \tag{2}$$

$$\omega([y_1] \otimes [[y_2, y_3]]) = y_2 *_2 \omega([y_1] \otimes [y_3]) - y_3 *_2 \omega([y_1] \otimes [y_2]) \tag{3}$$

Example 2.4. Let $Y \xrightarrow{\partial} Z$ be $nil(2)$-module. We can define the following (Id, Φ, ∂) quadratic module

$$\begin{tikzcd} & C \otimes C \arrow[dl, "Id"'] \arrow[d, "\Phi"] & \\ C \otimes C \arrow[r, "\Phi"'] & Y \arrow[r, "\partial"'] & Z \end{tikzcd}$$

Example 2.5. Let $(\omega, \delta, \partial)$ be a quadratic module. Then, we can define the quadratic module $(0, 0, \partial)$ with trivial quadratic map such as

$$1 \xrightarrow{0} Y \xrightarrow{\partial} Z$$

Example 2.6. If $\mathcal{L} = (X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-]))$ is a quadratic module, then $Im\delta$ is a Lie ideal of Y and we have that

$$Y/Im\delta \xrightarrow{\partial} Z$$

is an induced crossed module.

Definition 2.7. Let $\mathcal{L} = (\omega, \delta, \partial)$ and $\mathcal{L}' = (\omega', \delta', \partial')$ be two quadratic modules of Lie algebras. A morphism of quadratic modules from $\mathcal{L}$ to $\mathcal{L}'$ is illustrated by the following commutative diagram:

$$\begin{tikzcd} C \otimes C \arrow[r, "\omega"] \arrow[d, "\varphi \otimes \varphi"'] & X \arrow[r, "\delta"] \arrow[d, "f_2"] & Y \arrow[r, "\partial"] \arrow[d, "f_1"] & Z \arrow[d, "f_0"] \\ C' \otimes C' \arrow[r, "\omega'"'] & X' \arrow[r, "\delta'"'] & Y' \arrow[r, "\partial'"'] & Z' \end{tikzcd}$$

where (f_1, f_0) is a morphism of $nil(2)$ modules which induces $\varphi : C \to C'$ and also following equations are satisfied:

$$f_1(z *_1 y) = f_0(z) *'_1 f_1(y)$$

$$f_2(z *_3 x) = f_0(z) *'_3 f_2(x)$$

$$f_2(\omega([y_1] \otimes [y_2])) = \omega'([f_1(y_1)] \otimes [f_1(y_2)])$$

for all $z \in Z$, $y, y_1, y_2 \in Y$, and $x \in X$.

We will denote by $\mathbf{QM_L}$ the category of quadratic modules of Lie algebras and by $\mathbf{QM_L}/(Y \overset{\partial}{\to} Z)$ the subcategory of quadratic modules over fixed $nil(2)$ module (Y, Z, ∂).

Remark 2.8. We can define another canonically associated crossed module as $\widehat{\delta} : X \to Y \rtimes Z, \widehat{\delta}(x) = (\delta(x), 0_Z)$ with $(y, z) \bullet x = y *_2 x + z *_3 x$. $(X, Y \rtimes Z, \widehat{\delta})$ is a crossed modules because of

XMod$_L$1

$$\begin{aligned} \widehat{\delta}((y,z) \bullet x) &= \widehat{\delta}(y *_2 x + z *_3 x) \\ &= (\delta(y *_2 x + z *_3 x), 0_Z) \\ &= (\delta(y *_2 x) + \delta(z *_3 x), 0_Z) \\ &= ([y, \delta(x)] + z *_1 \delta(x) - 0_Z *_1 y, [z, 0_Z]) \\ &= [(y,z), (\delta(x), 0_Z)] \\ &= [(y,z), \widehat{\delta}(x)] \end{aligned}$$

XMod$_L$2

$$\begin{aligned} \widehat{\delta}(x') \bullet x'' &= (\delta(x'), 0_Z) \bullet x'' \\ &= \delta(x') *_2 x'' + 0_z *_3 x'' \\ &= [x', x''] \end{aligned}$$

for each $z \in Z$, $y \in Y$, and $x, x', x'' \in X$.

