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Image Curves on the Parallel-Like Surfaces in E^3

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In this paper, it's introduced the curves lying on parallel-like surface M^f of a surface M in Euclidean space. Taking into account the definition of the parallel-like surface it's obtained parametric expression of these curves and examinated Darboux frame for these curves which we call image curves. And finally the curves lying on the surfaces M and M^f are compared by considering their geodesic and normal curvatures, the geodesic torsion.

Keywords: *Darboux frame, parallel-like surfaces.*
AMS Subject Classification (2020): *53A04; 53A05.*

1. Introduction

Parallel-like surfaces were first described by Tarakcı and Hacısalihoğlu in 2002 and named with surfaces at a constant distance from the edge of regression on a surface [11]. The authors have obtained by considering a surface instead of a curve in the paper written by Hans Vogler in 1963. Hans Vogler have defined notion of curve at a constant distance from the edge of regression on a curve. In 2004, Tarakcı and Hacısalihoğlu have computed for parallel-like surfaces some properties and theorems given for parallel surfaces [12]. After this work, it's made many articles by different authors on parallel-like surfaces. In 2010, Sağlam and Kalkan have searched parallel-like surfaces in E_1^3 Minkowski space [10]. In 2015, Yurttançıkmaz and Tarakcı have established a relationship between focal surfaces and parallel-like surfaces. They were able to find the focal surfaces of a given surface by means of parallel-like surfaces [13]. In 2016, Çakmak and Tarakcı have examined parallel-like surfaces of surface of revolution and they have worked curves on these surfaces in another paper [1, 2]. In this study, the Darboux frame of curves lying on parallel-like surfaces in E^3 will be investigated.

The differential geometry of curves and surfaces has attracted the attention of geometers from past to present. Therefore, many studies have been done on this subject [3, 6]. The differential geometry of the curves lying on the surfaces, on the other hand, is important since the properties of the surface in question must also be taken into account when examining the differential geometric properties of the curve [4, 5, 7, 8]. In the theory of surfaces, the Darboux frame constructed at any non-umbilical point of the surface can be viewed as an analog of the Frenet frame. In this paper first, image curve on parallel-like surface of a surface M which denoted by M^f has been found and then, calculating Darboux frame for this image curve it has been compared the geodesic curvatures, the normal curvatures, the geodesic torsions of reference curve on M and its image curve on M^f and expressed the relationships between these two curves [14].

2. Preliminaries

Let α be a unit speed curve lying on the surface M in E^3 and s be arc length of the curve α, i.e. $\left\|\alpha^{'}(s)\right\| = 1$. Suppose that Z is a unit normal vector of the surface M and T is unit tangent vector field of the curve α.

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Considering the vector field Y defined by $Y = Z \times T$, set of $\{T, Y, Z\}$ create orthonormal frame which is called Darboux frame for partner of curve-surface (α, M).

Thus, the geodesic curvature κ_g, the normal curvature κ_n, the geodesic torsion t_r of the curve $\alpha(s)$ can be calculated as follows [9]

$$\kappa_g = \left\langle \alpha^{''}(s), Y \right\rangle \tag{2.1}$$

$$\kappa_n = \left\langle \alpha^{''}(s), Z_{\alpha(s)} \right\rangle \tag{2.2}$$

$$t_r = -\left\langle Z^{'}_{\alpha(s)}, Y \right\rangle. \tag{2.3}$$

Besides, the derivative formulas of the Darboux frame of (α, M) is given by

$$\begin{bmatrix} T^{'} \\ Y^{'} \\ Z^{'} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & t_r \\ -\kappa_n & -t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}. \tag{2.4}$$

In addition, given a arbitrary curve $\beta(s)$ on the surface M under the condition $\left\| \beta^{'}(s) \right\| = c$, the geodesic curvature κ_g, the normal curvature κ_n, the geodesic torsion t_r of the curve $\beta(s)$ can be calculated as follows

$$\kappa_g = \frac{1}{c^2} \left\langle \beta^{''}(s), Y \right\rangle \tag{2.5}$$

$$\kappa_n = \frac{1}{c^2} \left\langle \beta^{''}(s), Z_{\beta(s)} \right\rangle \tag{2.6}$$

$$t_r = -\frac{1}{c} \left\langle Z^{'}_{\beta(s)}, Y \right\rangle. \tag{2.7}$$

Furthermore, in the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M the followings are well-known

i) $\alpha(s)$ is a principal line $\Longleftrightarrow t_r = 0$,
ii) $\alpha(s)$ is an asymptotic curve $\Longleftrightarrow \kappa_n = 0$,
iii) $\alpha(s)$ is a geodesic curve $\Longleftrightarrow \kappa_g = 0$.

