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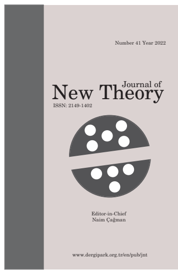

Soft A-Metric Spaces

Hande Poşul[1], Çiğdem Gündüz Aras[2], Servet Kütükcü[3]

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Abstract — This paper draws on the theory of soft A-metric space using soft points of soft sets and the concept of A-metric spaces. This new space has great importance as a new type of generalisation of metric spaces since it includes various known metric spaces. In this paper, we introduce the concept of soft A-metric space and examine the relations with known spaces. Then, we examine various basic properties of these spaces: soft Hausdorffness, a soft Cauchy sequence, and soft convergence.

Keywords — *Soft metric, A-metric, soft A-metric*

Mathematics Subject Classification (2020) — 54E45, 47H10

1. Introduction

Metric spaces have major importance in both mathematics and other sciences. The first study of metric spaces was initiated by Fréchet [1] at the beginning of the 20th century. Since that day, a great many generalisations of metric space have been obtained by different authors. Firstly, in 1963, 2-metric spaces were studied by Gahler [2]. In 1984, Dhage [3] introduced the notion of D-metric using basic modifications in the definition of 2-metric. After that, Mustafa and Sims [4] initiated the theory of G-metric since they found various mistakes in the definition of open sets in D-metric spaces. Later, because of the same reasons, Sedghi et al. [5] gave the theory of D^*-metric space. In 2012, Sedghi et al. [6] introduced the structure of S-metric spaces by modifying some conditions in the definition of D^*-metric spaces. Finally, Ahmed et al. [7] examined A-metric spaces as a general version of S-metric spaces.

Soft set theory was presented as a significant tool by Molodtsov [8] for dealing with uncertainties. Maji et al. [9] examined the primary properties of this space. Babitha and Sunil [10] investigated soft set relations and functions in this concept. Gündüz and Poşul [11] introduced the probabilistic soft sets. Many researchers applied this new concept to their studies [12–24].

The concept of soft metric space was studied by Das and Samanta [25] as a generalisation of metric spaces in 2013. This new metric caught the attention of authors, and many studies have been done on this topic [26–32].

In this study, we work on the notion of soft A-metric space. We design this theory using the soft points of soft sets and the concept of A-metric spaces. This study gives a new general form of metric spaces, and the resulting structure is a larger family from soft metric spaces. This paper is organised into 4 sections. In section 2, we recall some important definitions in soft set theory. In section 3, we

[1]handeposul@kilis.edu.tr (Corresponding Author); [2]caras@kocaeli.edu.tr; [3]skutukcu@omu.edu.tr
[1]Department of Mathematics, Faculty of Science, Kilis 7 Aralık University, Kilis, Türkiye
[2]Department of Mathematics, Faculty of Arts and Sciences, Kocaeli University, Kocaeli, Türkiye
[3]Department of Mathematics, Faculty of Science, Ondokuz Mayıs University, Samsun, Türkiye

introduce the concept of soft A-metric space as a new generalisation of metric spaces and examine the relations of soft metric spaces, soft S-metric spaces and soft A-metric spaces. After that, we present various important properties of this space: soft Hausdorffness, being a soft Cauchy sequence, soft convergence, and soft completeness. In section 4, we describe our results and point to the studies that can be done about this new theory.

2. Preliminaries

This section provides various basic definitions and properties before moving on to the main topic.

Definition 2.1. [8] Consider that X is an initial universe, E is the set of all the parameters, and $P(X)$ is the power set of X. Define a mapping $F{:}E \to P(X)$. Then, an ordered pair (F, E) is called a soft set over X. In that case, it can be thought that if (F, E) is a soft set over X, then it is a parameterized family of subsets of the set X.

From here, assume that X is an initial universe, E is the set of all the parameters, $P(X)$ is the power set of X, and (F, E) and (G, E) are soft sets over X.

Definition 2.2. [12] (F, E) is a soft subset of (G, E), if $F(a) \subseteq G(a)$, for every $a \in E$. This is written by $(F, E)\widetilde{\subseteq}(G, E)$. In addition, (G, E) is a soft superset of (F, E).

Definition 2.3. [12] (F, E) and (G, E) are soft equal, if $(F, E)\widetilde{\subseteq}(G, E)$ and $(G, E)\widetilde{\subseteq}(F, E)$.

Definition 2.4. [24] A soft set (H, E) is called the soft intersection of (F, E) and (G, E) over X, if $H(a) = F(a) \cap G(a)$, for every $a \in E$. This is written by $(H, E) = (F, E)\widetilde{\cap}(G, E)$.

Definition 2.5. [24] A soft set (U, E) is called the soft union of (F, E) and (G, E) over X, if $U(a) = F(a) \cup G(a)$, for every $a \in E$. This is written by $(U, E) = (F, E)\widetilde{\cup}(G, E)$.

Definition 2.6. [9] A soft set (F, E) is null soft set over X, if $F(a) = \varnothing$, for every $a \in E$. This is written by Φ.

