Communications in Advanced Mathematical Sciences
Vol. 5, No. 3, 150-160, 2022
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1177174

$\mathscr{I}_2$-Uniform Convergence of Double Sequences of Functions In 2-Normed Spaces

Sevim Yegül Güzey[1], Erdinç Dündar[2]*, Mukaddes Arslan[3]

Abstract
In this work, we discuss various types of $\mathscr{I}_2$-uniform convergence and equi-continuous for double sequences of functions. Also, we introduce the concepts of $\mathscr{I}_2$-uniform convergence, $\mathscr{I}_2^*$-uniform convergence, $\mathscr{I}_2$-uniformly Cauchy sequences and $\mathscr{I}_2^*$-uniformly Cauchy sequences for double sequences of functions in 2-normed spaces. Then, we show the relationships between these new concepts.

Keywords: Double sequence of functions, $\mathscr{I}$-Convergence, Uniformly convergence, 2-normed spaces.
2010 AMS: Primary 40A05, Secondary 40A30, 40A35, 46A70

[1] *Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey, ORCID: 0000-0001-8301-0252*
[2] *Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey, ORCID: 0000-0002-0545-7486*
[3] *Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey, ORCID: 0000-0002-5798-670X*
***Corresponding author**: carlos.raposo@ufba.br
Received: 19 September 2022, **Accepted:** 30 September 2022, **Available online:** 30 September 2022

1. Introduction

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [15] and Schoenberg [34].

The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [27] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of $\mathbb{N}$ [15, 16]. Das et al. [8] introduced the concept of $\mathscr{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Gökhan et al. [20] introduced the notions of pointwise and uniform statistical convergence of double sequences of real-valued functions. Gezer and Karakuş [19] investigated $\mathscr{I}$-pointwise and $\mathscr{I}$-uniform convergence and $\mathscr{I}^*$-pointwise and $\mathscr{I}^*$-uniform convergence of function sequences. Also, they examined the relationships between them. Baláz et al. [5] investigated $\mathscr{I}$-convergence and $\mathscr{I}$-continuity of real functions. Balcerzak et al. [6] studied statistical convergence and ideal convergence for sequences of functions. Dündar and Altay [10, 11] studied the concepts of $\mathscr{I}_2$-pointwise and $\mathscr{I}_2$-uniform convergence and $\mathscr{I}_2^*$- pointwise and $\mathscr{I}_2^*$-uniform convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [12] investigated some results of $\mathscr{I}_2$-convergence of double sequences of functions.

The concept of 2-normed spaces was initially introduced by Gähler [17, 18] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [24] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Şahiner et al. [36] and Gürdal [26] studied $\mathscr{I}$-convergence in 2-normed spaces. Gürdal and Açık [25] investigated $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [32] presented various kinds of statistical convergence and $\mathscr{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathscr{I}$-equistatistically convergence and study $\mathscr{I}$-equistatistically convergence of sequences of functions. Recently, Savaş and Gürdal [33] concerned with $\mathscr{I}$-convergence of sequences of functions in random 2-normed spaces and introduce the concepts of ideal uniform convergence and ideal pointwise convergence in the topology induced by random 2-normed spaces, and gave some basic properties of these concepts. Arslan and Dündar [1, 2] investigated the concepts of $\mathscr{I}$-convergence, $\mathscr{I}^*$-convergence, $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences of functions in 2-normed spaces and showed relationships between them. Yegül and Dündar [39] studied statistical convergence of sequence of functions in 2-normed spaces. Also, Dündar et al. [13] investigated $\mathscr{I}$-uniform convergence of sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [7, 28, 29, 30, 35, 37]).

2. Definitions and Notations

Now, we recall the concept of 2-normed space, ideal convergence and some fundamental definitions and notations (See [1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 19, 21, 22, 23, 24, 25, 26, 27, 31, 32, 36, 38]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x,y\| = \|y,x\|$.

(iii) $\|\alpha x,y\| = |\alpha|\|x,y\|$, $\alpha \in \mathbb{R}$.

(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

The pair $(X,\|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1|; \;\; x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

A sequence (x_n) in 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim\limits_{n\to\infty} \|x_n - L,y\| = 0$, for every $y \in X$. In such a case, we write $\lim\limits_{n\to\infty} x_n = L$ and call L the limit of (x_n).

Let $X \neq \emptyset$. A class $\mathscr{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathscr{I}$, (ii) $A,B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$, (iii) $A \in \mathscr{I}$, $B \subset A$ implies $B \in \mathscr{I}$.

$\mathscr{I}$ is called a nontrivial ideal if $X \notin \mathscr{I}$. A nontrivial ideal $\mathscr{I}$ in X is called admissible if $\{x\} \in \mathscr{I}$, for each $x \in X$.

Throughout the paper, we let $\mathscr{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.

Let $\mathscr{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then, $\mathscr{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathscr{I}_f$ convergence is the usual convergence.