3. Parallel-like Surfaces

Definition 3.1. Let M and M^f be two surfaces in E^3 Euclidean space and Z_P be a unit normal vector and $T_P M$ be tangent space at point P of the surface M and $\{X_P, Y_P\}$ be an orthonormal bases of $T_P M$. Take a unit vector $E_P = d_1 X_P + d_2 Y_P + d_3 Z_P$, where $d_1, d_2, d_3 \in \mathbb{R}$ are constant and $d_1^2 + d_2^2 + d_3^2 = 1$. If there is a function f defined by,

$$f : M \to M^f, \qquad f(P) = P + rE_P$$

where $r \in \mathbb{R}$, then the surface M^f is called parallel-like surface of the surface M.

Here, if $d_1 = d_2 = 0$, then $E_P = Z_P$ and so M and M^f are parallel surfaces. Now, we represent parametrization for parallel-like surface of the surface M . Let (ϕ, U) be a parametrization of M, so we can write that

$$\begin{aligned} \phi \ &: \ U \subset E^2 \to M. \\ & \quad (u, v) \qquad \phi(u, v) \end{aligned}$$

In the case $\{\phi_u, \phi_v\}$ is a bases of $T_P M$, then we can write that $E_P = d_1\phi_u + d_2\phi_v + d_3 Z_P$. Where, ϕ_u, ϕ_v are respectively partial derivatives of ϕ according to u and v. Since $M^f = \{f(P) : \ f(P) = P + rE_P\}$, a parametric representation of M^f is

$$\psi(u, v) = \phi(u, v) + rE(u, v).$$

Thus, it's obtained

$$M^f = \{\psi(u, v) : \ \psi(u, v) = \phi(u, v) + r\left(d_1\phi_u(u, v) + d_2\phi_v(u, v) + d_3 Z(u, v)\right)\}$$

and if we get $rd_1 = \lambda_1, rd_2 = \lambda_2, rd_3 = \lambda_3$, then we have

$$M^f = \left\{ \begin{array}{c} \psi(u,v): \ \psi(u,v) = \phi(u,v) + \lambda_1\phi_u(u,v) + \lambda_2\phi_v(u,v) + \lambda_3 Z(u,v), \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = r^2 \end{array} \right\}$$

Calculation of ψ_u and ψ_v gives us that

$$\begin{aligned} \psi_u &= \phi_u + \lambda_1\phi_{uu} + \lambda_2\phi_{vu} + \lambda_3 Z_u \\ \psi_v &= \phi_v + \lambda_1\phi_{uv} + \lambda_2\phi_{vv} + \lambda_3 Z_v \end{aligned} \tag{3.1}$$

Here $\phi_{uu}, \phi_{vu}, \phi_{uv}, \phi_{vv}, Z_u, Z_v$ are calculated as like as [11]. Suppose that parameter curves are curvature lines of M and let u and v be arc length of these curves. Thus, following equations are obtained

$$\begin{aligned} \phi_{uu} &= -\kappa_1 Z \\ \phi_{vv} &= -\kappa_2 Z \\ \phi_{uv} &= \phi_{vu} = 0 \\ Z_u &= \kappa_1\phi_u \\ Z_v &= \kappa_2\phi_v \end{aligned} \tag{3.2}$$

From 3.1 and 3.2, we find

$$\begin{aligned} \psi_u &= (1+\lambda_3\kappa_1)\,\phi_u - \lambda_1\kappa_1 Z \\ \psi_v &= (1+\lambda_3\kappa_2)\,\phi_v - \lambda_2\kappa_2 Z \end{aligned}$$

and $\{\psi_u, \psi_v\}$ be a bases of $\chi(M^f)$. If we denote by Z^f unit normal vector of M^f, then Z^f is

$$Z^f = \frac{[\psi_u, \psi_v]}{\|[\psi_u, \psi_v]\|} = \frac{\lambda_1\kappa_1(1+\lambda_3\kappa_2)\,\phi_u + \lambda_2\kappa_2(1+\lambda_3\kappa_1)\,\phi_v + (1+\lambda_3\kappa_1)(1+\lambda_3\kappa_2)\,Z}{\sqrt{\lambda_1^2\kappa_1^2(1+\lambda_3\kappa_2)^2 + \lambda_2^2\kappa_2^2(1+\lambda_3\kappa_1)^2 + (1+\lambda_3\kappa_1)^2(1+\lambda_3\kappa_2)^2}}$$

where, κ_1, κ_2 are principal curvatures of the surface M. If

$$A = \sqrt{\lambda_1^2\kappa_1^2(1+\lambda_3\kappa_2)^2 + \lambda_2^2\kappa_2^2(1+\lambda_3\kappa_1)^2 + (1+\lambda_3\kappa_1)^2(1+\lambda_3\kappa_2)^2}$$

we can write

$$Z^f = \frac{\lambda_1\kappa_1(1+\lambda_3\kappa_2)}{A}\phi_u + \frac{\lambda_2\kappa_2(1+\lambda_3\kappa_1)}{A}\phi_v + \frac{(1+\lambda_3\kappa_1)(1+\lambda_3\kappa_2)}{A}Z$$

Here in case of $\kappa_1 = \kappa_2$ and $\lambda_3 = -\frac{1}{\kappa_1} = -\frac{1}{\kappa_2}$ since ψ_u and ψ_v are not linear independent, M^f is not regular surface. We will not consider this case [11].