3. Quasi-Quadratic Modules of Lie Algebras

Definition 3.1. A quasi-quadratic module of Lie algebras is a semi-exact sequence

$$C \xrightarrow{\delta} D \xrightarrow{\partial} E$$

of E-Lie algebras together with a μ quadratic map

$$\mu([-] \otimes [-]) : B \otimes B \to C$$

where $B = D^{cr}/[D^{cr}, D^{cr}]$, such that $\mathcal{QQM}_L1$, $\mathcal{QQM}_L2$, $\mathcal{QQM}_L3$ and $\mathcal{QQM}_L4$ hold:

$\mathcal{QQM}_L1$:

$$\delta\mu([d_1] \otimes [d_2]) = \Phi([d_1] \otimes [d_2]) = \partial(d_1) *_1 d_2 - [d_1, d_2]$$

$\mathcal{QQM}_L2$:

$$\begin{aligned} \mu([[d_1, d_2]] \otimes [d_3]) &= \partial(d_1) *_3 \mu([d_2] \otimes [d_3]) + \mu([d_1] \otimes [[d_2, d_3]])| \\ &\quad -\partial(d_2) *_3 \mu([d_1] \otimes [d_3]) - \mu([d_2] \otimes [[d_1, d_3]]) \end{aligned}$$

$\mathcal{QQM}_L3$:

$$\mu([d_1] \otimes [[d_2, d_3]]) = d_2 *_2 \mu([d_1] \otimes [d_3]) - d_3 *_2 \mu([d_1] \otimes [d_2])$$

$\mathcal{QQM}_L4$:

$$[\mu([d_1] \otimes [d_2]), \partial(d_1) *_3 (d_2 *_2 c)] = \mu([\partial(d_1) *_1 [d_2, \delta c]] \otimes [\delta\mu([d_1] \otimes [d_2])])$$

for all $d_1, d_2, d_3 \in D$ and $c \in C$.

Quasi-quadratic module morphisms are defined in the same way as quadratic module morphisms. We will denote the category of quasi-quadratic module of Lie algebras by $\mathcal{QQM}_L$.

Lemma 3.2. Any quadratic module is a quasi-quadratic module.

PROOF. Let $\mathcal{L} = \left(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\right)$ be a quadratic module of Lie algebras. We need to show that it satisfies the quasi-quadratic module axioms. As can be seen, the quasi-quadratic module $\mathcal{QQM}_L1$ axiom and the quadratic module $\mathbf{QM_L}1$ axiom are the same. In addition, $\mathcal{QQM}_L2$ and $\mathcal{QQM}_L3$ axioms are provided due to Lemma 2.3. Therefore, we only need to provide the axiom $\mathcal{QQM}_L4$.

$\mathcal{QQM}_L4$:

$$\begin{aligned}
[\mu([d_1] \otimes [d_2]), \partial(d_1) *_3 (d_2 *_2 c)] &= \mu([\delta(\partial(d_1) *_3 (d_2 *_2 c))] \otimes [\delta\mu([d_1] \otimes [d_2])]) \\
&= \mu([\partial(d_1) *_1 \delta(d_2 *_2 c)] \otimes [\delta\mu([d_1] \otimes [d_2])]) \\
&= \mu([\partial(d_1) *_1 [d_2, \delta(x)]] \otimes [\partial(d_1) *_1 d_2 - [d_1, d_2]])
\end{aligned}$$

for all $d_1, d_2 \in D$ and $c \in C$. □

Then, we will construct a quadratic module of Lie algebras associated with a quasi-quadratic module of Lie algebras:

Let $\mathcal{L} = \left(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\right)$ be quasi-quadratic module and I be defined as the Lie subalgebra of C generated by the elements of the form:

- $d\#_1 c = \partial(d) *_3 c - \mu([d] \otimes [\delta(c)]) - \mu([\delta(d)] \otimes [c])$
- $c_1 \#_2 c_2 = [c_2, c_1] - \mu([\delta(c_1)] \otimes [\delta(c_2)])$

$d \in D$ and $c, c_1, c_2 \in C$.

Lemma 3.3. For any quasi-quadratic module $\mathcal{L} = \left(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\right)$, the Lie subalgebra I of C is an E-invariant Lie ideal.

PROOF. I is an E-invariant Lie ideal of C:

$$\begin{aligned}
e *_3 (d\#_1 c) &= e *_3 (\partial(d) *_3 c - \mu([d] \otimes [\delta(c)]) - \mu([\delta(c)] \otimes [d])) \\
&= e *_3 (\partial(d) *_3 c) - e *_3 \mu([d] \otimes [\delta(c)]) - e *_3 \mu([\delta(c)] \otimes [d]) \\
&= [e, \partial(d)] *_3 c + \partial(d) *_3 (e *_3 c) - e *_3 \mu([d] \otimes [\delta(c)]) - e *_3 \mu([\delta(c)] \otimes [d]) \\
&= \partial(e *_1 d) *_3 c - \mu([e *_1 d] \otimes [\delta(c)]) - \mu([\delta(c)] \otimes [e *_1 d]) \\
&\quad + \partial(d) *_3 (e *_3 c) - \mu([d] \otimes [\delta(e *_3 c)]) - \mu([\delta(e *_3 c)] \otimes [d]) \\
&= (e *_1 d\#_1 c) + (d\#_1 e *_3 c)
\end{aligned}$$