Throughout the paper we take $\mathscr{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathscr{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

$\mathscr{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i,j \geq m(A) \Rightarrow (i,j) \notin A)\}$. Then $\mathscr{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathscr{I}_2$ is strongly admissible if and only if $\mathscr{I}_2^0 \subset \mathscr{I}_2$.

We say that an admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1,A_2,...\}$ belonging to $\mathscr{I}_2$, there exists a countable family of sets $\{B_1,B_2,...\}$ such that $A_j \Delta B_j \in \mathscr{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathscr{I}_2$ (hence $B_j \in \mathscr{I}_2$ for each $j \in \mathbb{N}$).

Let $X \neq \emptyset$. A non empty class $\mathscr{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathscr{F}$, (ii) $A,B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$, (iii) $A \in \mathscr{F}$, $A \subset B$ implies $B \in \mathscr{F}$.

If $\mathscr{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class

$$\mathscr{F}(\mathscr{I}) = \{M \subset X : (\exists A \in \mathscr{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathscr{I}$.

A sequence (x_n) in 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be $\mathscr{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ and each nonzero $z \in X$, $A(\varepsilon,z) = \{n \in \mathbb{N} : \|x_n - L,z\| \geq \varepsilon\} \in \mathscr{I}$. In this case, we write $\mathscr{I} - \lim\limits_{n\to\infty} \|x_n - L,z\| = 0$ or $\mathscr{I} - \lim\limits_{n\to\infty} \|x_n,z\| = \|L,z\|$.

A sequence (x_n) in 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be $\mathscr{I}^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathscr{F}$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $\lim\limits_{k\to\infty} \|x_{m_k} - L,z\| = 0$, for each nonzero $z \in X$.

Throughout the paper, we let X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f,g be two functions from X to Y.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is equi-continuous on X if

$$(\forall z\in X)\,(\forall\varepsilon>0)\,(\exists\delta>0)\,(\forall x,x_0\in X)\,\|x-x_0,z\|_X<\delta\Rightarrow\|f_n(x)-f_n(x_0)\|_\infty<\varepsilon.$$

The sequence $\{f_n\}$ is said to be $\mathscr{I}$-uniformly convergent to f (on X) if and only if

$$(\forall z\in Y)\,(\forall\varepsilon>0)\,(\exists M\in\mathscr{I})\,(\forall n\in\mathbb{N}\backslash M)\,(\forall x\in X)\,\|f_n(x)-f(x),z\|_Y\le\varepsilon.$$

We write $f_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}} f$.

The sequence of functions $\{f_n\}$ is said to be $\mathscr{I}^*$-uniformly convergent to f on X, if for every $\varepsilon>0$ there exists a set $K\in\mathscr{F}(\mathscr{I})$ $(\mathbb{N}\backslash K\in\mathscr{I})$ and $\exists n_0=n_{0(\varepsilon)}\in K$ such that for all $n\ge n_0$, $n\in K$ and for each nonzero $z\in Y$, $\|f_n(x)-f(x),z\|<\varepsilon$, for each $x\in X$ and in this case, we write $f_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}^*} f$.

$\{f_n\}$ is said to be $\mathscr{I}$-uniformly Cauchy if for every $\varepsilon>0$ there exists $s=s(\varepsilon)\in\mathbb{N}$ such that for each nonzero $z\in Y$,

$$\{n\in\mathbb{N}:\|f_n(x)-f_s(x),z\|\ge\varepsilon\}\in\mathscr{I},\ for\ each\ \ x\in X. \tag{2.1}$$

The sequence of functions $\{f_n\}$ is said to be $\mathscr{I}^*$-uniformly Cauchy sequence, if there exist a set $M\in\mathscr{F}(\mathscr{I}), M=\{m_1<m_2<...<m_k<...\}\subset\mathbb{N}$ such that for every $\varepsilon>0$ there is an $k_0=k_0(\varepsilon)$ such that for each nonzero $z\in Y$, $\|f_{m_k}(x)-f_{m_p}(x),z\|<\varepsilon$, for each $x\in X$ and for all $k,p>k_0$.

Throughout the paper, we let $\mathscr{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, X and Y be two 2-normed spaces, $\{f_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$, $\{g_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ and $\{h_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ be three double sequences of functions, f, g and k be three functions from X to Y.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be convergent (pointwise) to f if, for each point $x\in X$ and every $\varepsilon>0$, there exists a positive integer $k_0=k_0(x,\varepsilon)$ such that for all $m,n\ge k_0$ implies $\|f_{mn}(x)-f(x),z\|<\varepsilon$, for every $z\in Y$. In this case we write $f_{mn}\overset{\|.,.\|_Y}{\to} f$.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be uniformly convergent to f if, for every $\varepsilon>0$ there exists a positive integer $k_0=k_0(\varepsilon)$ such that for all $m,n\ge k_0$ implies $\|f_{mn}(x)-f(x),z\|<\varepsilon$, for all $x\in X$ and every $z\in Y$. In this case we write $f_{mn}\overset{\|.,.\|_Y}{\rightrightarrows} f$.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be $\mathscr{I}_2$-convergent (pointwise sense) to f if, for each $x\in X$ and every $\varepsilon>0$, $A(\varepsilon,z)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\ge\varepsilon\}\in\mathscr{I}_2$, for each nonzero $z\in Y$.