4. Darboux Frame of Curves on Parallel-like Surfaces

Let $\alpha(s)$ be first parameter curve of the surface M. In this study, since parameter curves are regarded as curvature lines, $\alpha(s)$ is also curvature line. Because principal directions relating to different curvature lines of the surface M are orthogonal, we can take as

$$\phi_u = \alpha^{'}(s) = T$$

and

$$\phi_v = Y.$$

Under these conditions, we can use Darboux frame $\{T, Y, Z\}$ in place of orthonormal frame $\{\phi_u, \phi_v, Z\}$. If we consider definition of parallel-like surface of the surface M, parametric representation of the curve β which is image of the curve α is

$$\beta(s) = \alpha(s) + \lambda_1 T + \lambda_2 Y + \lambda_3 Z. \tag{4.1}$$

Now, we calculate Darboux frame $\{T^f, Y^f, Z^f\}$ for partner of curve-surface (β, M^f). It is clear that

$$T^f = \frac{\beta^{'}(s)}{\|\beta^{'}(s)\|}$$

If we take derivative according to s of eq.4.1, we find

$$\beta^{'}(s)=\alpha^{'}(s)+\lambda_1 T^{'}+\lambda_2 Y^{'}+\lambda_3 Z^{'}$$

and if considering that $\alpha(s)$ is a principal line and so $t_r=0$ equations 2.4 are substituted in this equation, we obtain

$$\beta^{'}(s)=(1-\kappa_g\lambda_2-\kappa_n\lambda_3)\,T+\kappa_g\lambda_1 Y+\kappa_n\lambda_1 Z \tag{4.2}$$

where $\left\|\beta^{'}(s)\right\|=c=\sqrt{(1-\kappa_g\lambda_2-\kappa_n\lambda_3)^2+\lambda_1^2\left(\kappa_g^2+\kappa_n^2\right)}$. Thus, we find

$$T^f=\frac{(1-\kappa_g\lambda_2-\kappa_n\lambda_3)}{c}T+\frac{\kappa_g\lambda_1}{c}Y+\frac{\kappa_n\lambda_1}{c}Z. \tag{4.3}$$

Moreover, we know already that

$$Z^f=\frac{\lambda_1\kappa_1(1+\lambda_3\kappa_2)}{A}T+\frac{\lambda_2\kappa_2(1+\lambda_3\kappa_1)}{A}Y+\frac{(1+\lambda_3\kappa_1)(1+\lambda_3\kappa_2)}{A}Z. \tag{4.4}$$

For orthonormal frame $\{T^f, Y^f, Z^f\}$, if we consider that $Y^f=Z^f\times T^f$, we get

$$\begin{aligned} Y^f &= \left[\frac{\kappa_n\lambda_1\lambda_2(\kappa_2+\lambda_3 K)-\kappa_g\lambda_1\left(1+\lambda_3 H+\lambda_3^2 K\right)}{Ac}\right]T \\ &+\left[\frac{\left(1+\lambda_3 H+\lambda_3^2 K\right)(1-\kappa_g\lambda_2-\kappa_n\lambda_3)-\kappa_n\lambda_1^2(\kappa_1+\lambda_3 K)}{Ac}\right]Y \\ &+\left[\frac{\kappa_g\lambda_1^2(\kappa_1+\lambda_3 K)-\lambda_2(\kappa_1+\lambda_3 K)(1-\kappa_g\lambda_2-\kappa_n\lambda_3)}{Ac}\right]Z \end{aligned} \tag{4.5}$$

where $K=\kappa_1\kappa_2$, $H=\kappa_1+\kappa_2$ are Gauss curvature and mean curvature of the surface M, respectively.

Now, we calculate the geodesic curvature κ_g^f, the normal curvature κ_n^f, the geodesic torsion t_r^f of the curve $\beta(s)$. We will use to calculate these curvatures following equations

$$\kappa_g^f = \frac{1}{c^2}\left\langle \beta^{''}(s), Y^f\right\rangle \tag{4.6}$$

$$\kappa_n^f = \frac{1}{c^2}\left\langle \beta^{''}(s), Z^f\right\rangle \tag{4.7}$$

$$t_r^f = -\frac{1}{c}\left\langle \left(Z^f\right)^{'}, Y^f\right\rangle \tag{4.8}$$

Firstly we find vector $\beta^{''}(s)$. If we take derivative of eq.4.2 according to s and use equations 2.4, we obtain

$$\begin{aligned} \beta^{''}(s) &= \left(-\lambda_2\kappa_g^{'}-\lambda_3\kappa_n^{'}-\lambda_1\kappa_g^2-\lambda_1\kappa_n^2\right)T \\ &+\left(\lambda_1\kappa_g^{'}+\kappa_g(1-\kappa_g\lambda_2-\kappa_n\lambda_3)\right)Y \\ &+\left(\lambda_1\kappa_n^{'}+\kappa_n(1-\kappa_g\lambda_2-\kappa_n\lambda_3)\right)Z. \end{aligned} \tag{4.9}$$