Definition 2.7. [9] A soft set (F, E) is absolute soft set over X, if $F(a) = X$, for every $a \in E$. This is written by $\widetilde{X}$.

Definition 2.8. [24] A soft set (K, E) is called the soft difference of (F, E) and (G, E) over X, if $K(a) = F(a)\backslash G(a)$, for every $a \in E$. This is written by $(K, E) = (F, E)\widetilde{\backslash}(G, E)$.

Definition 2.9. [24] Consider that a mapping $F^c : E \to P(X)$ defined by $F^c(a) = X\backslash F(a)$, for every $a \in E$. Then, $(F, E)^c = (F^c, E)$ is called the soft complement of (F, E).

Definition 2.10. [15] Let $\widetilde{\tau}$ be the collection of soft sets over X. $\widetilde{\tau}$ is called a soft topology on X, if the followings hold:

i. Φ and $\widetilde{X}$ belong to $\widetilde{\tau}$.

ii. The intersection of any two soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

iii. The union of any number of soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The ordered triplet $(X, \widetilde{\tau}, E)$ is called a soft topological space over X.

Definition 2.11. [15] Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. Then, elements of $\widetilde{\tau}$ are called soft open sets in X. Moreover, (F, E) is a soft closed set in X, if $(F, E)^c$ belongs to $\widetilde{\tau}$.

Definition 2.12. [25] A soft set (F, E) is called a soft point, if $F(a) = \{x\}$ and $F(a') = \varnothing$, for the element $a \in E$ and for every $a' \in E\backslash\{a\}$. The soft point is written by (x_a, E) or x_a. Note that every soft set can be defined as a union of soft points.

From now on, the collection of all soft points of the absolute soft set will be denoted by $SP(\widetilde{X})$.

Definition 2.13. [25] Let x_a and $y_{a'}$ be soft points over X. It is said to be x_a and $y_{a'}$ are equal soft points, if $x = y$ and $a = a'$.

Definition 2.14. [25] Let x_a be a soft point over X. If $x_a(a)$ is an element of $F(a)$, i.e., $\{x\} \subseteq F(a)$, then x_a belongs to (F,E). This is written by $x_a \widetilde{\in} (F,E)$.

Proposition 2.15. [25] The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it.

Proposition 2.16. [25] Let x_a be a soft point over X. Then,

i. $x_a \widetilde{\in} (F,E) \Leftrightarrow x_a \widetilde{\notin} (F,E)^c$.

ii. $x_a \widetilde{\in} (F,E) \widetilde{\cup} (G,E) \Leftrightarrow x_a \widetilde{\in} (F,E)$ or $x_a \widetilde{\in} (G,E)$.

iii. $x_a \widetilde{\in} (F,E) \widetilde{\cap} (G,E) \Leftrightarrow x_a \widetilde{\in} (F,E)$ and $x_a \widetilde{\in} (G,E)$.

Remark 2.17. [25] The collection of all soft points of (F,E) will be expressed by $SP(F,E)$.

Definition 2.18. [25] Consider that $\mathbb{R}$ is the set of real numbers. In addition, the collection of all the non-empty bounded subset of $\mathbb{R}$ stands for $B(\mathbb{R})$. A soft real set is also denoted by (F,E), where F is a mapping from E to $B(\mathbb{R})$. If (F,E) has a only one element, then it is a soft real number and this is written by $\widetilde{r}, \widetilde{s}, \widetilde{p}$ etc. In this study, the soft real number $\widetilde{r}$ satisfies $\widetilde{r}(a) = r$, for all $a \in E$.

Definition 2.19. [25] Consider soft real numbers $\widetilde{r}$ and $\widetilde{s}$. Then, for all $a \in E$, the followings hold:

i. $\widetilde{r} \widetilde{\leq} \widetilde{s}$, if $\widetilde{r}(a) \leq \widetilde{s}(a)$.

ii. $\widetilde{r} \widetilde{\geq} \widetilde{s}$, if $\widetilde{r}(a) \geq \widetilde{s}(a)$.

iii. $\widetilde{r} \widetilde{<} \widetilde{s}$, if $\widetilde{r}(a) < \widetilde{s}(a)$.

iv. $\widetilde{r} \widetilde{>} \widetilde{s}$, if $\widetilde{r}(a) > \widetilde{s}(a)$.

Definition 2.20. [25] Let $\mathbb{R}(E)^*$ be the set of all the positive soft real numbers. A soft metric on $\widetilde{X}$ is a mapping $d : SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^*$ that satisfies the following conditions: for every soft points $x_a, y_b, z_c \in SP(\widetilde{X})$,

i. $d(x_a, y_b) \geq \widetilde{0}$.

ii. $d(x_a, y_b) = \widetilde{0}$ if and only if $x_a = y_b$.

iii. $d(x_a, y_b) = d(y_b, x_a)$.

iv. $d(x_a, z_c) \leq d(x_a, y_b) + d(y_b, z_c)$.

Then, the ordered triplet $\left(\widetilde{X}, d, E\right)$ is called a soft metric space.