This can be expressed by the formula

$$(\forall z\in Y)\ (\forall x\in X)\ (\forall\varepsilon>0)\ (\exists H\in\mathscr{I}_2)\ (\forall(m,n)\notin H)\ \|f_{mn}(x)-f(x),z\|<\varepsilon.$$

In this case, we write $\mathscr{I}_2-\lim\limits_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|$ *or* $f_{mn}\overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}_2} f$.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X,\|.,.\|)$ is said to be $\mathscr{I}_2^*$-convergent (pointwise sense) to f, if there exists a set $M\in\mathscr{F}(\mathscr{I}_2)$ $(i.e., H=\mathbb{N}\times\mathbb{N}\setminus M\in\mathscr{I}_2)$ such that for each $x\in X$, each nonzero $z\in Y$ and all $(m,n)\in M$ $\lim\limits_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|$ and we write $\mathscr{I}_2^*-\lim\limits_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|$ or $f_{mn}\overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}_2^*} f$.

A double sequence of functions $\{f_{mn}\}$ is said to be $\mathscr{I}_2$-Cauchy sequence, if for every $\forall\varepsilon>0$ and each $x\in X$ there exist $s=s(\varepsilon,x)$, $t=t(\varepsilon,x)\in\mathbb{N}$ such that

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f_{st}(x),z\|\ge\varepsilon\}\in\mathscr{I}_2,$$

for each nonzero $z\in Y$.

A double sequence of functions $\{f_{mn}\}$ is said to be $\mathscr{I}_2^*$- Cauchy sequence, if there exists a set $M\in\mathscr{F}(\mathscr{I}_2)$ $(i.e., H=\mathbb{N}\times\mathbb{N}\setminus M\in\mathscr{I}_2)$ and for every $\varepsilon>0$ and each $x\in X$, $k_0=k_0(\varepsilon,x)\in\mathbb{N}$ such that for all $(m,n),(s,t)\in M$ and each $z\in Y$ $\|f_{mn}(x)-f_{st}(x),z\|<\varepsilon$, whenever $m,n,s,t>k_0$. In this case, we write $\lim\limits_{m,n,s,t\to\infty}\|f_{mn}(x)-f_{st}(x),z\|=0$.

Now we begin with quoting the lemmas due to Yegül and Dündar [40, 41, 42] which are needed throughout the paper.

Lemma 2.1 ([41]). *For each $x\in X$ and each nonzero $z\in Y$,*

$$\mathscr{I}_2^*-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|\ implies\ \mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|.$$

Lemma 2.2 ([41]). *Let $\mathscr{I} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal having the property (AP2). For each $x \in X$ and each nonzero $z \in Y$,*

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Lemma 2.3 ([42]). *If $\{f_{mn}\}$ is $\mathscr{I}_2$-convergent if and only if it is $\{f_{mn}\}$ is $\mathscr{I}_2$-Cauchy double sequence in 2-normed spaces.*

Lemma 2.4 ([40]). *Let D be a compact subset of X and f and f_{mn}, $(m,n = 1,2,\ldots)$, be continuous functions on D. Then, $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows} f$ on D if and only if $\lim_{m,n\to\infty} c_{mn} = 0$, where $c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\|$.*

3. Main Results

In this paper, we define concepts of $\mathscr{I}_2$-uniform convergence, $\mathscr{I}_2^*$-uniform convergence, $\mathscr{I}_2$-uniformly Cauchy and $\mathscr{I}_2^*$-uniformly Cauchy sequence of functions and investigate relationships between them and some properties such as continuity in 2-normed spaces.

Definition 3.1. *The double sequence $\{f_{mn}\}$ is said to be $\mathscr{I}_2$-uniformly convergent to f (on X) if for every $\varepsilon > 0$ and each nonzero $z \in Y$,*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \in \mathscr{I}_2, \text{ for each } x \in X.$$

This can be written by the formula

$$(\forall z \in Y)\,(\forall \varepsilon > 0)\,(\exists M \in \mathscr{I}_2)\,(\forall m,n \in \mathbb{N}\backslash M)\,(\forall x \in X)\,\|f_{mn}(x) - f(x), z\|_Y \leq \varepsilon.$$

We write $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$.

