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Conformable Fractional Elzaki Decomposition Method of Conformable Fractional Space-Time Fractional Telegraph Equations

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Abstract — Conformable space-time fractional linear telegraph equations are examined using a new method known as conformable fractional Elzaki decomposition method. The suggested method combines the Adomian decomposition method with the conformable fractional Elzaki transform. It is found that numerical simulations confirm the effectiveness and reliability of the proposed method.

Subject Classification (2020): 65R10.

1. Introduction

The appearance of fractional calculus is based on a question that Leibniz asked L'Hospital on 30 September, 1695. Since 1695, the mathematicians have developed in fractional derivatives and produced derivatives of various orders. Recently, we have observed that fractional analysis allows an elegant modelling of a lot of interdisciplinary problems [1-7]. Until recently, the fractional derivative definitions such as Grunwald–Letnikov, Riesz, Riemann–Liouville, Caputo [2-3, 8] have been widely used in the solution methods to obtain the approximate solutions of differential equations. Since these derivative definitions include integral operators, the calculations are extremely challenging. Besides, analytical solutions usually can not be obtained in the models using these derivative definitions and to solve these equations scientists sometimes benefit from numerical methods.

Different fractional-order models are utilized in engineering and the applied sciences because these models provide a more accurate description of real-world scenarios. Various researchers have already utilized conformable fractional derivatives in numerous disciplines. [9]. The conformable fractional operator [3, 10-12] overcomes certain limitations of the existing fractional operators and provides traditional calculus with properties including the mean value theorem, the chain rule, the product of two functions, the derivative of the quotient of two functions, and Rolle's theorem.

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The telegraph equation has been improved by Oliver Heaviside in the 1880s, which defines the distance and time on an electric transmission line with voltage and current. The telegraph equation is usually implemented in the investigation of electric signals, as well as wave propagations in the wave phenomena and cable transmission line. The telegraph equation has a lot of applications in areas such as radio frequency, wireless signals, telephone lines, and microwave transmission [13]. Many numerical and analytical methods have been utilised to solve fractional-order telegraph equations, such as Laplace transform (LT) [14], homotopy perturbation method (HPM) [15], variational iteration method (VIM) [16] . Recently, Keskin and Oturanc has extended FRDM for FDEs, where they showed that FRDTM can simply obtain the exact solution for both linear and nonlinear FDEs [17-18]. In the literature, there are a lot of numerical and analytical methods such as conformable variational iteration method (C-VIM) [19], conformable fractional reduced differential transform method (CFRDTM) [19], conformable homotopy analysis method (C-HAM) [19], conformable fractional differential transform method (CFDTM) [20], conformable fractional adomian decomposition method (CFADM) [21] , and conformable modified homotopy perturbation method (CMHPM) [21]. The main motivation of writing this paper is to suggest a new method which is called conformable fractional Elzaki decomposition method (CFEDM) to obtain numerical solutions for the conformable time-fractional linear telegraph equations.

In this study, CFEDM is applied to solve the following types of the conformable time-fractional linear telegraph equations.

In this study, CFEDM is applied to solve the following types of the conformable time-fractional linear telegraph equations.

1) One-dimensional space-time conformable fractional telegraph equation is introduced by

$$\frac{\partial^{2\mu} w(x,t)}{\partial t^{2\mu}} + 2\alpha \frac{\partial^{\mu} w(x,t)}{\partial t^{\mu}} + \beta^2 w(x,t) = \frac{\partial^{2\vartheta} w(x,t)}{\partial t^{2\vartheta}} + h(x,t), 0 < \vartheta, \mu \leq 1. \tag{1}$$

with the initial and boundary conditions

$$w(x,0) = \Phi_1(x), w_t(x,0) = \Phi_2(x), w(0,t) = \Phi_3(t), w_x(0,t) = \Phi_4(t). \tag{2}$$