Definition 2.21. [25] Let $\left(\widetilde{X}, d, E\right)$ be a soft metric space, $\left\{x_{a_k}^k\right\}$ be a soft sequence of soft points in $\left(\widetilde{X}, d, E\right)$ and y_b is a soft point over $\widetilde{X}$. Then,

i. $\left\{x_{a_k}^k\right\}$ is called a soft convergent sequence, if for $\widetilde{\varepsilon} > \widetilde{0}$, there exists a natural number k_0 such that $d\left(x_{a_k}^k, y_b\right) < \widetilde{\varepsilon}$, for each natural number $k \geq k_0$. Moreover, it is said that $\left\{x_{a_k}^k\right\}$ converges to y_b.

ii. $\left\{x_{a_k}^k\right\}$ is called a soft Cauchy sequence, if for $\widetilde{\varepsilon} > \widetilde{0}$, there exists a natural number k_0 such that $d\left(x_{a_k}^k, x_{a_m}^m\right) < \widetilde{\varepsilon}$, for each natural numbers $k, m \geq k_0$.

iii. If every soft Cauchy sequence is soft convergent in a soft metric space, then this space is called soft complete metric space.

Definition 2.22. [25] Let $\left(\widetilde{X}, d, E\right)$ be a soft metric space. For a soft real number $\widetilde{r} > \widetilde{0}$ and a soft point $x_a \in SP(\widetilde{X})$, the soft open ball $B\left(x_a, \widetilde{r}\right)$ and soft closed ball $\mathbf{B}\left(x_a, \widetilde{r}\right)$ with center x_a and a radius $\widetilde{r}$ are defined as follows:

$$\begin{aligned} B\left(x_a, \widetilde{r}\right) &= \left\{y_b \in SP(\widetilde{X}) : d(y_b, x_a) < \widetilde{r}\right\} \\ \mathbf{B}\left(x_a, \widetilde{r}\right) &= \left\{y_b \in SP(\widetilde{X}) : d\left(y_b, x_a\right) \leq \widetilde{r}\right\} \end{aligned}$$

Definition 2.23. [25] A soft metric space $\left(\widetilde{X}, d, E\right)$ is soft Hausdorff space, if for every different soft points x_a, y_b in $SP(\widetilde{X})$, there exist two soft open balls $B\left(x_a, \widetilde{r}\right)$ and $B\left(y_b, \widetilde{r}\right)$ such that their soft intersection is null soft set.

Definition 2.24. [32] A soft S-metric on $SP(\widetilde{X})$ is a mapping $S : \left(SP(\widetilde{X})\right)^3 \to [0, \infty)$ that satisfies the following conditions: for every soft points x_a, y_b, z_c, t_d in $SP(\widetilde{X})$,

i. $S\left(x_a, y_b, z_c\right) = \widetilde{0} \Leftrightarrow x_a = y_b = z_c$.

ii. $S\left(x_a, y_b, z_c\right) \leq S(x_a, x_a, t_d) + S\left(y_b, y_b, t_d\right) + S\left(z_c, z_c, t_d\right)$.

The ordered pair (X, S) is called a soft S-metric space.

Definition 2.25. [7] Let $X \neq \varnothing$ be a set and $n \geqslant 2$ be a natural number. A A-metric on X is a mapping $A : X^n \to [0, \infty)$ that satisfies the following conditions: for every $x_i \in X, i = 1, 2, ..., n$,

i. $A\left(x_1, x_2, \ldots, x_{n-1}, x_n\right) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_n$.

ii. $A\left(x_1, x_2, \ldots, x_{n-1}, x_n\right) \leq A\left(x_1, x_1, \ldots, x_1, a\right) + A\left(x_2, x_2, \ldots, x_2, a\right) + ... + A\left(x_n, x_n, \ldots, x_n, a\right).$

The ordered pair (X, A) is called a A-metric space.

3. Soft A-Metric Spaces

This section presents the theory of soft A-metric space, which uses soft points of soft sets and A-metric spaces. In this study, $\mathbb{R}(E)^*$ stands for the set of all the positive soft real numbers.

Definition 3.1. If a mapping which is defined from $\left(SP(\widetilde{X})\right)^n$ to $\mathbb{R}(E)^*$ satisfies the followings, then it is said to be a soft A-metric on $SP(\widetilde{X})$, where $n \geqslant 2$ is a natural number: for each soft points $x_{ia_i}, y_b \in SP(\widetilde{X}), i = 1, 2, ..., n$,

S1. $A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{n-1a_{n-1}}, x_{na_n}\right) = \widetilde{0} \Leftrightarrow x_{1a_1} = x_{2a_2} = \ldots = x_{na_n}$.

S2. $$\begin{aligned} A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{n-1a_{n-1}}, x_{na_n}\right) \leq\ & A\left(x_{1a_1}, x_{1a_1}, \ldots, x_{1a_1}, y_b\right) + A\left(x_{2a_2}, x_{2a_2}, \ldots, x_{2a_2}, y_b\right) \\ & + ... + A\left(x_{na_n}, x_{na_n}, \ldots, x_{na_n}, y_b\right). \end{aligned}$$

Then, the ordered triplet $(\widetilde{X}, A, E)$ is said to be a soft A-metric space.