Furthermore we find vector $\left(Z^f\right)^{'}$. If we take derivative of eq.4.4 according to s and use equations 2.4, we obtain

$$\begin{aligned} \left(Z^f\right)^{'} &= \left(\frac{\begin{array}{c} A\lambda_1\left(\kappa_1^{'}+\lambda_3 K^{'}\right)-B\lambda_1(\kappa_1+\lambda_3 K) \\ -\kappa_g\lambda_2 A(\kappa_2+\lambda_3 K)-\kappa_n A\left(1+\lambda_3 H+\lambda_3^2 K\right)\end{array}}{A^2}\right)T \\ &+\left(\frac{A\lambda_2\left(\kappa_2^{'}+\lambda_3 K^{'}\right)-B\lambda_2(\kappa_2+\lambda_3 K)+\kappa_g\lambda_1 A(\kappa_1+\lambda_3 K)}{A^2}\right)Y \\ &+\left(\frac{A\left(\lambda_3 H^{'}+\lambda_3^2 K^{'}\right)-B\left(1+\lambda_3 H+\lambda_3^2 K\right)+\kappa_n\lambda_1 A(\kappa_1+\lambda_3 K)}{A^2}\right)Z \end{aligned} \tag{4.10}$$

where

$$B = A^{'} = \frac{1}{A}[\lambda_1^2\kappa_1\kappa_1^{'} + \lambda_2^2\kappa_2\kappa_2^{'} + \lambda_3\left(\lambda_1^2\kappa_1^{'} + \lambda_2^2\kappa_2^{'}\right)K + \lambda_3\left(\lambda_1^2\kappa_1 + \lambda_2^2\kappa_2\right)K^{'} + \left(\lambda_1^2+\lambda_2^2\right)\lambda_3^2 KK^{'} + \left(1+\lambda_3 H+\lambda_3^2 K\right)\left(\lambda_3 H^{'} + \lambda_3^2 K^{'}\right)].$$

So, if we substitute equations 4.5 and 4.9 into eq.4.6, we obtain

$$\begin{aligned} \kappa_g^f = \frac{1}{Ac^3}\{&\kappa_g\lambda_1\left(\lambda_2\kappa_g^{'}+\lambda_3\kappa_n^{'}\right)\left(1+\lambda_3 H+\lambda_3^2 K\right) \\ &+\left(1+\lambda_3 H+\lambda_3^2 K\right)\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right)\left(\lambda_1\kappa_g^{'}+\kappa_g\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right)\right) \\ &-\kappa_n\lambda_2\left(\kappa_2+\lambda_3 K\right)\left(\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right)^2+\lambda_1\left(\lambda_2\kappa_g^{'}+\lambda_3\kappa_n^{'}\right)\right) \\ &+\lambda_1^2\left(\kappa_n^2+\kappa_g^2\right)\left(\kappa_g\left(1+\lambda_3 H+\lambda_3^2 K\right)-\kappa_n\lambda_2\left(\kappa_2+\lambda_3 K\right)\right) \\ &-\kappa_n^{'}\lambda_1\lambda_2\left(\kappa_2+\lambda_3 K\right)\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right) \\ &+\lambda_1^3\left(\kappa_1+\lambda_3 K\right)\left(\kappa_n^{'}\kappa_g-\kappa_n\kappa_g^{'}\right)\}. \end{aligned} \tag{4.11}$$

Also, if we substitute equations 4.4 and 4.9 into eq.4.7, we obtain

$$\begin{aligned} \kappa_n^f = \frac{1}{Ac^2}\{&\lambda_1\left(\kappa_1+\lambda_3 K\right)\left(-\lambda_2\kappa_g^{'}-\lambda_3\kappa_n^{'}-\lambda_1\left(\kappa_n^2+\kappa_g^2\right)\right) \\ &+\left(\lambda_1\kappa_n^{'}+\kappa_n\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right)\right)\left(1+\lambda_3 H+\lambda_3^2 K\right) \\ &+\lambda_2\left(\kappa_2+\lambda_3 K\right)\left(\lambda_1\kappa_g^{'}+\kappa_g\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right)\right)\}. \end{aligned} \tag{4.12}$$