2) Two-dimensional conformable fractional-order telegraph equation is given by

$$\frac{\partial^{2\mu} w(x,y,t)}{\partial t^{2\mu}} + 2\alpha \frac{\partial^{\mu} w(x,y,t)}{\partial t^{\mu}} + \beta^2 w(x,y,t) = \frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2} + h(x,y,t),$$

$$0 < \mu \leq 1, \vartheta = 1. \tag{3}$$

with the initial and boundary conditions

$$w(x,y,0) = \xi_1(x,y), w_t(x,y,0) = \xi_2(x,y). \tag{4}$$

3) Three-dimensional conformable fractional-order telegraph equation is introduced by

$$\frac{\partial^{2\mu} w(x,y,z,t)}{\partial t^{2\mu}} + 2\alpha \frac{\partial^{\mu} w(x,y,z,t)}{\partial t^{\mu}} + \beta^2 w(x,y,z,t) = \frac{\partial^2 w(x,y,z,t)}{\partial x^2} + \frac{\partial^2 w(x,y,z,t)}{\partial y^2}$$

$$+ \frac{\partial^2 w(x,y,z,t)}{\partial z^2} + h(x,y,z,t), 0 < \mu \leq 1, \vartheta = 1. \tag{5}$$

with the initial and boundary conditions

$$w(x,y,z,0) = \kappa_1(x,y,z), w_t(x,y,0) = \kappa_2(x,y,z). \tag{6}$$

In this study, the symbol D^{μ} represents the conformable fractional derivative operator.

2. Preliminaries

In this section, the definitions of conformable fractional calculus and Elzaki transform that should be utilized in the current study are presented.

Definition 1 [11-12, 22]. Given a function $f:[0,\infty)\to\mathbb{R}$. Then, the conformable fractional derivative of f order α is defined by

$$T_\alpha(f)(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha})-f(x)}{\varepsilon}, \tag{7}$$

for all $x>0, \alpha\in(0,1]$.

Theorem 1 [11, 23]. Let $\alpha\in(0,1]$ and f,g be α −differentiable at a point $x>0$. Then it is obtained as

(i) $T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$, for all $a,b\in\mathbb{R}$, (8)

(ii) $T_\alpha(x^p)=px^{p-1}$, for all $p\in\mathbb{R}$, (9)

(iii) $T_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$, (10)

(iv) $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$, (11)

(v) $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$. (12)

(vi) If f is differentiable, then $T_\alpha(f)(t)=t^{1-\alpha}\frac{d}{dt}f(t)$. (13)

Definition 2 [12]. Let f be an n −times differentiable at x. Then, the conformable fractional derivative of f order α is defined by:

$$T_\alpha(f)(x)=\lim_{\varepsilon\to 0}\frac{f^{([\alpha]-1)}\left(x+\varepsilon x^{([\alpha]-\alpha)}\right)-f^{([\alpha]-1)}(x)}{\varepsilon}, \tag{14}$$

for all $x>0, \alpha\in(n,n+1]$, $[\alpha]$ is the smallest integer greater than or equal to α.

Theorem 2 [12]. Let f be an n −times differentiable at x. Then

$$T_\alpha\big(f(x)\big)=x^{[\alpha]-\alpha}f^{[\alpha]}(x), \tag{15}$$

for all $x>0, \alpha\in(n,n+1]$.

Definition 3 [23]. The Mittag-Leffler function E_a is given as follows:

$$E_a(z)=\sum_{n=0}^{\infty}\frac{z^a}{\Gamma(na+1)}. \tag{16}$$

Definition 4 [12]. The conformable fractional exponential function is defined for every $t\geq 0$ by

$$E_\alpha(c,t) = exp\left(c\frac{t^\alpha}{\alpha}\right), \tag{17}$$

where $c \in \mathbb{R}$ and $0 < \alpha \leq 1$.

Definition 5 [24]. Let $0 < \alpha \leq 1, f: [0,\infty) \to \mathbb{R}$ be real valued function. The conformable fractional Elzaki transform of order α of f is defined by

$$E_\alpha[f(t)](v) = \int_0^\infty v\, E_\alpha\left[-\frac{1}{v}, t\right] f(t) d_\alpha t, v > 0. \tag{18}$$

The Elzaki transform for the conformable fractional-order derivative is described by

$$E_\alpha[T_\alpha f(t)](v) = \frac{1}{v} E_\alpha[f(t)](v) - vf(0). \tag{19}$$

Theorem 3. Let $F_\alpha[v] = E_\alpha[f(t)](\mathrm{v})$ exists for $v > 0$. Then, it is obtained as

1. If c is a constant, then

$$E_\alpha[c] = v^2, \tag{20}$$

2. If w is a constant, then

$$E_\alpha[t^w] = \alpha^{\frac{w}{\alpha}} \Gamma\left(1 + \frac{w}{\alpha}\right) v^{2+\frac{w}{\alpha}}. \tag{21}$$