Remark 3.2. Note that if $n = 3$ is taken in the definition of soft A-metric spaces, then the definition of the soft S-metric spaces is obtained. Similarly, if $n = 2$ is taken in the definition of soft A-metric spaces, then the definition of the soft metric spaces is obtained. Therefore, soft A-metric space is a general version of soft S-metric spaces and soft metric spaces. In other words,

i. For $n = 3$, every soft A-metric space is a soft S-metric space.

ii. For $n = 2$, every soft A-metric space is a soft metric space.

Example 3.3. Let $E \neq \varnothing$ be a set of parameters, $E \subset \mathbb{R}$ and d be an ordinary metric on a non-empty set $X \subset \mathbb{R}$. Then, $d_A(x_{ia_i}, y_{ib_i}) = |a_i - b_i| + d(x_i, y_i), i = 1, 2, ..., n$, is a soft metric [32]. Now, we define a mapping $A : \left(SP(\widetilde{X})\right)^n \to \mathbb{R}(E)^*$ as follow:

$$A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{n-1a_{n-1}}, x_{na_n}\right) = d_A\left(x_{1a_1}, x_{na_n}\right) + d_A\left(x_{2a_2}, x_{na_n}\right) + \ldots + d_A\left(x_{n-1a_{n-1}}, x_{na_n}\right)$$

for all $x_{ia_i} \in SP(\widetilde{X})$ and $i = 1, 2, ..., n$. Then, A is a soft A-metric on $SP(\widetilde{X})$. For this, let's show that the condition S2 is satisfied:

$$\begin{aligned}
A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{na_n}\right) &= d_A\left(x_{1a_1}, x_{na_n}\right) + d_A\left(x_{2a_2}, x_{na_n}\right) + \ldots + d_A(x_{n-1a_{n-1}}, x_{na_n}) \\
&= |a_1 - a_n| + |a_2 - a_n| + ... + |a_{n-1} - a_n| + d(x_1, x_n) + d(x_2, x_n) \\
&\quad + ... + d(x_{n-1}, x_n) \\
&\leq |a_1 - b| + |b - a_n| + |a_2 - b| + |b - a_n| + ... + |a_{n-1} - b| + |b - a_n| \\
&\quad + d(x_1, y) + d(y, x_n) + d(x_2, y) + d(y, x_n) + ... + d(x_{n-1}, y) + d(y, x_n) \\
&\leq |a_1 - b| + |a_1 - b| + ... + |a_1 - b| + d(x_1, y) + d(x_1, y) + ... + d(x_1, y) \\
&\quad + |a_2 - b| + |a_2 - b| + ... + |a_2 - b| + d(x_2, y) + d(x_2, y) + ... + d(x_2, y) \\
&\quad + ... + |a_n - b| + |a_n - b| + ... + |a_n - b| + d(x_n, y) + d(x_n, y) \\
&\quad + ... + d(x_n, y) \\
&= A\left(x_{1a_1}, x_{1a_1}, \ldots, x_{1a_1}, y_b\right) + A\left(x_{2a_2}, x_{2a_2}, \ldots, x_{2a_2}, y_b\right) \\
&\quad + ... + A\left(x_{na_n}, x_{na_n}, \ldots, x_{na_n}, y_b\right).
\end{aligned}$$

Remark 3.4. It is obvious that every one of soft A-metrics is a family of parametrized A-metric. Namely, if we consider a soft A-metric space $\left(\widetilde{X}, A, E\right)$, then (X, A_a) is an A-metric space, for every a in E. But it is not true converse of this statement. Here, A_a stands for the $A-$metric for only parameter a and (X, A_a) is a crisp $A-$metric space.

Example 3.5. Let $E = \mathbb{R}$ and $\left(X, \widetilde{A}\right)$ be an A-metric space. Define a mapping

$$A : \left(SP(\widetilde{X})\right)^n \to \mathbb{R}(E)^*$$

$$A\left(x_{1a_1}, x_{2a_2}, \ldots, x_{na_n}\right) = \widetilde{A}\left(x_1, x_2, \ldots, x_{n-1}, x_n\right)^{1+|a_1-a_2|+|a_1-a_3|+...+|a_1-a_n|}$$

for all $x_{ia_i} \in SP(\widetilde{X})$ and $i = 1, 2, ..., n$. Then, for every $a \in \mathbb{R}$, A_a is an A-metric on X, but A is not a soft A-metric on $SP(\widetilde{X})$.