Theorem 3.2. *For each $x \in X$ and each nonzero $z \in Y$,*

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows} f \text{ implies } f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f.$$

Proof. Let $\varepsilon > 0$ be given. Since

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

for each $x \in X$ and each nonzero $z \in Y$, therefore, there exists a positive integer $k_0 = k_0(\varepsilon)$ such that $\|f_{mn}(x) - f(x), z\| < \varepsilon$, whenever $m,n \geq k_0$. This implies that for each nonzero $z \in Y$,

$$\begin{aligned} A(\varepsilon, z) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| < \varepsilon\} \\ &\subset ((\mathbb{N}\times\{1,2,..,(k_0-1)\}) \cup (\{1,2,..,(k_0-1)\}\times\mathbb{N})). \end{aligned}$$

Since $\mathscr{I}_2$ be a strongly admissible ideal, therefore

$$((\mathbb{N}\times\{1,2,..,(k_0-1)\}) \cup (\{1,2,..,(k_0-1)\}\times\mathbb{N})) \in \mathscr{I}_2.$$

Hence, it is clear that $A(\varepsilon, z) \in \mathscr{I}_2$ and consequently we have

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f.$$

□

Theorem 3.3. *Let D be a compact subset of X and f, $\{f_{mn}\}$, $m,n = 1,2,\ldots$ be continuous functions on D. Then,*

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$$

on D if and only if for each nonzero $z \in Y$,

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|c_{mn}(x), z\| = 0,$$

where

$$c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\|.$$

Proof. Assume that $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$ on D. Since f and $\{f_{mn}\}$ be continuous functions on D, so $(f_{mn}(x) - f(x))$ is continuous on D, for each $m,n \in \mathbb{N}$. By $\mathscr{I}_2-$uniform convergence, for every $\varepsilon > 0$ and each nonzero $z \in Y$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathscr{I}_2,$$

for each $x \in D$. Hence, for every $\varepsilon > 0$ and each nonzero $z \in Y$, it is clear that

$$c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2},$$

for each $x \in D$. Thus, we have

$$\mathscr{I}_2 - \lim_{m,n\to\infty} c_{mn} = 0.$$

Now, conversely, suppose that $\mathscr{I}_2 - \lim_{m,n\to\infty} c_{mn} = 0$. For every $\varepsilon > 0$ and each nonzero $z \in Y$, we let following sets

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}$$

and

$$B(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\},$$

for each $x \in D$. Since $\mathscr{I}_2 - \lim_{m,n\to\infty} c_{mn} = 0$, then $A(\varepsilon) \in \mathscr{I}_2$. Now, we let $(m,n) \in A^c(\varepsilon)$. Since for every $\varepsilon > 0$ and each nonzero $z \in Y$,

$$\|f_{mn}(x) - f(x), z\| \leq \max_{x\in D} \|f_{mn}(x) - f(x), z\| < \varepsilon,$$

for each $x \in D$, then $(m,n) \in B^c(\varepsilon)$ and so, $A^c(\varepsilon) \subset B^c(\varepsilon)$. Hence, we have $B(\varepsilon) \subset A(\varepsilon)$ and so, $B(\varepsilon) \in \mathscr{I}_2$. This proves the theorem. □

Definition 3.4. *The sequence of functions $\{f_{mn}\}$ is said to be $\mathscr{I}_2^*$-uniformly convergent to f on X, if for every $\varepsilon > 0$ there exists a set $K \in \mathscr{F}(\mathscr{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash K \in \mathscr{I}_2$) and $\exists n_0 = n_0(\varepsilon) \in K$ such that for all $m,n \geq n_0$, $(m,n) \in K$ and for each nonzero $z \in Y$,*

$$\|f_{mn}(x) - f(x), z\| < \varepsilon,$$

for each $x \in X$ and in this case, we write $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} f$.

Theorem 3.5. *Let $\{f_{mn}\}$ be a sequence of continuous functions and f be function from X to Y. If $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} f$, then f is continuous on X.*

Proof. Assume $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} f$ on X. Then, for every $\varepsilon > 0$, there exists a set $K \in \mathscr{F}(\mathscr{I}_2)$ $(i.e.,\ H = \mathbb{N}\times\mathbb{N}\setminus K \in \mathscr{I}_2)$ and $k_0 = k_0(\varepsilon), l_0 = l_0(\varepsilon) \in \mathbb{N}$ such that

$$\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{3},\ \ (m,n \in K)$$

for each nonzero $z \in Y$, each $x \in X$ and all $m > k_0, n > l_0$. Now, we let $x_0 \in X$ is arbitrary. Since $\{f_{k_0 l_0}\}$ is continuous at $x_0 \in X$, there is a $\delta > 0$ such that for each nonzero $z \in Y$,

$$\|x - x_0, z\| < \delta$$

implies

$$\|f_{k_0 l_0}(x) - f_{k_0 l_0}(x_0), z\| < \frac{\varepsilon}{3}.$$

Then, for all $x \in X$ for which $\|x - x_0, z\| < \delta$, we have

$$\begin{aligned} \|f(x) - f(x_0), z\| &\leq \|f(x) - f_{k_0 l_0}(x_0), z\| + \|f_{k_0 l_0}(x) - f_{k_0 l_0}(x_0), z\| \\ &+ \|f_{k_0 l_0}(x) - f(x_0), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

for each nonzero $z \in Y$. Since $x_0 \in X$ is arbitrary, f is continuous on X. □

Theorem 3.6. *Let $\mathscr{I} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property $(AP2)$, D be a compact subset of X and $\{f_{mn}\}$ be a sequence of continuous function on D. Assume that $\{f_{mn}\}$ be monotonic decreasing on D, i.e.,*

$$f_{(m+1),(n+1)}(x) \leq f_{mn}(x), (m,n = 1,2,...)$$

for every $x \in D$, f is continuous and for each nonzero $z \in Y$,

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

on D. Then,

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$$

on D.