3. Conformable Fractional Elzaki Decomposition Method (CFEDM)

Now to present the fundamental idea of CFEDM, we consider the conformable fractional order nonlinear partial differential equation:

$$\frac{\partial^\mu u(x,t)}{\partial t^\mu} + Ru(x,t) + Nu(x,t) = f(x,t), t > 0, n-1 < \mu \leq n, \tag{22}$$

where R indicates the linear operator, N denotes the nonlinear operator, $f(x,t)$ symbolizes source term, and $\frac{\partial^\mu u(x,t)}{\partial t^\mu}$ is the conformable fractional derivative operator μ.

Now, by performing conformable Elzaki transform on Eq. (22) and using initial condition, we have

$$\frac{1}{v} E_\mu[u(x,t)] - vu(x,0) + E_\mu[Ru(x,t) + Nu(x,t)] = E_\mu[f(x,t)]. \tag{23}$$

If we simplify the Eq. (23), we get

$$E_\mu[u(x,t)] = v^2 u(x,0) + vE_\mu[f(x,t)] - vE_\mu[Ru(x,t) + Nu(x,t)]. \tag{24}$$

On applying inverse conformable Elzaki transform to Eq. (24), we get

$$u(x,t) = H(x,t) - {E_\mu}^{-1}\{vE_\mu[Ru(x,t) + Nu(x,t)]\}, \tag{25}$$

where $H(x,t)$ is obtained from initial condition and non-homogeneous term. Now, assume that, the infinite series solution is of the form:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \tag{26}$$

By employing Eqs. (25)-(26), we have

$$\sum_{m=0}^{\infty} u_m(x,t) = H(x,t) - E_\mu{}^{-1}\left(vE_\mu\left[R\sum_{m=0}^{\infty} u_m(x,t) + \sum_{m=0}^{\infty} A_m\right]\right). \tag{27}$$

where A_m is the Adomian polynomial and which denotes the nonlinear term $Nu(x,t)$. By comparing both sides of Eq. (27), we get

$$u_0(x,t) = H(x,t), \tag{28}$$

$$u_1(x,t) = -E_\mu{}^{-1}\left(vE_\mu[Ru_0(x,t) + A_0]\right), \tag{29}$$

$$u_2(x,t) = -E_\mu{}^{-1}\left(vE_\mu[Ru_1(x,t) + A_1]\right), \tag{30}$$

$$\vdots$$

Similarly, we obtain the general recursive relation by

$$u_{m+1}(x,t) = -E_\mu{}^{-1}\left(vE_\mu[Ru_m(x,t) + A_m]\right), m \geq 1. \tag{31}$$

Finally, the approximate solution $u(x,t)$ is given by

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \tag{32}$$

4. Convergence Analysis

Theorem 4.1. Let's assume that A is a Banach space. Then, the expansion result of $u(x,t)$ converges uncertainty; there becomes $\rho, 0 < \rho < 1$, so that $\|u_i(x,t)\| \leq \rho\|u_{i-1}(x,t)\|$, for all i ∈ N.

Proof. Consider the subsequent succession

$$H_i(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_i(x,t). \tag{33}$$

It is vital to verify that successions of i-th partial sums $H_i(x,t)$ are a Cauchy series in Banach space. In this regard, we consider the following:

$$\|H_{i+1}(x,t) - H_i(x,t)\| \leq \|u_{i+1}(x,t)\| \leq \rho\|u_i(x,t)\| \leq \rho^2\|u_{i-1}(x,t)\| \leq \cdots \leq \rho^{i+1}\|u_0(x,t)\|. \tag{34}$$

For every $i, j \in N$, $i \leq j$, it is obtained as

$$\left\|H_i(x,t) - H_j(x,t)\right\| \leq \left\|H_{j+1}(x,t) - H_j(x,t)\right\| + \left\|H_{j+2}(x,t) - H_{j+1}(x,t)\right\| + \cdots$$

$$+\|H_i(x,t) - H_{i+1}(x,t)\|. \tag{35}$$

Using the triangle inequality, then the inequality (35) transforms into the inequality (36):

$$\left\|H_i(x,t) - H_j(x,t)\right\| \leq \left\|H_{j+1}(x,t) - H_j(x,t)\right\| + \left\|H_{j+2}(x,t) - H_{j+1}(x,t)\right\|$$

$$+ \cdots + \|H_i(x,t) - H_{i+1}(x,t)\|. \tag{36}$$

The inequality (36) can be represented as follows:

$$\left\|H_i(x,t) - H_j(x,t)\right\| \le \rho^{j+1}\|u_0(x,t)\| + \rho^{j+2}\|u_0(x,t)\| + \cdots + \rho^{i}\|u_0(x,t)\|. \tag{37}$$