Lemma 3.6. Let A be a soft A-metric on $SP(\widetilde{X})$. Then,

$$A\left(x_a, x_a, \ldots, x_a, y_b\right) = A\left(y_b, y_b, \ldots, y_b, x_a\right)$$

PROOF. Because of conditions S1 and S2 in the definition of soft A-metrics,

$$\begin{aligned}
A\left(x_a, x_a, \ldots, , x_a, y_b\right) &\leq (n-1)\, A\left(x_a, x_a, \ldots, x_a, x_a\right) + A\left(y_b, y_b, \ldots, y_b, x_a\right) \\
&= A\left(y_b, y_b, \ldots, y_b, x_a\right)
\end{aligned}$$

Thus,

$$A\left(x_a, x_a, \ldots, , x_a, y_b\right) \leq A\left(y_b, y_b, \ldots, y_b, x_a\right) \tag{1}$$

Similarly,

$$\begin{aligned}
A\left(y_b, y_b, \ldots, y_b, x_a\right) &\leq (n-1)\, A\left(y_b, y_b, \ldots y_b, y_b\right) + A\left(x_a, x_a, \ldots, x_a, y_b\right) \\
&= A\left(x_a, x_a, \ldots, x_a, y_b\right)
\end{aligned}$$

Therefore,

$$A\left(y_b, y_b, \ldots, y_b, x_a\right) \leq A\left(x_a, x_a, \ldots, x_a, y_b\right) \tag{2}$$

Hence, from inequality (1) and (2),

$$A\left(x_a, x_a, \ldots, x_a, y_b\right) = A\left(y_b, y_b, \ldots, y_b, x_a\right)$$

□

Definition 3.7. Let A be a soft A-metric on $SP(\widetilde{X})$. The soft open ball $B_A\left(x_a, \widetilde{r}\right)$ is defined as follows:

$$B_A\left(x_a, \widetilde{r}\right) = \left\{y_b \in SP(\widetilde{X}) : A\left(y_b, y_b, \ldots, y_b, x_a\right) < \widetilde{r}\right\}$$

where $x_a \in SP(\widetilde{X})$ is the center of the soft open ball and the non-negative soft real number $\widetilde{r}$ is the radius of the soft open ball. Moreover,

$$\mathbf{B}_A\left(x_a, \widetilde{r}\right) = \left\{y_b \in SP(\widetilde{X}) : A\left(y_b, y_b, \ldots, y_b, x_a\right) \leq \widetilde{r}\right\}$$

is the soft closed ball with the center x_a and the radius $\widetilde{r}$.

Example 3.8. Let $n = 5$ in the definition of soft A-metric spaces, $E = \mathbb{Z}$, and $X = \mathbb{R}^n$. Denote

$$\begin{aligned} A\left(x_{1a_1}, x_{2a_2}, x_{3a_3}, x_{4a_4}, x_{5a_5}\right) &= \left|a_1 - a_5\right| + \left|a_2 - a_5\right| + \left|a_3 - a_5\right| + \left|a_4 - a_5\right| + d\left(x_1, x_5\right) \\ &\quad + d\left(x_2, x_5\right) + d\left(x_3, x_5\right) + d\left(x_4, x_5\right) \end{aligned}$$

for all $x_{ia_i} \in SP(\widetilde{X}), i = 1, 2, ..., 5$. Then, for $\theta = (0, 0, \ldots, 0) \in \mathbb{R}^5$,

$$\begin{aligned} B_A\left(\theta_0, \widetilde{9}\right) &= \left\{y_b \in SP(\widetilde{X}) : A\left(y_b, y_b, y_b, y_b, \theta_0\right) < \widetilde{9}\right\} \\ &= \left\{y_b \in SP(\widetilde{X}) : 4\left|b\right| + 4d\left(y, \theta\right) < \widetilde{9}\right\} \\ &= \left\{y_b \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{9}}{4} - \left|b\right|\right\} \\ &= \left\{y_b \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{9} - 4\left|b\right|}{4}\right\} \\ &= \left\{y_0 \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{9}}{4}\right\} \cup \left\{y_1 \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{5}}{4}\right\} \\ &\quad \cup \left\{y_2 \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{1}}{4}\right\} \cup \left\{y_{-1} \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{5}}{4}\right\} \\ &\quad \cup \left\{y_{-2} \in SP(\widetilde{X}) : d\left(y, \theta\right) < \frac{\widetilde{1}}{4}\right\} \end{aligned}$$

Definition 3.9. Let $(\widetilde{X}, A, E)$ be a soft A-metric space and (F, E) be a soft set on X. If, for all $x_a \in (F, E)$, there exists a $\widetilde{r} > \widetilde{0}$ such that $B_A\left(x_a, \widetilde{r}\right) \subset SP\left(F, E\right)$, then (F, E) is said to be a soft open set in $\left(\widetilde{X}, A, E\right)$.

Proposition 3.10. The soft open ball $B_A\left(x_a, \widetilde{r}\right)$ is a soft open set in a soft A-metric space $\left(\widetilde{X}, A, E\right)$.