And finally, if we substitute equations 4.5 and 4.10 into eq.4.8, we obtain

$$\begin{aligned} t_r^f = -\frac{1}{A^3c^2}\{&\kappa_n\kappa_g A\lambda_1\left(\left(1+\lambda_3 H+\lambda_3^2 K\right)^2-\lambda_2^2\left(\kappa_2+\lambda_3 K\right)^2\right) \\ &-A\lambda_2\left(\kappa_2+\lambda_3 K\right)\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right)\left(\kappa_n\lambda_1\left(\kappa_1+\lambda_3 K\right)+\left(\lambda_3 H^{'}+\lambda_3^2 K^{'}\right)\right) \\ &+A\lambda_1^2\left(\kappa_1^{'}+\lambda_3 K^{'}\right)\left(\kappa_n\lambda_2\left(\kappa_2+\lambda_3 K\right)-\kappa_g\left(1+\lambda_3 H+\lambda_3^2 K\right)\right) \\ &+A\lambda_2\left(\kappa_2^{'}+\lambda_3 K^{'}\right)\left(1+\lambda_3 H+\lambda_3^2 K\right)\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right) \\ &+A\kappa_g\lambda_1\left(\kappa_1+\lambda_3 K\right)\left(1+\lambda_3 H+\lambda_3^2 K\right)\left(1-\kappa_g\lambda_2-\kappa_n\lambda_3\right) \\ &+A\lambda_1\lambda_2\left(\kappa_g^2-\kappa_n^2\right)\left(\kappa_2+\lambda_3 K\right)\left(1+\lambda_3 H+\lambda_3^2 K\right) \\ &-\kappa_n A\lambda_1^2\lambda_2\left(\kappa_1+\lambda_3 K\right)\left(\kappa_2^{'}+\lambda_3 K^{'}\right) \\ &+A\kappa_g\lambda_1^2\left(\kappa_1+\lambda_3 K\right)\left(\lambda_3 H^{'}+\lambda_3^2 K^{'}\right)\}. \end{aligned} \tag{4.13}$$

Theorem 4.1. *Let M be a surface in E^3 and M^f be parallel-like surface of the surface M that formed along directions of E_P lying in plane $Sp\{Y,Z\}$, i.e. $\lambda_1=0$. Recall that the curve β on the surface M^f is image curve of the curve α lying on M, then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface $\left(\beta, M^f\right)$ are as follows*

$$\kappa_g^f = \frac{\kappa_g\left(1+\lambda_3 H+\lambda_3^2 K\right)-\lambda_2\kappa_n\left(\kappa_2+\lambda_3 K\right)}{Ac} \tag{4.14}$$

$$\kappa_n^f = \frac{\kappa_n\left(1+\lambda_3 H+\lambda_3^2 K\right)+\lambda_2\kappa_g\left(\kappa_2+\lambda_3 K\right)}{Ac} \tag{4.15}$$

$$\begin{aligned} t_r^f = -\frac{\lambda_2}{A^3c}\{&A\left(\kappa_2^{'}+\lambda_3 K^{'}\right)\left(1+\lambda_3 H+\lambda_3^2 K\right) \\ &-A\left(\kappa_2+\lambda_3 K\right)\left(\lambda_3 H^{'}+\lambda_3^2 K^{'}\right)\}. \end{aligned} \tag{4.16}$$

Proof. If we substitute $\lambda_1 = 0$ in equations 4.11, 4.12, 4.13, we can easily hold equations 4.14, 4.15, 4.16. $\square$

Corollary 4.1. *Let the curve α lying on M be geodesic curve. Providing $\lambda_1 = 0$, the curve β which is image curve of the curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Corollary 4.2. *Let the curve α lying on M be asymptotic curve. Providing $\lambda_1 = 0$, the curve β which is image curve of the curve α is a geodesic curve if and only if α is a geodesic curve.*

Theorem 4.2. *Let M be a surface in E^3 and M^f be parallel-like surface of the surface M that formed along directions of E_P lying in plane $Sp\{T, Z\}$, i.e. $\lambda_2 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface (β, M^f) are as follows*

$$\begin{aligned} \kappa_g^f &= \frac{1}{Ac^3}\{\kappa_g\lambda_1\left(1+\lambda_3 H+\lambda_3^2 K\right)\left(\lambda_3\kappa_n^{'}+\lambda_1\kappa_g^2+\lambda_1\kappa_n^2\right) \\ &\quad +\kappa_g^{'}\lambda_1\left(1-\kappa_n\lambda_3\right)\left(1+\lambda_3 H+\lambda_3^2 K\right) \\ &\quad +\kappa_g\left(1-\kappa_n\lambda_3\right)^2\left(1+\lambda_3 H+\lambda_3^2 K\right) \\ &\quad +\lambda_1^3\left(\kappa_1+\lambda_3 K\right)\left(\kappa_n^{'}\kappa_g-\kappa_n\kappa_g^{'}\right)\}. \end{aligned} \tag{4.17}$$

$$\begin{aligned} \kappa_n^f &= \frac{1}{Ac^2}\{\lambda_1\left(\kappa_1+\lambda_3 K\right)\left(-\lambda_3\kappa_n^{'}-\lambda_1\kappa_g^2-\lambda_1\kappa_n^2\right) \\ &\quad +\left(1+\lambda_3 H+\lambda_3^2 K\right)\left(\lambda_1\kappa_n^{'}+\kappa_n\left(1-\kappa_n\lambda_3\right)\right)\}. \end{aligned} \tag{4.18}$$

$$\begin{aligned} t_r^f &= -\frac{1}{A^3c^2}\{A\kappa_n\kappa_g\lambda_1\left(1+\lambda_3 H+\lambda_3^2 K\right)^2 \\ &\quad +A\kappa_g\lambda_1\left(1-\kappa_n\lambda_3\right)\left(\kappa_1+\lambda_3 K\right)\left(1+\lambda_3 H+\lambda_3^2 K\right) \\ &\quad -A\kappa_g\lambda_1^2\left(1+\lambda_3 H+\lambda_3^2 K\right)\left(\kappa_1^{'}+\lambda_3 K^{'}\right) \\ &\quad +A\kappa_g\lambda_1^2\left(\kappa_1+\lambda_3 K\right)\left(\lambda_3 H^{'}+\lambda_3^2 K^{'}\right)\}. \end{aligned} \tag{4.19}$$