The simple calculation enables us to write the inequality (37) as

$$\left\|H_i(x,t) - H_j(x,t)\right\| \le \rho^{j+1}\left(1 + \rho + \rho^2 + \cdots + \rho^{i-j-1}\right)\|u_0(x,t)\|, \tag{38}$$

where $\left(\frac{1-\rho^{i-j}}{1-\rho}\right) = 1 + \rho + \rho^2 + \cdots + \rho^{i-j-1}$.

Thus, inequality (38) is obtained as

$$\left\|H_i(x,t) - H_j(x,t)\right\| \le \rho^{j+1}\left(\frac{1-\rho^{i-j}}{1-\rho}\right)\|u_0(x,t)\|. \tag{39}$$

Hence it is acquired as $0 < \rho < 1$, and $1 - \rho^{i-j} \le 1$.

Using inequality (39), we have

$$\left\|H_i(x,t) - H_j(x,t)\right\| \le \frac{\rho^{i+1}}{1-\rho}\|u_0(x,t)\|. \tag{40}$$

Since $u_0(x,t)$ is bounded, it is obtained as

$$\lim_{i,j\to\infty}\left\|H_i(x,t) - H_j(x,t)\right\| = 0. \tag{41}$$

Thus, $\{H_i\}$ is a Cauchy series in Banach space. Hence, Eq. (32) converges.

5. Applications

Example 5.1 Consider the conformable time-fractional linear telegraph equation (CTFLTE) **[25]**

$$\frac{\partial^{2\mu} w(x,t)}{\partial t^{2\mu}} + 2\frac{\partial^{\mu} w(x,t)}{\partial t^{\mu}} + w(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2}, 0 < \mu \le 1, t \ge 0, \tag{42}$$

with the initial condition

$$w(x,0) = e^x, w_t(x,0) = -2e^x. \tag{43}$$

Now, by performing conformable Elzaki transform on Eq. (42), then we get

$$\frac{1}{v^2}E_\mu\{w(x,t)\} - w(x,0) - v w_t(x,0) + E_\mu\left[2\frac{\partial^{\mu} w(x,t)}{\partial t^{\mu}} + w(x,t) - \frac{\partial^2 w(x,t)}{\partial x^2}\right] = 0. \tag{44}$$

If we simplify the Eq. (44), then we have

$$E_\mu\{w(x,t)\} = v^2\, w(x,0) + v^3 w_t(x,0) - v^2 E_\mu\left[2\frac{\partial^{\mu} w(x,t)}{\partial t^{\mu}} + w(x,t) - \frac{\partial^2 w(x,t)}{\partial x^2}\right]. \tag{45}$$

Applying the inverse conformable Elzaki transform,

$$w(x,t) = {E_\mu}^{-1}\left[v^2\, w(x,0) + v^3 w_t(x,0) - v^2 E_\mu\left[2\frac{\partial^{\mu} w(x,t)}{\partial t^{\mu}} + w(x,t) - \frac{\partial^2 w(x,t)}{\partial x^2}\right]\right]. \tag{46}$$

Using the ADM procedure, we obtain

$$w_0(x,t) = {E_\mu}^{-1}[v^2\, w(x,0) + v^3 w_t(x,0)] = {E_\mu}^{-1}[v^2\, e^x - 2e^x v^3] = e^x - 2e^x\frac{t^\mu}{\mu}, \tag{47}$$

$$w_{s+1}(x,t) = -E_\mu{}^{-1}\left[v^2 E_\mu\left[2\frac{\partial^\mu w_s(x,t)}{\partial t^\mu} + w_s(x,t) - \frac{\partial^2 w_s(x,t)}{\partial x^2}\right]\right], \qquad s = 0,1,2,\dots \tag{48}$$