PROOF. Let $y_b \in B_A(x_a, \widetilde{r})$. Then, $A(y_b, y_b, \ldots, y_b, x_a) < \widetilde{r}$. Let $\widetilde{d} = A(x_a, x_a, \ldots, x_a, y_b)$ and $\widetilde{r}'(e) = \frac{\widetilde{r}(e) - \widetilde{d}}{n-1}$, for all $e \in E$. We claim that $B_A(y_b, \widetilde{r}') \subset B_A(x_a, \widetilde{r})$. For this, let $z_c \in B_A(y_b, \widetilde{r}')$. Then, $A(z_c, z_c, \ldots, z_c, y_b) < \widetilde{r}'$. Owing to the condition S2 in the definition of soft A-metrics,

$$\begin{aligned} A(z_c, z_c, \ldots, z_c, x_a) &\leq A(z_c, z_c, \ldots, z_c, y_b) + A(z_c, z_c, \ldots, z_c, y_b) \\ &\quad + \ldots + A(z_c, z_c, \ldots, z_c, y_b) + A(x_a, x_a, \ldots, x_a, y_b) \\ &= (n-1) A(z_c, z_c, \ldots, z_c, y_b) + A(x_a, x_a, \ldots, x_a, y_b) \\ &< (n-1) \widetilde{r}' + \widetilde{d} \\ &= \widetilde{r} \end{aligned}$$

Then, $z_c \in B_A(x_a, \widetilde{r})$ and so, $B_A(y_b, \widetilde{r}') \subset B_A(x_a, \widetilde{r})$. □

Theorem 3.11. Every soft A-metric space produces a soft topology as follows:

$$\tau = \left\{ (F, E) : \text{ For every } x_a \in SP(\widetilde{X}), \text{ there exists a } \widetilde{r} > \widetilde{0} \text{ such that } B_A(x_a, \widetilde{r}) \subset SP(F, E) \right\}$$

This topology is said to be soft topology produced by soft A-metric.

PROOF. Firstly, we will show that the intersection of two open soft sets is also a soft open set. Let us consider the soft open sets (F, E) and (G, E). Let $x_a \in (F, E) \widetilde{\cap} (G, E)$. Then, since $x_a \in (F, E)$ and $x_a \in (G, E)$, there exists a $\widetilde{r_1} > \widetilde{0}$ such that $B_A(x_a, \widetilde{r_1}) \subset SP(F, E)$ and there exists a $\widetilde{r_2} > \widetilde{0}$ such that $B_A(x_a, \widetilde{r_2}) \subset SP(G, E)$. Take $\widetilde{r}(e) = \min\{\widetilde{r_1}(e), \widetilde{r_2}(e)\}$, for all $e \in E$. Hence, $B_A(x_a, \widetilde{r}) \subset B_A(x_a, \widetilde{r_1})$ and $B_A(x_a, \widetilde{r}) \subset B_A(x_a, \widetilde{r_2})$. Then, we have

$$x_a \in B_A(x_a, \widetilde{r}) \subset B_A(x_a, \widetilde{r_1}) \cap B_A(x_a, \widetilde{r_2}) \subset SP(F, E) \cap SP(G, E)$$

Thus, $(F, E) \widetilde{\cap} (G, E)$ is a soft open set. Secondly, we will show that the arbitrary union of soft open sets is also a soft open set. Let (F_λ, E) be a soft open set, for all λ in I, an index set. Let $x_a \in \widetilde{\bigcup_\lambda} (F_\lambda, E)$. Then, $x_a \in (F_{\lambda_0}, E)$, for a λ_0 in I. Since (F_{λ_0}, E) is a soft open set, there exists a $\widetilde{r} > \widetilde{0}$ such that $B_A(x_a, \widetilde{r}) \subset SP(F_{\lambda_0}, E)$. Then, we have

$$x_a \in B_A(x_a, \widetilde{r}) \subset SP(F_{\lambda_0}, E) \subset \bigcup_\lambda SP(F_\lambda, E)$$

Hence, $\widetilde{\bigcup_\lambda} (F_\lambda, E)$ is a soft open set. In addition, obviously, Φ and $\widetilde{X}$ are soft open sets. Therefore, τ is a soft topology. □

Theorem 3.12. Every soft A-metric space is a soft Hausdorff space. Namely, for every different soft points $x_a,\ y_b \in SP(\widetilde{X})$, there exist two soft open balls such that their soft intersection is null soft set.

PROOF. Let $x_a,\ y_b \in SP(\widetilde{X})$ and $x_a \neq y_b$. Then, $A(x_a, x_a, \ldots, x_a, y_b) > \widetilde{0}$. For a soft real number $\widetilde{r},\ \widetilde{0} < \widetilde{r} < \widetilde{1},\ A(x_a, x_a, \ldots, x_a, y_b) = \widetilde{r}$. Now, consider the soft open balls $B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right)$ and $B_A\left(y_b, \frac{\widetilde{r}}{2}\right)$. We claim that $B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right) \cap B_A\left(y_b, \frac{\widetilde{r}}{2}\right)$ is null soft set. For this, we suppose that $B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right) \cap B_A\left(y_b, \frac{\widetilde{r}}{2}\right) \neq \emptyset$. Then, there exists a $z_c \in SP(\widetilde{X})$ such that $z_c \in B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right) \cap B_A\left(y_b, \frac{\widetilde{r}}{2}\right)$. Since $z_c \in B_A\left(x_a, \frac{\widetilde{r}}{2(n-1)}\right)$ and $z_c \in B_A\left(y_b, \frac{\widetilde{r}}{2}\right)$, then $A(z_c, z_c, \ldots, z_c, x_a) < \frac{\widetilde{r}}{2(n-1)}$ and $A(z_c, z_c, \ldots, z_c, y_b) < \frac{\widetilde{r}}{2}$, respectively. Because of the condition S2 of the definition of soft A-metrics,