Proof. Let

$$g_{mn} = f_{mn} - f \tag{3.1}$$

be a sequence of functions on D. Since $\{f_{mn}\}$ is continuous and monotonic decreasing and f is continuous on D, then $\{g_{mn}\}$ is continuous and monotonic decreasing on D. Since

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in D$ and nonzero $z \in Y$, then by (3.1),

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = 0$$

on D and since $\mathscr{I}_2$ satisfies the condition $(AP2)$ then, by Lemma 2.2, for each nonzero $z \in Y$, we have

$$\mathscr{I}_2^* - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = 0,$$

for each $x \in D$. Hence, for every $\varepsilon > 0$ and each $x \in D$ there exists $K_x \in \mathscr{F}(\mathscr{I}_2)$ such that

$$0 \leq g_n(x) < \frac{\varepsilon}{2}, \ ((m,n),(m(x) = m(x,\varepsilon), n(x) = n(x,\varepsilon)) \in K_x)$$

for $m \geq m(x)$ and $n \geq n(x)$, $(m,n) \in K_x$. Since $\{g_{mn}\}$ is continuous at $x \in D$, for every $\varepsilon > 0$ there exists an open set $A(x)$ which contains x such that for each nonzero $z \in Y$,

$$\|g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x), z\| \leq \frac{\varepsilon}{2},$$

for all $t \in A(x)$. Then, for every $\varepsilon > 0$, by monotonicity for each nonzero $z \in Y$, we have

$$\begin{aligned} 0 \leq g_{mn}(x) \leq g_{mn}(t) \leq g_{m(x)n(x)}(t) &= g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x) + g_{m(x)n(x)}(x) \\ &\leq \|g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x), z\| + g_{m(x)n(x)}(x) \end{aligned}$$

for every $t \in A(x)$ and for all $m \geq m(x)$, $n \geq n(x)$ and for each $x \in D$. Since $D \subset \bigcup_{x\in D} A(x)$ and D is a compact set, by the Heine-Borel theorem D has a finite open covering such that

$$D \subset A(x_1) \cup A(x_2) \cup A(x_3) ... \cup A(x_i).$$

Now, let

$$K = K_{x_1} \cap K_{x_2} \cap K_{x_3} \cap ... \cap K_{x_i}$$

and define

$$M = max\{m(x_1), m(x_2), m(x_3), ..., m(x_i)\},$$

$N = max\{n(x_1), n(x_2), n(x_3), ..., n(x_i)\}$.

Since for every K_{x_i} belong to $\mathscr{F}(\mathscr{I}_2)$, we have $K \in \mathscr{F}(\mathscr{I}_2)$. Then, when all $(m,n) \geq (M,N)$

$$0 \leq g_{mn}(t) < \varepsilon, \, (m,n) \in K,$$

for every $t \in A(x)$. So

$$g_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} 0,$$

on D. Since $\mathscr{I}$ is an admissible ideal

$$g_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} 0$$

on D and by (3.1) we have

$$f_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$$

on D. □

Definition 3.7. *The sequence $\{f_{mn}\}_{n\in\mathbb{N}}$ is equi-continuous on X if*

$$(\forall z \in X)\,(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x, x_0 \in X)\,\|x - x_0, z\|_X < \delta \Rightarrow \|f_{mn}(x) - f_{mn}(x_0), z\|_\infty < \varepsilon.$$

Theorem 3.8. *Let $\mathscr{I} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, X and Y be two 2-normed spaces with $dimY < \infty$. Assume that $f_{mn} \overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}_2} f$ on X, where $f_{mn} : X \to Y$, $m,n \in \mathbb{N}$ are equi-continuous on X and $f : X \to Y$, then f is continuous on X. If X is compact then, we have $f_n \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$ on X.*

Proof. First we will prove that f is continuous on X. Let $x_0 \in X$ and $\varepsilon > 0$. By the equi-continuity of f_{mn}'s there exists $\delta > 0$ and for each nonzero $z \in Y$ such that

$$\|f_{mn}(x) - f_{mn}(x_0), z\| < \frac{\varepsilon}{3}$$

for every $m,n \in \mathbb{N}$, $x \in B_\delta(x_0)$ ($B_\delta(x_0)$ stands for an open ball in X with center x_0 and radius δ.) Since $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$. The set

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x_0) - f(x_0), z\| \geq \frac{\varepsilon}{3}\right\} \cup \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{3}\right\}$$

is in $\mathscr{I}_2$ and is different from $\mathbb{N}\times\mathbb{N}$. Hence, for each nonzero $z \in Y$, there exists $(m,n) \in \mathbb{N}\times\mathbb{N}$ such that

$$\|f_{mn}(x_0) - f(x_0), z\| < \frac{\varepsilon}{3} \text{ and } \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{3}.$$