For $s = 0$ in Eq. (48), we obtain

$$w_1(x,t) = -E_\mu{}^{-1}\left[v^2 E_\mu\left[2\frac{\partial^\mu w_0(x,t)}{\partial t^\mu} + w_0(x,t) - \frac{\partial^2 w_0(x,t)}{\partial x^2}\right]\right], \tag{49}$$

$$w_1(x,t) = -E_\mu{}^{-1}\left[4e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)v^{2+\frac{\mu-1}{\mu}+2}\right] = \frac{4e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)t^{3\mu-1}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)}. \tag{50}$$

We get the subsequent terms, recursively

$$w_2(x,t) = -E_\mu{}^{-1}\left[v^2 E_\mu\left[2\frac{\partial^\mu w_1(x,t)}{\partial t^\mu} + w_1(x,t) - \frac{\partial^2 w_1(x,t)}{\partial x^2}\right]\right]$$

$$= \frac{-8e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)t^{5\mu-2}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)}, \tag{51}$$

$$w_3(x,t) = -E_\mu{}^{-1}\left[v^2 E_\mu\left[2\frac{\partial^\mu w_2(x,t)}{\partial t^\mu} + w_2(x,t) - \frac{\partial^2 w_2(x,t)}{\partial x^2}\right]\right]$$

$$= \frac{16e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)(5\mu-2)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)},$$

$$\times \frac{\mu^{\frac{5\mu-3}{\mu}}\Gamma\left(1+\frac{5\mu-3}{\mu}\right)t^{7\mu-3}}{\mu^{\frac{7\mu-3}{\mu}}\Gamma\left(1+\frac{7\mu-3}{\mu}\right)}. \tag{52}$$

$$\vdots$$

Proceeding in a similar way, we obtain

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots = e^x - 2e^x\frac{t^\mu}{\mu}$$

$$+\frac{4e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)t^{3\mu-1}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)} - \frac{8e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)t^{5\mu-2}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)}$$

$$+\frac{16e^x \mu^{\frac{\mu-1}{\mu}}\Gamma\left(1+\frac{\mu-1}{\mu}\right)(3\mu-1)(5\mu-2)\mu^{\frac{3\mu-2}{\mu}}\Gamma\left(1+\frac{3\mu-2}{\mu}\right)\mu^{\frac{5\mu-3}{\mu}}\Gamma\left(1+\frac{5\mu-3}{\mu}\right)t^{7\mu-3}}{\mu^{\frac{3\mu-1}{\mu}}\Gamma\left(1+\frac{3\mu-1}{\mu}\right)\mu^{\frac{5\mu-2}{\mu}}\Gamma\left(1+\frac{5\mu-2}{\mu}\right)\mu^{\frac{7\mu-3}{\mu}}\Gamma\left(1+\frac{7\mu-3}{\mu}\right)}$$

$$+\cdots \tag{53}$$

Subsituting $\mu = 1$ in Eq. (53), then CFEDM solution is reduced as

$$w(x,t) = e^x \left[1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \cdots\right]. \tag{54}$$

This result is evaluated to the exact solution in a closed form:

$$w(x,t) = e^{x-2t}. \tag{55}$$

The CFEDM solutions of $w(x,t)$ is found to be in excellent agreement with the exact solution of problem. For more understanding the results for the variable $w(x,t)$ of Example 4.1 are plotted in Figure 1. In Figure 1, we observe that this solution is higher accuracy.