Proof. If we substitute $\lambda_2 = 0$ in equations 4.11, 4.12, 4.13, we can easily hold equations 4.17, 4.18, 4.19. $\square$

Corollary 4.3. *Providing $\lambda_2 = 0$, the curve β lying on M^f is a geodesic curve if and only if α lying on M is a geodesic curve.*

Corollary 4.4. *Providing $\lambda_2 = 0$, the curve β lying on M^f is a principal line if and only if α lying on M is a geodesic curve.*

Corollary 4.5. *Let the curve α lying on M be geodesic curve. Providing $\lambda_2 = 0$, the curve β which is image curve of the curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Theorem 4.3. *Let M be a surface in E^3 and M^f be parallel-like surface of the surface M that formed along directions of E_P lying in plane $Sp\{T, Y\}$, i.e. $\lambda_3 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface (β, M^f) are as follows*

$$\begin{aligned} \kappa_g^f &= \frac{1}{Ac^3}\{\left(-\lambda_2\kappa_g^{'}-\lambda_1\kappa_g^2-\lambda_1\kappa_n^2\right)\left(\kappa_n\kappa_2\lambda_1\lambda_2-\kappa_g\lambda_1\right) \\ &\quad +\left(\lambda_1\kappa_g^{'}+\kappa_g\left(1-\kappa_g\lambda_2\right)\right)\left(1-\kappa_g\lambda_2-\kappa_n\kappa_1\lambda_1^2\right) \\ &\quad +\left(\lambda_1\kappa_n^{'}+\kappa_n\left(1-\kappa_g\lambda_2\right)\right)\left(\kappa_g\kappa_1\lambda_1^2-\lambda_2\kappa_2\left(1-\kappa_g\lambda_2\right)\right)\}. \end{aligned} \tag{4.20}$$

$$\begin{aligned} \kappa_n^f &= \frac{1}{Ac^2}\{\lambda_1\kappa_1\left(-\lambda_2\kappa_g^{'}-\lambda_1\kappa_g^2-\lambda_1\kappa_n^2\right) \\ &\quad +\lambda_2\kappa_2\left(\lambda_1\kappa_g^{'}+\kappa_g\left(1-\kappa_g\lambda_2\right)\right) \\ &\quad +\lambda_1\kappa_n^{'}+\kappa_n\left(1-\kappa_g\lambda_2\right)\}. \end{aligned} \tag{4.21}$$

$$t_r^f = -\frac{1}{A^3c^2}\{\left(A\lambda_1\kappa_1^{'} - B\lambda_1\kappa_1 - A\kappa_g\kappa_2\lambda_2 - A\kappa_n\right)(\kappa_n\kappa_2\lambda_1\lambda_2 - \kappa_g\lambda_1) + \left(A\kappa_2^{'}\lambda_2 - B\kappa_2\lambda_2 + A\kappa_g\kappa_1\lambda_1\right)(1 - \kappa_g\lambda_2 - \kappa_n\kappa_1\lambda_1^2) + (A\kappa_n\kappa_1\lambda_1 - B)\left[\kappa_g\kappa_1\lambda_1^2 - \lambda_2\kappa_2(1 - \kappa_g\lambda_2)\right]\}. \tag{4.22}$$

Proof. If we substitute $\lambda_3 = 0$ in equations 4.11, 4.12, 4.13, we can easily hold equations 4.20, 4.21, 4.22. ☐

Corollary 4.6. *Let the curve α lying on M be asymptotic curve. Providing $\lambda_3 = 0$, the curve β which is image curve of the curve α is a geodesic curve if and only if α is a geodesic curve.*

Corollary 4.7. *Let the curve α lying on M be geodesic curve. Providing $\lambda_3 = 0$, the curve β which is image curve of the curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Theorem 4.4. *Let M be a surface in E^3 and M^f be parallel-like surface of the surface M that formed along vector field Z, i.e. $\lambda_1 = \lambda_2 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface (β, M^f) are as follows*

$$\kappa_g^f = \frac{\kappa_g}{1 - \kappa_n\lambda_3} \tag{4.23}$$

$$\kappa_n^f = \frac{\kappa_n}{1 - \kappa_n\lambda_3} \tag{4.24}$$

$$t_r^f = 0. \tag{4.25}$$

Proof. If we substitute $\lambda_1 = \lambda_2 = 0$ in equations 4.11, 4.12, 4.13, we can easily hold equations 4.23, 4.24, 4.25. ☐