$$\begin{aligned} A(x_a, x_a, \ldots, x_a, y_b) &\leq A(x_a, x_a, \ldots, x_a, z_c) + A(x_a, x_a, \ldots, x_a, z_c) + \\ &\quad + \ldots + A(x_a, x_a, \ldots, x_a, z_c) + A(y_b, y_b, \ldots, y_b, z_c) \\ &= (n-1) A(x_a, x_a, \ldots, x_a, z_c) + A(y_b, y_b, \ldots, y_b, z_c) \\ &< (n-1) \frac{\widetilde{r}}{2(n-1)} + \frac{\widetilde{r}}{2} \\ &= \widetilde{r} \end{aligned}$$

Since this is a contradiction, the claim is true. Then, soft A-metric spaces are soft Hausdorff spaces. □

Definition 3.13. Let $\left(\widetilde{X}, A, E\right)$ be a soft A-metric space, $\{x_{a_k}^k\}$ be a soft sequence of soft points in $\left(\widetilde{X}, A, E\right)$, and y_b is a soft point of over $\widetilde{X}$. Then,

i. $\{x_{a_k}^k\}$ is called a soft convergent sequence, if for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a natural number k_0 such that $A\left(x_{a_k}^k, x_{a_k}^k, \dots, x_{a_k}^k, y_b\right) < \widetilde{\varepsilon}$, for each natural number $k \geq k_0$. This is denoted by $\lim\limits_{k\to\infty} x_{a_k}^k = y_b$. Moreover, it is said that $\{x_{a_k}^k\}$ converges to y_b.

ii. $\{x_{a_k}^k\}$ is called a soft Cauchy sequence, if for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a natural number k_0 such that $A\left(x_{a_k}^k, x_{a_k}^k, \dots, x_{a_k}^k, x_{a_m}^m\right) < \widetilde{\varepsilon}$, for each natural numbers $k, m \geq k_0$.

iii. If every soft Cauchy sequence is soft convergent in a soft A-metric space, then this space is said to be soft complete A-metric space.

Lemma 3.14. Let $\left(\widetilde{X}, A, E\right)$ be a soft A-metric space. Every soft convergent sequence in this space converges a unique soft point.

Proof. Let $\{x_{a_k}^k\}$ be a soft sequence of soft points in $\left(\widetilde{X}, A, E\right)$ and it soft converges to both y_b and z_c. Then, for each $\widetilde{\varepsilon} > \widetilde{0}$, there exist $k_1, k_2 \in \mathbb{N}$ such that

$$A\left(x_{a_k}^k, x_{a_k}^k, \dots, x_{a_k}^k, y_b\right) < \frac{\widetilde{\varepsilon}}{2(n-1)}$$

for each natural number $k \geq k_1$, and

$$A\left(x_{a_k}^k, x_{a_k}^k, \dots, x_{a_k}^k, z_c\right) < \frac{\widetilde{\varepsilon}}{2}$$

for each natural number $k \geq k_2$. We take $k_0 = \max\{k_1, k_2\}$. Then, for each natural number $k \geq k_0$, from Lemma 3.6 and the condition S2 in the definition of soft A-metric spaces,

$$\begin{aligned} A(y_b, y_b, \dots, y_b, z_c) &\leq (n-1) A\left(y_b, y_b, \dots, y_b, x_{a_k}^k\right) + A\left(z_c, z_c, \dots, z_c, x_{a_k}^k\right) \\ &= (n-1) A\left(x_{a_k}^k, x_{a_k}^k, \dots, x_{a_k}^k, y_b\right) + A\left(x_{a_k}^k, x_{a_k}^k, \dots, x_{a_k}^k, z_c\right) \\ &< (n-1) \frac{\widetilde{\varepsilon}}{2(n-1)} + \frac{\widetilde{\varepsilon}}{2} \\ &= \widetilde{\varepsilon} \end{aligned}$$

Thus, we get $A(y_b, y_b, \dots, y_b, z_c) = \widetilde{0}$ and this means that $y_b = z_c$. □

Lemma 3.15. Let $\left(\widetilde{X}, A, E\right)$ be a soft A-metric space. In this space, every soft convergent sequence is a soft Cauchy sequence.