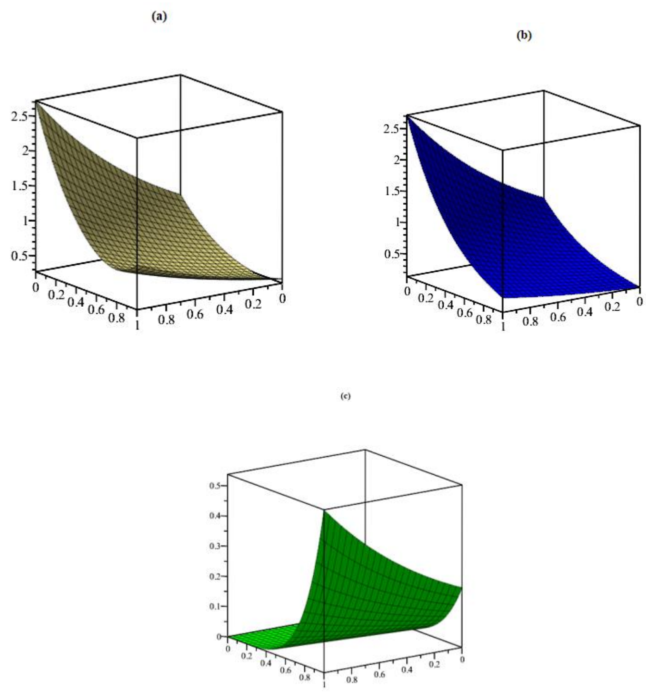

Fig. 1. **(a)** Nature of CFEDM solution for $w(x,t)$ **(b)** Nature of exact solution for $w(x,t)$ **(c)** Nature of absolute error=$|u_{exact} - u_{CFEDM}|$ in Ex. 5.1 at $\mu = 1$.

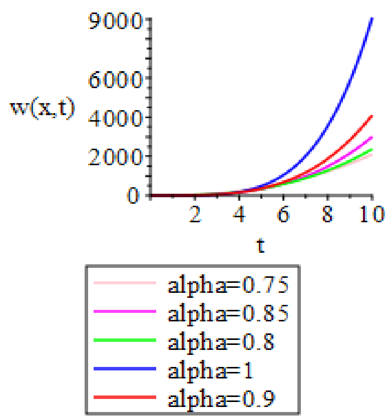

Fig. 2. Nature of CFEDM solution for $w(x,t)$ in Ex. 5.1 at $x = 0.5$ with distinct μ.

Table 1. Numerical solution of $w(x,t)$ for CTFLTE by CFEDM in Ex. 5.1 at with distinct values of x and t for diverse μ.

x	t	$\mu = 0.75$	$\mu = 0.8$	$\mu = 0.85$	$\mu = 0.9$	$\mu = 1$
0.1	0.001	1.3×10^{-2}	8.4×10^{-3}	5.0×10^{-3}	2.6×10^{-3}	2.9×10^{-16}
	0.002	2.0×10^{-2}	1.3×10^{-2}	8.5×10^{-3}	4.6×10^{-3}	9.4×10^{-15}
	0.003	2.6×10^{-2}	1.8×10^{-2}	1.1×10^{-2}	6.3×10^{-3}	7.1×10^{-14}
	0.004	3.1×10^{-2}	2.2×10^{-2}	1.4×10^{-2}	7.9×10^{-3}	3.0×10^{-13}
	0.005	3.5×10^{-2}	2.5×10^{-2}	1.6×10^{-2}	9.4×10^{-3}	9.1×10^{-13}
0.2	0.001	1.4×10^{-2}	9.3×10^{-2}	5.5×10^{-3}	2.9×10^{-3}	3.2×10^{-16}
	0.002	2.2×10^{-2}	1.5×10^{-2}	9.4×10^{-3}	5.1×10^{-3}	1.0×10^{-14}
	0.003	2.9×10^{-2}	2.0×10^{-2}	1.2×10^{-2}	7.0×10^{-3}	7.9×10^{-14}
	0.004	3.4×10^{-2}	2.4×10^{-2}	1.5×10^{-2}	8.8×10^{-3}	3.3×10^{-13}
	0.005	3.9×10^{-2}	2.8×10^{-2}	1.8×10^{-2}	1.0×10^{-2}	1.0×10^{-12}
0.3	0.001	1.6×10^{-2}	1.0×10^{-2}	0.6×10^{-3}	3.2×10^{-3}	3.5×10^{-16}
	0.002	2.5×10^{-2}	1.6×10^{-2}	1.0×10^{-2}	5.6×10^{-3}	1.1×10^{-14}
	0.003	3.2×10^{-2}	2.2×10^{-2}	1.4×10^{-2}	7.8×10^{-3}	8.7×10^{-14}
	0.004	3.8×10^{-2}	2.7×10^{-2}	1.7×10^{-2}	9.7×10^{-3}	3.6×10^{-13}
	0.005	4.3×10^{-2}	3.1×10^{-2}	2.0×10^{-2}	1.1×10^{-2}	1.1×10^{-12}
0.4	0.001	1.7×10^{-2}	1.1×10^{-2}	6.7×10^{-3}	3.5×10^{-3}	3.9×10^{-16}
	0.002	2.7×10^{-2}	1.8×10^{-2}	1.1×10^{-2}	6.2×10^{-3}	1.2×10^{-14}
	0.003	3.5×10^{-2}	2.4×10^{-2}	1.5×10^{-2}	8.6×10^{-3}	9.6×10^{-14}
	0.004	4.2×10^{-2}	3.0×10^{-2}	1.9×10^{-2}	1.0×10^{-2}	4.0×10^{-13}
	0.005	4.8×10^{-2}	3.4×10^{-2}	2.2×10^{-2}	1.2×10^{-2}	1.2×10^{-12}
0.5	0.001	1.9×10^{-2}	1.2×10^{-2}	7.5×10^{-3}	3.9×10^{-3}	4.3×10^{-16}
	0.002	3.0×10^{-2}	2.0×10^{-2}	1.2×10^{-2}	6.9×10^{-3}	1.4×10^{-14}
	0.003	3.9×10^{-2}	2.7×10^{-2}	1.7×10^{-2}	9.5×10^{-3}	1.0×10^{-13}
	0.004	4.7×10^{-2}	3.3×10^{-2}	2.1×10^{-2}	1.1×10^{-2}	4.4×10^{-13}
	0.005	5.3×10^{-2}	3.8×10^{-2}	2.4×10^{-2}	1.4×10^{-2}	1.3×10^{-12}