Corollary 4.8. *Providing $\lambda_1 = \lambda_2 = 0$, the curve β lying on M^f is a geodesic curve if and only if α lying on M is a geodesic curve.*

Corollary 4.9. *Providing $\lambda_1 = \lambda_2 = 0$, the curve β lying on M^f is an asymptotic curve if and only if α lying on M is an asymptotic curve.*

Corollary 4.10. *Providing $\lambda_1 = \lambda_2 = 0$, the curve β lying on M^f is a principal line if and only if α lying on M is a principal line.*

Theorem 4.5. *Let M be a surface in E^3 and M^f be parallel-like surface of the surface M that formed along vector field Y, i.e. $\lambda_1 = \lambda_3 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface (β, M^f) are as follows*

$$\kappa_g^f = \frac{(\kappa_g - \lambda_2\kappa_n\kappa_2)}{(1 - \kappa_g\lambda_2)(\lambda_2^2\kappa_2^2 + 1)^{\frac{1}{2}}} \tag{4.26}$$

$$\kappa_n^f = \frac{(\kappa_n + \lambda_2\kappa_g\kappa_2)}{(1 - \kappa_g\lambda_2)(\lambda_2^2\kappa_2^2 + 1)^{\frac{1}{2}}} \tag{4.27}$$

$$t_r^f = -\frac{\kappa_2^{'}\lambda_2}{(\lambda_2^2\kappa_2^2 + 1)(1 - \kappa_g\lambda_2)}. \tag{4.28}$$

Proof. If we substitute $\lambda_1 = \lambda_3 = 0$ in equations 4.11, 4.12, 4.13, we can easily hold equations 4.26, 4.27, 4.28. ☐

Corollary 4.11. *Let the curve α lying on M be asymptotic curve. Providing $\lambda_1 = \lambda_3 = 0$, the curve β which is image curve of the curve α is a geodesic curve if and only if α is a geodesic curve.*

Corollary 4.12. *Let the curve α lying on M be geodesic curve. Providing $\lambda_1 = \lambda_3 = 0$, the curve β which is image curve of the curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Theorem 4.6. *Let M be a surface in E^3 and M^f be parallel-like surface of the surface M that formed along vector field T, i.e. $\lambda_2 = \lambda_3 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface (β, M^f) are as follows*

$$\kappa_g^f = \frac{1}{Ac^3}\left\{\kappa_g\lambda_1^2\left(\kappa_g^2 + \kappa_n^2 + \kappa_1\left(\lambda_1\kappa_n^{'} + \kappa_n\right)\right) + \left(\lambda_1\kappa_g^{'} + \kappa_g\right)(1 - \kappa_n\kappa_1\lambda_1^2)\right\} \tag{4.29}$$

$$\kappa_n^f = \frac{1}{Ac^2}\left\{\kappa_n + \lambda_1\kappa_n^{'} - \lambda_1^2\kappa_1\left(\kappa_g^2 + \kappa_n^2\right)\right\} \tag{4.30}$$

$$t_r^f = -\frac{1}{A^2c^2}\left\{\kappa_g\lambda_1\left(\kappa_n + \kappa_1 - \kappa_1^{'}\lambda_1\right)\right\}. \tag{4.31}$$

Proof. If we substitute $\lambda_2 = \lambda_3 = 0$ in equations 4.11, 4.12, 4.13, we can easily hold equations 4.29, 4.30, 4.31. □

Corollary 4.13. *Let the curve α lying on M be asymptotic curve. Providing $\lambda_2 = \lambda_3 = 0$, the curve β which is image curve of the curve α is a geodesic curve if and only if α is a geodesic curve.*

Corollary 4.14. *Let the curve α lying on M be geodesic curve. Providing $\lambda_2 = \lambda_3 = 0$, the curve β which is image curve of the curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Corollary 4.15. *Providing $\lambda_2 = \lambda_3 = 0$, if the curve α lying on M is a geodesic curve, the curve β lying on M^f is a principal line.*

Example 4.1. Consider the paraboloid surface given by parameterization $\phi : E^2 \to E^3$,

$$\phi(u, v) = \left(u \cos v, u \sin v, u^2\right).$$

Using the definition of the parallel-like surfaces, the parametric expression of the parallel-like surface of the paraboloid is found in the form

$$\psi(u, v) = \begin{pmatrix} \left(u + \lambda_1 - \frac{2u\lambda_3}{\sqrt{1+4u^2}}\right) \cos v - u\lambda_2 \sin v, \\ \left(u + \lambda_1 - \frac{2u\lambda_3}{\sqrt{1+4u^2}}\right) \sin v + u\lambda_2 \cos v, \\ u^2 + 2u\lambda_1 + \frac{\lambda_3}{\sqrt{1+4u^2}} \end{pmatrix}.$$

Here, if it is taken as $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ the graphs of the paraboloid and its parallel-like surface in the same coordinate system are as follows