Moreover,

$$\begin{aligned}
e *_3 (c_1 \#_2 c_2) &= e *_3 ([c_2, c_1] - \mu([\delta(c_1)] \otimes [\delta(c_2)])) \\
&= e *_3 [c_2, c_1] - e *_3 \mu([\delta(c_1)] \otimes [\delta(c_2)]) \\
&= [e *_3 c_2, c_1] + [c_2, e *_3 c_1] - \mu([e *_1 \delta(c_1)] \otimes [\delta(c_2)]) - \mu([\delta(c_1)] \otimes [e *_1 \delta(c_2)]) \\
&= [c_2, e *_3 c_1] - \mu([\delta(e *_3 c_1)] \otimes [\delta(c_2)]) + [e *_3 c_2, c_1] - \mu([\delta(c_1)] \otimes [\delta(e *_3 c_2)]) \\
&= (e *_3 c_1 \#_2 c_2) + (c_1 \#_2 e *_3 c_2)
\end{aligned}$$

□

For any quasi-quadratic module $\mathcal{L} = \left(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-] \otimes [-])\right)$, we can define the quotient Lie algebra $C^{cr} = C/I$ and quotient homomorphism:

$$\begin{aligned}
\delta^{cr} : C^{cr} = C/I &\rightarrow D \\
(c + I) &\mapsto \delta^{cr}((c + I)) = \delta(c)
\end{aligned}$$

Lemma 3.4. Let $\mathcal{L} = \big(C \overset{\delta}{\to} D \overset{\partial}{\to} E, \mu([-]\otimes[-])\big)$ be quasi-quadratic module, then

1-) the induced map gives a quadratic module

$$\mathcal{L}^{cr} = \big(\, C^{cr} \xrightarrow{\delta^{cr}} D \xrightarrow{\partial} E\ , \mu^{cr}([-]\otimes[-])\big)$$

where the quadratic map $\mu^{cr}([-]\otimes[-])$ is the composition $q\mu = \mu^{cr}$,

$$B \otimes B \xrightarrow{\quad\mu\quad} C \xrightarrow{\quad q\quad} C^{cr}$$

for all $d_1, d_2 \in D$; $\mu^{cr}([d_1]\otimes[d_2]) = (\mu([d_1]\otimes[d_2]) + I)$. All these data are summarized diagrammatically as follows:

$$\begin{tikzcd}
 & B\otimes B \arrow[dl, "\mu"'] \arrow[d, "\Phi"] & \\
C \arrow[r, "\delta"'] \arrow[d, "q"'] & D \arrow[r, "\partial"'] & E \\
C^{cr} \arrow[ur, "\delta^{cr}"'] & &
\end{tikzcd}$$

2-) Let $\mathcal{L}' = \big(C' \overset{\delta'}{\to} D \overset{\partial}{\to} E, \mu([-]\otimes[-])\big)$ be an object in $\mathbf{QM}_L/(D \overset{\partial}{\to} E)$. If $\varphi : C \longrightarrow C'$ is a morphism of quasi-quadratic module over $\big(D \overset{\partial}{\to} E\big)$, then φ determines a unique Lie morphism $\varphi' : C^{cr} \longrightarrow C'$ such that $\varphi' q = \varphi$:

$$\begin{tikzcd}
C \arrow[d, "q"'] \arrow[dr, "\varphi"] & \\
C^{cr} \arrow[r, "\varphi'"'] & C'
\end{tikzcd}$$

Proof. 1-) For $d\#_1 c$ and $c_1\#_2 c_2$ elements in C, $\delta(d\#_1 c) = \delta(c_1\#_2 c_2) = 0$;

$$\begin{aligned}
\delta(d\#_1 c) &= \delta(\partial(d) *_3 c - \mu([d]\otimes[\delta(c)]) - \mu([\delta(c)]\otimes[d])) \\
&= \delta(\partial(d) *_3 c) - \delta(\mu([d]\otimes[\delta(c)])) - \delta(\mu([\delta(c)]\otimes[d])) \\
&= \partial(d) *_1 \delta(c) - \partial(d) *_1 \delta(c) + [d, \delta(c)] - \delta(d *_2 c) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\delta(c_1\#_2 c_2) &= \delta([c_2, c_1] - \mu([\delta(c_1)]\otimes[\delta(c_2)])) \\
&= \delta([c_2, x_1]) - \delta(\mu([\delta(c_1)]\otimes[\delta(c_2)])) \\
&= \delta([c_2, x_1]) - \delta(\delta(c_2) *_2 c_1) \\
&= \delta([c_2, x_1]) - \delta([c_2, x_1]) \\
&= 0
\end{aligned}$$

We have $\delta(I) = 0$, for all $d \in D$ and $c, c_1, c_2 \in C$.