Example 5.2 Consider the conformable space-fractional linear telegraph equation (CSFLTE) [26-27]

$$\frac{\partial^{2\mu} w(x,t)}{\partial x^{2\mu}} = \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} + w(x,t), 1 < \mu \leq 2, t \geq 0, \tag{56}$$

with the initial condition

$$w(0,t) = e^{-t}, w_x(0,t) = e^{-t}. \tag{57}$$

Now, by performing conformable Elzaki transform on Eq. (56), then we get

$$\frac{1}{v^2} E_\mu\{w(x,t)\} - w(0,t) - v w_x(0,t) = E_\mu\left[\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} + w(x,t)\right]. \tag{58}$$

If we simplify the Eq. (58), then we have

$$E_\mu\{w(x,t)\} = v^2 w(0,t) + v^3 w_x(0,t) + v^2 E_\mu\left[\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} + w(x,t)\right]. \tag{59}$$

Applying the inverse conformable Elzaki transform,

$$w(x,t) = E_\mu{}^{-1}\left[v^2\, w(0,t) + v^3 w_t(0,t) + v^2 E_\mu\left[\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} + w(x,t)\right]\right]. \tag{60}$$

Using the ADM procedure, we obtain

$$w_0(x,t) = E_\mu{}^{-1}[v^2\, w(0,t) + v^3 w_t(0,t)] = E_\mu{}^{-1}[v^2\, e^{-t} + v^3 e^{-t}] = e^{-t} + e^{-t}\frac{x^\mu}{\mu}. \tag{61}$$

$$w_{s+1}(x,t) = E_\mu{}^{-1}\left[v^2 E_\mu\left[\frac{\partial^2 w_s(x,t)}{\partial t^2} + \frac{\partial w_s(x,t)}{\partial t} + w_s(x,t)\right]\right], \qquad s = 0,1,2,\ldots \tag{62}$$

For $s = 0$ in Eq. (62), we obtain

$$w_1(x,t) = E_\mu{}^{-1}\left[v^2 E_\mu\left[\frac{\partial^2 w_0(x,t)}{\partial t^2} + \frac{\partial w_0(x,t)}{\partial t} + w_0(x,t)\right]\right], \tag{63}$$

$$w_1(x,t) = E_\mu{}^{-1}[v^4\, e^{-t} + e^{-t} v^5] = e^{-t}\frac{x^{2\alpha}}{2!\,\alpha^2} + e^{-t}\frac{x^{3\alpha}}{3!\,\alpha^3}. \tag{64}$$