Thus, for each nonzero $z \in Y$ we have

$$\begin{aligned} \|f(x_0) - f(x), z\| &\leq \|f(x_0) - f_{mn}(x_0), z\| + \|f_{mn}(x_0) - f_{mn}(x), z\| + \|f_{mn}(x) - f(x), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

so f is continuous on X. We assume that X is compact. Let $\varepsilon > 0$. Since X is compact, it follows that f is uniformly continuous and f_{mn}'s are equi-uniformly continuous on X. So, pick $\delta > 0$ such that for any $x, x' \in X$ with

$$\|x - x', z\| < \delta,$$

then, by equi-uniformly and uniformly continuous for each nonzero $z \in Y$, we have

$$\|f_{mn}(x) - f_{mn}(x'), z\| < \frac{\varepsilon}{3} \text{ and } \|f(x) - f(x'), z\| < \frac{\varepsilon}{3}.$$

By the compactness of X, we can choose a finite subcover

$$B_{x_1}(\delta), B_{x_2}(\delta), B_{x_3}(\delta), ..., B_{x_k}(\delta)$$

from the cover $\{B_x(\delta)\}_{x\in X}$ of X. Using $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$ pick a set $M \in \mathscr{I}_2$ such that for each nonzero $z \in Y$,

$$\|f_{mn}(x_i) - f(x_i), z\| < \frac{\varepsilon}{3}, \ \ i \in \{1,2,...,k\},$$

for all $m,n \notin M$. Let $m,n \notin M$ and $x \in X$. Thus, $x \in B_{x_i}(\delta)$ for since $i \in \{1,2,...,k\}$. Hence, for each nonzero $z \in Y$ we have

$$\begin{aligned} \|f_{mn}(x) - f(x), z\| &\leq \|f_{mn}(x) - f_{mn}(x_i), z\| + \|f_{mn}(x_i) - f(x_i), z\| + \|f(x_i) - f(x), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

and so $f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$ on X. □

Definition 3.9. *$\{f_{mn}\}$ is said to be $\mathscr{I}_2$-uniformly Cauchy if for every $\varepsilon > 0$ there exists $s = s(\varepsilon) \in \mathbb{N}$, $t = t(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\} \in \mathscr{I}_2, \ for\ each\ \ x \in X. \tag{3.2}$$

Now, we give $\mathscr{I}_2$-Cauchy criteria for $\mathscr{I}_2$-uniformly convergence in 2-normed space.

Theorem 3.10. *Let $\mathscr{I}_2 \subset 2^{\mathbb{N}} \times \mathbb{N}$ be a strongly admissible ideal with the property $(AP2)$ and let $\{f_{mn}\}$ be a sequence of bounded function on X. Then, $\{f_{mn}\}$ is $\mathscr{I}_2$-uniformly convergent if and only if it is $\mathscr{I}_2$-uniformly Cauchy sequence on X.*

Proof. Assume that $\{f_{mn}\}$ $\mathscr{I}_2$-uniformly convergent to a function f defined on X. Let $\varepsilon > 0$. Then, for each nonzero $z \in Y$, we have

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}\right\} \notin \mathscr{I}_2$$

for each $x \in X$. We can select an $m(\varepsilon), n(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{m(\varepsilon)n(\varepsilon)}(x) - f(x), z\| < \frac{\varepsilon}{2}\right\} \notin \mathscr{I}_2,$$

for each $x \in X$. The triangle inequality yields that for each nonzero $z \in Y$

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{m(\varepsilon)n(\varepsilon)}(x), z\| < \varepsilon\} \notin \mathscr{I}_2,$$

for each $x \in X$. Since ε is arbitrary, $\{f_{mn}\}$ is $\mathscr{I}_2$-uniformly Cauchy on X.

Conversely, assume that $\{f_{mn}\}$ is $\mathscr{I}_2$-uniformly Cauchy on X. Let $x \in X$ be fixed. By (3.2) for every $\varepsilon > 0$ there is an $s = s(\varepsilon)$ and $t = t(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| < \varepsilon\} \notin \mathscr{I}_2.$$

Hence, $\{f_{mn}\}$ is $\mathscr{I}_2$-Cauchy, so by Lemma 2.3 we have that $\{f_{mn}\}$ is $\mathscr{I}_2$-convergent to f.Then, $f_{mn} \overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}_2} f$ on X.