Moreover, this construct needs to satisfy the $\mathbf{QM_L}$ axioms:

$\mathbf{QM_L 1}$ The axiom $\mathbf{QM_L 1}$ is provided directly.

$\mathbf{QM_L 2}$:

$$\begin{aligned}
\delta^{cr}\mu^{cr}([d_1]\otimes[d_2]) &= \delta^{cr}(\mu([d_1]\otimes[d_2]) + I) \\
&= \delta\mu([d_1]\otimes[d_2]) \\
&= \partial(d_1) *_1 d_2 - [d_1, d_2]
\end{aligned}$$

$\mathbf{QM_L 3}$:

$$\begin{aligned}
\mu^{cr}([\delta^{cr}(c+I)]\otimes[d] + [d]\otimes[\delta^{cr}(c+I)]) &= \mu^{cr}([\delta(c)]\otimes[d] + [d]\otimes[\delta(c)]) \\
&= (\mu([\delta(c)]\otimes[d] + [d]\otimes[\delta(c)]) + I) \\
&= \mu([\delta(c)]\otimes[d]) + \mu([d]\otimes[\delta(c)]) + I \\
&= \partial(y) *_3 c + I
\end{aligned}$$

$\mathbf{QM_L 4}$:

$$\begin{aligned}\mu^{cr}([\delta^{cr}(c_1+I)]\otimes[\delta^{cr}(c_2+I)]) &= \mu^{cr}([\delta(c_1)]\otimes[\delta(c_2)]) \\ &= (\mu([\delta(c_1)]\otimes[\delta(c_2)])+I) \\ &= [c_2,c_1]+I\end{aligned}$$

for each $d, d_1, d_2 \in D$, $c, c_1, c_2 \in C$.

2-) It is clear since all elements in the form $d\#_1 c$ and $c_1\#_2 c_2$ vanish.

$\square$

Thus, we can define a functor from the association of the quadratic module $\mathcal{L}^{cr} = \big(C^{cr} \xrightarrow{\delta^{cr}} D \xrightarrow{\partial} E, \mu^{cr}([-]\otimes[-])\big)$ to a quasi-quadratic module $\mathcal{L} = \big(C \xrightarrow{\delta} D \xrightarrow{\partial} E, \mu([-]\otimes[-])\big)$ as follows:

$$(-)^{cr} : \mathcal{QQM}_L \to \mathbf{QM_L}$$

Therefore, a morphism

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "f_2"'] & D \arrow[r, "\partial"] \arrow[d, "f_1"] & E \arrow[d, "f_0"] \\ C'' \arrow[r, "\delta''"'] & D'' \arrow[r, "\partial''"'] & E'' \end{tikzcd}$$

gives a (f^{cr}, f_1, f_0) of the associated quadratic modules. In more detail, $f^{cr} : C^{cr} \longrightarrow C''^{cr}$ is well defined by $f^{cr}(c+I) = f_2(c)+I''$, since $f_2(d\#_1 c) = f_1(d)\#_1 f_2(c)$ and $f_2(c_1\#_2 c_2) = f_2(c_1)\#_2 f_2(c_2)$). Thus, it satisfies the functorial rules. As a result, we get the following adjunction

$$\begin{tikzcd} \mathcal{QQM}_L \arrow[rr, bend left, "(-)^{cr}"] & & \mathbf{QM_L} \arrow[ll, hook, bend left] \end{tikzcd}$$

4. Conclusion

In this paper, the category of quasi-quadratic modules of Lie algebras has been introduced. It is concluded that the existence of an adjunction between this category and that of quadratic modules of Lie algebras is also valid as in the category of 2-crossed modules of groups and Lie algebras. Thus, it can be expected that quasi-quadratic modules are a useful tool in the construction of some categorical content, such as coproduct objects in the category of quadratic modules for Lie algebras. Furthermore, in future research, the answer to the question of which object corresponds to “quasi” in the category of crossed squares, which is another two-dimensional analogous of crossed modules, can be searched.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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