Proof. A soft sequence $\{x_{a_k}^k\}$ of soft points in $\left(\widetilde{X}, A, E\right)$ soft converges to y_b. Then, for each $\widetilde{\varepsilon} > \widetilde{0}$, there exist $k_1, k_2 \in \mathbb{N}$ such that

$$A\left(x_{a_k}^k, x_{a_k}^k, \dots, x_{a_k}^k, y_b\right) < \frac{\widetilde{\varepsilon}}{2(n-1)}$$

for each natural number $k \geq k_1$, and

$$A\left(x_{a_m}^m, x_{a_m}^m, \dots, x_{a_m}^m, y_b\right) < \frac{\widetilde{\varepsilon}}{2}$$

for each natural number $m \geq k_2$. We take $k_0 = \max\{k_1, k_2\}$. Then, for each natural numbers $k, m \geq k_0$, from the condition S2 in the definition of soft A-metric spaces,

$$\begin{aligned} A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_{a_m}^m\right) &\leq (n-1) A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_b\right) + A\left(x_{a_m}^m, x_{a_m}^m, \ldots, x_{a_m}^m, y_b\right) \\ &< (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + \frac{\widetilde{\varepsilon}}{2} = \widetilde{\varepsilon} \end{aligned}$$

Therefore, $\{x_{a_k}^k\}$ is a soft Cauchy sequence. □

Lemma 3.16. Let $\left(\widetilde{X}, A, E\right)$ be a soft A-metric space and $\{x_{a_k}^k\}$ and $\left\{y_{b_k}^k\right\}$ be soft sequences of soft points in this space. If $\{x_{a_k}^k\}$ converges to x_a, and $\left\{y_{b_k}^k\right\}$ converges to y_b, then

$$\lim_{k\to\infty} A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) = A\left(x_a, x_a, \ldots, x_a, y_b\right)$$

Proof. Since $\lim_{k\to\infty} x_{a_k}^k = x_a$, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a $k_1 \in \mathbb{N}$ such that

$$A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_a\right) < \frac{\widetilde{\varepsilon}}{2(n-1)}$$

for each natural number $k \geq k_1$. Similarly, since $\lim_{k\to\infty} y_{b_k}^k = y_b$, for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a $k_2 \in \mathbb{N}$ such that

$$A\left(y_{b_k}^k, y_{b_k}^k, \ldots, y_{b_k}^k, y_b\right) < \frac{\widetilde{\varepsilon}}{2(n-1)}$$

for each natural number $k \geq k_2$. If we take $k_0 = \max\{k_1, k_2\}$, then for every natural number $k \geq k_0$, from the condition S2 in the definition of soft A-metric spaces,

$$\begin{aligned} A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) &\leq (n-1) A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_a\right) + A\left(y_{b_k}^k, y_{b_k}^k, \ldots, y_{b_k}^k, x_a\right) \\ &\leq (n-1) A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, x_a\right) + (n-1) A\left(y_{b_k}^k, y_{b_k}^k, \ldots, y_{b_k}^k, y_b\right) \\ &\quad + A\left(x_a, x_a, \ldots, x_a, y_b\right) \\ &< (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + A\left(x_a, x_a, \ldots, x_a, y_b\right) \end{aligned}$$

Thus,

$$A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) - A\left(x_a, x_a, \ldots, x_a, y_b\right) < \widetilde{\varepsilon} \tag{3}$$

Similarly, from Lemma 3.6 and the condition S2 in the definition of soft A-metric spaces,

$$\begin{aligned} A\left(x_a, x_a, \ldots, x_a, y_b\right) &\leq (n-1) A\left(x_a, x_a, \ldots, x_a, x_{a_k}^k\right) + A\left(y_b, y_b, \ldots, y_b, x_{a_k}^k\right) \\ &\leq (n-1) A\left(x_a, x_a, \ldots, x_a, x_{a_k}^k\right) + (n-1) A\left(y_b, y_b, \ldots, y_b, y_{b_k}^k\right) \\ &\quad + A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) \\ &= (n-1) A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k x_a\right) + (n-1) A\left(y_{b_k}^k, y_{b_k}^k, \ldots, y_{b_k}^k, y_b\right) \\ &\quad + A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) \\ &< (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + (n-1)\frac{\widetilde{\varepsilon}}{2(n-1)} + A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) \end{aligned}$$

Hence,

$$A\left(x_a, x_a, \ldots, x_a, y_b\right) - A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) < \widetilde{\varepsilon} \tag{4}$$

Hence, from inequalities (3) and (4),

$$\left|A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) - A\left(x_a, x_a, \ldots, x_a, y_b\right)\right| < \widetilde{\varepsilon}$$

Therefore, $\lim_{k\to\infty} A\left(x_{a_k}^k, x_{a_k}^k, \ldots, x_{a_k}^k, y_{b_k}^k\right) = A\left(x_a, x_a, \ldots, x_a, y_b\right).$ □

4. Conclusion

This study looked into soft A-metric space which is built by soft points of soft sets and A-metric spaces. Soft A-metric space is the general form of soft S-metric spaces, and it is valuable in this respect. Moreover, it is a generalisation of soft metric spaces. Therefore, soft A-metric spaces are a larger family of soft metric spaces. Many studies can be done on soft A-metric spaces, and important results can be obtained. Especially various well-known fixed point studies and fixed circle studies in this concept will contribute to science. In all these respects, this study presents a new line of vision to generalised metric spaces.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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