Now we shall show that this convergence must be uniform. Note that since $\mathscr{I}_2$ satisfy the condition $(AP2)$, by (3.2) there is a $K \notin \mathscr{I}_2$ such that for each nonzero $z \in Y$,

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon, \ ((m,n),(s,t) \in K) \tag{3.3}$$

for all $m,n,s,t \geq N$ and $N = N(\varepsilon) \in \mathbb{N}$ and for each $x \in X$. By (3.3) for $s,t \to \infty$ and each nonzero $z \in Y$,

$$\|f_{mn}(x) - f(x), z\| < \varepsilon, \ \ ((m,n) \in K)$$

for all $n,m > N$ and each $x \in X$. This shows that

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2^*} f$$

on X. Since $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal we have

$$f_{mn} \overset{\|.,.\|_Y}{\rightrightarrows}_{\mathscr{I}_2} f$$

on X. □

Definition 3.11. *Let $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and $\{f_{mn}\}$ be a double sequence of function on X. $\{f_{mn}\}$ is said to be $\mathscr{I}_2^*$-uniformly Cauchy sequence, if there exist a set $K \in \mathscr{F}(\mathscr{I}_2), (i.e., H = \mathbb{N}\times\mathbb{N} \setminus K \in \mathscr{I}_2)$, for every $\varepsilon > 0$ and each $x \in X$, $k_0 = k_0(\varepsilon, x)$ such that for all $((m,n),(s,t)) \in K$ and each nonzero $z \in Y$,*

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

whenever $m,n,s,t, > k_0$. In this case, we write

$$\lim_{m,n,s,t\to\infty} \|f_{mn}(x) - f_{st}(x), z\| = 0.$$

Theorem 3.12. *If $\{f_{mn}\}$ is a $\mathscr{I}_2^*$-uniformly Cauchy sequence then it is $\mathscr{I}_2$-uniformly Cauchy sequence in 2-normed spaces.*

Proof. Let $\{f_{mn}\}$ be a $\mathscr{I}_2^*$-uniformly Cauchy sequence in 2-normed spaces then, by definition there exists the set $K \in \mathscr{F}(\mathscr{I}_2), (i.e., H = \mathbb{N}\times\mathbb{N} \setminus K \in \mathscr{I}_2)$ such that for every $\varepsilon > 0$ and for each nonzero $z \in Y$, $k_0 = k_0(\varepsilon)$ and $((m,n),(s,t)) \in K$

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

for each $x \in X$ and $m,n,s,t > k_0$. Let $N = N(\varepsilon, z)$. Then for $\varepsilon > 0$ and for each nonzero $z \in Y$, we have

$$\|f_{mn}(x) - f_N(x), z\| < \varepsilon,$$

for each $x \in X$ and $m,n > k_0$. Now put $H = \mathbb{N}\times\mathbb{N} \setminus K$. It is clear that $H \in \mathscr{I}_2$ and

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_{mn}(x) - f_N(x)\| \geq \varepsilon\} \subset H \cup K.$$

Since $\mathscr{I}_2$ is an admissible ideal then $H \cup K \in \mathscr{I}_2$. Hence, for every $\varepsilon > 0$ we find $N = N(\varepsilon, z)$ such that $A(\varepsilon, z) \in \mathscr{I}_2$, i.e., $\{f_{mn}\}$ is $\mathscr{I}_2$-uniformly Cauchy sequence. □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous referees for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] M. Arslan, E. Dündar, *$\mathscr{I}$-Convergence and $\mathscr{I}$-Cauchy Sequence of Functions In 2-Normed Spaces*, Konuralp Journal of Mathematics, **6**(1) (2018), 57–62.

[2] M. Arslan, E. Dündar, *On $\mathscr{I}$-Convergence of sequences of functions in 2-normed spaces*, Southeast Asian Bulletin of Mathematics, **42** (2018), 491–502.

[3] M. Arslan, E. Dündar, *Rough convergence in 2-normed spaces*, Bulletin of Mathematical Analysis and Applications, **10**(3) (2018), 1–9.

[4] M. Arslan, E. Dündar, On Rough Convergence in 2-Normed Spaces and Some Properties Filomat **33**(16) (2019), 5077–5086.

[5] V. Baláz, J. Červeňanský, P. Kostyrko, T. Šalát, *I-convergence and I-continuity of real functions*, Acta Mathematica, Faculty of Natural Sciences, Constantine the Philosopher University, Nitra, **5** (2004), 43–50.

[6] M. Balcerzak, K. Dems, A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328**(1) (2007), 715-729.

[7] H. Çakallı and S. Ersan, *New types of continuity in 2-normed spaces,* Filomat, **30**(3) (2016), 525–532.

[8] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, *$\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences*, Math. Slovaca, **58** (2008), No. 5, 605–620.

[9] P. Das, P. Malik, *On extremal $\mathscr{I}$-limit points of double sequences*, Tatra Mt. Math. Publ. **40** (2008), 91–102.

[10] E. Dündar, B. Altay, *$\mathscr{I}_2$-convergence of double sequences of functions*, Electronic Journal of Mathematical Analysis and Applications, **3**(1) (2015), 111–121.

[11] E. Dündar, B. Altay, *$\mathscr{I}_2$-uniform convergence of double sequences of functions*, Filomat, **30**(5) (2016), 1273–1281.

[12] E. Dündar, *On some results of $\mathscr{I}_2$-convergence of double sequences of functions*, Mathematical Analysis Sciences and Applications E-notes, **3**(1) (2015), 44–52.