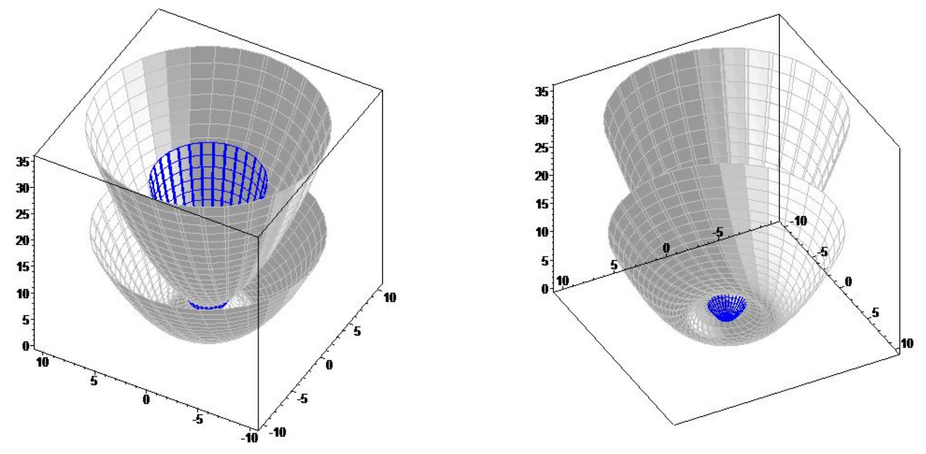

Figure 1. Paraboloid(blue) and its parallel-like surface(gray)

We will examine the image curve on the parallel-like surface of the first parameter curve of paraboloid. Since the first and second fundamental form coefficients of the paraboloid surface are $F = f = 0$, the parameter curves to be taken on this surface will be the curvature lines. First parameter curve of the paraboloid surface M is

$$\alpha(u) = \phi(u, v_0) = \left(u \cos v_0, u \sin v_0, u^2\right)$$

and image curve on parallel-like surface M^f of α is as follows

$$\begin{aligned} \beta(u) &= \psi(u, v_0) = \phi(u, v_0) + \lambda_1 \phi_u(u, v_0) + \lambda_2 \phi_v(u, v_0) + \lambda_3 Z(u, v_0) \\ &= \begin{pmatrix} \left(u + \lambda_1 - \frac{2u\lambda_3}{\sqrt{1+4u^2}}\right) \cos v - u\lambda_2 \sin v, \\ \left(u + \lambda_1 - \frac{2u\lambda_3}{\sqrt{1+4u^2}}\right) \sin v + u\lambda_2 \cos v, \\ u^2 + 2u\lambda_1 + \frac{\lambda_3}{\sqrt{1+4u^2}} \end{pmatrix}. \end{aligned}$$

Thus, the first parameter curve passing through from the point $P = \phi(u_0, v_0) = \phi\left(\sqrt{2}, \frac{\pi}{4}\right) = (1, 1, 2)$ on the paraboloid and its image curve on the parallellike surface are obtained respectively as

$$\alpha(u) = \phi\left(u, \frac{\pi}{4}\right) = \left(\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}, u^2\right)$$

and

$$\beta(u) = \left(\begin{array}{c} \frac{1}{\sqrt{2}}\left(u + \lambda_1 - u\lambda_2 - \frac{2u\lambda_3}{\sqrt{1+4u^2}}\right), \\ \frac{1}{\sqrt{2}}\left(u + \lambda_1 + u\lambda_2 - \frac{2u\lambda_3}{\sqrt{1+4u^2}}\right), \\ u^2 + 2u\lambda_1 + \frac{\lambda_3}{\sqrt{1+4u^2}} \end{array}\right).$$

Here, by giving special values to $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ parametric expressions and the graphs of these curves α and β on the surfaces M and M^f have been found as follows

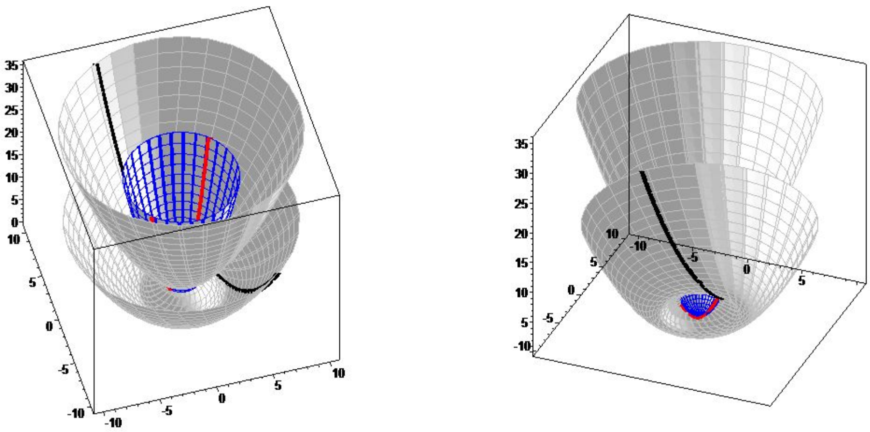

Figure 2. Paraboloid(blue),its parallel-like surface(gray),parameter curve on paraboloid(red) and its image curve on paralel-like surface(black)

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