[13] E. Dündar, M. Arslan, S. Yegül, *On $\mathscr{I}$-Uniform convergence of sequences of functions in 2-normed spaces*, Rocky Mountain Journal of Mathematics **50**(5) (2020), 1637–1646.

[14] E. Dündar, U. Ulusu, N. Pancaroğlu, *Strongly $\mathscr{I}_2$-lacunary convergence and $\mathscr{I}_2$-lacunary Cauchy double sequences of sets*, The Aligarh Bulletin of Mathematisc, **35**(1-2) (2016), 1-15.

[15] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[16] J.A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[17] S. Gähler, *2-metrische Räume und ihre topologische struktur*, Math. Nachr. **26** (1963), 115–148.

[18] S. Gähler, *2-normed spaces*, Math. Nachr. **28** (1964), 1–43.

[19] F. Gezer, S. Karakuş, *$\mathscr{I}$ and $\mathscr{I}^*$ convergent function sequences*, Math. Commun. **10** (2005), 71-80.

[20] A. Gökhan, M. Güngör and M. Et, *Statistical convergence of double sequences of real-valued functions*, Int. Math. Forum, **2(8)** (2007), 365-374.

[21] H. Gunawan, M. Mashadi, *On n-normed spaces*, Int. J. Math. Math. Sci. **27** (10) (2001), 631–639.

[22] H. Gunawan, M. Mashadi, *On finite dimensional 2-normed spaces*, Soochow J. Math. **27** (3) (2001), 321–329.

[23] M. Gürdal, S. Pehlivan, *The statistical convergence in 2-Banach spaces*, Thai J. Math. **2** (1) (2004), 107–113.

[24] M. Gürdal, S. Pehlivan, *Statistical convergence in 2-normed spaces*, Southeast Asian Bull. Math. **33** (2009), 257–264.

[25] M. Gürdal, I. Açık, *On $\mathscr{I}$-Cauchy sequences in 2-normed spaces*, Math. Inequal. Appl. **11**(2) (2008), 349–354.

[26] M. Gürdal, *On ideal convergent sequences in 2-normed spaces*, Thai J. Math. **4**(1) (2006), 85–91.

[27] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathscr{I}$-convergence*, Real Anal. Exchange **26** (2) (2000), 669–686.

[28] M. Mursaleen, S.A. Mohiuddine, *On ideal convergence in probabilistic normed spaces*, Math. Slovaca **62**(1) (2012), 49–62.

[29] M. Mursaleen, A. Alotaibi, *On $\mathscr{I}$-convergence in random 2-normed spaces*, Math. Slovaca **61**(6) (2011), 933–940.

[30] F. Nuray, W.H. Ruckle, *Generalized statistical convergence and convergence free spaces*, J. Math. Anal. Appl. **245** (2000), 513–527.

[31] F. Nuray, E. Dündar, U. Ulusu, *Wijsman $\mathscr{I}_2$-convergence of double sequences of cosed sets*, Pure and Applied Mathematics Letters, **2** (2014), 35-39.

[32] S. Sarabadan, S. Talebi, *Statistical convergence and ideal convergence of sequences of functions in 2-normed spaces*, Int. J. Math. Math. Sci. **2011** (2011), 10 pages. doi:10.1155/2011/517841.

[33] E. Savaş, M. Gürdal, *Ideal Convergent Function Sequences in Random 2-Normed Spaces,* Filomat, **30**(3) (2016), 557–567.

[34] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.

[35] A. Sharma, K. Kumar, *Statistical convergence in probabilistic 2-normed spaces,* Mathematical Sciences, **2**(4) (2008), 373–390.

[36] A. Şahiner, M. Gürdal, S. Saltan, H. Gunawan, *Ideal convergence in 2-normed spaces*, Taiwanese J. Math. **11**(5) (2007), 1477–1484.

[37] B.C. Tripathy, M. Sen, S. Nath, *$\mathscr{I}$-convergence in probabilistic n-normed space*, Soft Comput. **16**(6) (2012), 1021–1027.

[38] U. Ulusu, F. Nuray, E. Dündar, *$\mathscr{I}$-limit and $\mathscr{I}$-cluster points for functions defined on amenable semigroups*, Fundamental Journal of Mathematics and Applications, **4**(2) (2021), 45-48.

[39] S. Yegül, E. Dündar, *On Statistical convergence of sequences of functions in 2-normed spaces*, Journal of Classical Analysis, **10**(1) (2017), 49–57.

[40] S. Yegül, E. Dündar, *Statistical convergence of double sequences of functions and some properties in 2-normed spaces*, Facta Universitatis, Series Mathematics and Informatics, **33**(5) (2018), 705–719.

[41] S. Yegül, E. Dündar, *$\mathscr{I}_2$-convergence of double sequences of functions in 2-normed spaces*, Universal Journal of Mathematics and Applications **2**(3) (2019) 130–137.

[42] S. Yegül, E. Dündar, *On $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy double sequences of functions in 2-normed spaces*, Facta Universitatis Series Mathematics and Informatics **35**(3) (2020) 801–814.