We get the subsequent terms, recursively

$$w_2(x,t) = E_\mu{}^{-1}\left[v^2 E_\mu\left[\frac{\partial^2 w_1(x,t)}{\partial t^2} + \frac{\partial w_1(x,t)}{\partial t} + w_1(x,t)\right]\right] = e^{-t}\frac{x^{4\alpha}}{4!\,\alpha^4} + e^{-t}\frac{x^{5\alpha}}{5!\,\alpha^5}, \tag{65}$$

$$w_3(x,t) = E_\mu{}^{-1}\left[v^2 E_\mu\left[\frac{\partial^2 w_2(x,t)}{\partial t^2} + \frac{\partial w_2(x,t)}{\partial t} + w_2(x,t)\right]\right] = e^{-t}\frac{x^{6\alpha}}{6!\,\alpha^6} + e^{-t}\frac{x^{7\alpha}}{7!\,\alpha^7}. \tag{66}$$

$$\vdots$$

Proceeding in a similar way, we obtain

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots = e^{-t} + e^{-t}\frac{x^\mu}{\mu} + e^{-t}\frac{x^{2\alpha}}{2!\,\alpha^2}$$

$$+e^{-t}\frac{x^{3\mu}}{3!\,\mu^3} + e^{-t}\frac{x^{4\mu}}{4!\,\mu^4} + e^{-t}\frac{x^{5\mu}}{5!\,\mu^5} + e^{-t}\frac{x^{6\mu}}{6!\,\mu^6} + e^{-t}\frac{x^{7\mu}}{7!\,\mu^7} + \cdots \tag{67}$$

Subsituting $\mu = 1$ in Eq. (67), then CFEDM solution is reduced as

$$w(x,t) = e^{-t}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right]. \tag{68}$$

This result is evaluated to the exact solution in a closed form:

$$w(x,t) = e^{x-t}. \tag{69}$$

The CFEDM solutions of $w(x,t)$ is found to be in excellent agreement with the exact solution of problem. For more understanding the results for the variable $w(x,t)$ of Example 5.2 are plotted in Figure 3. In Figure 3, we conclude that this solution is higher accuracy.

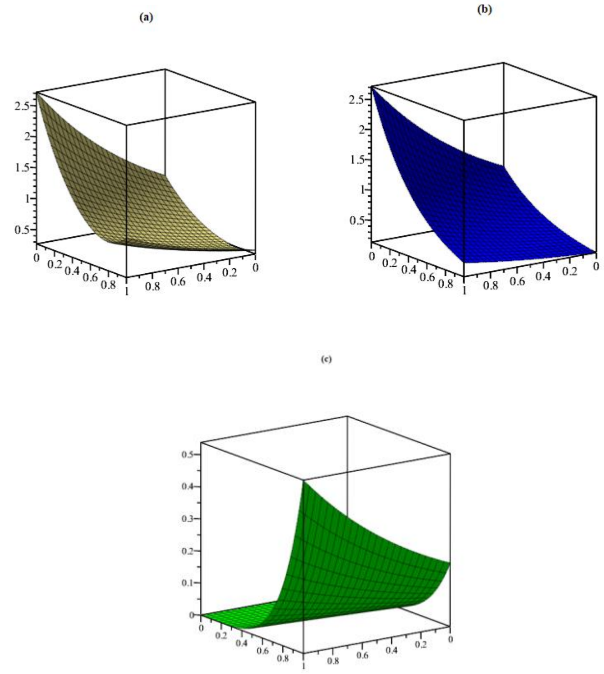

Fig. 3. (a) Nature of CFEDM solution for $w(x,t)$ **(b)** Nature of exact solution for $w(x,t)$ **(c)** Nature of absolute error=$|u_{exact} - u_{CFEDM}|$ in Ex. 5.2 at $\mu = 1$.

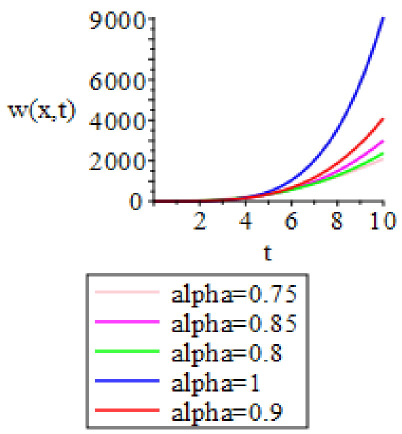

Fig. 4. Nature of CFEDM solution for $w(x,t)$ in Ex. 5.2 at $x = 0.5$ with distinct μ.

Table 2. Numerical solution of $w(x,t)$ for CSFLTE by CFEDM in Ex. 4.2 at with distinct values of x and t for diverse μ.

x	t	$\mu = 0.75$	$\mu = 0.8$	$\mu = 0.85$	$\mu = 0.9$	$\mu = 1$
0.1	0.001	1.3×10^{-2}	8.4×10^{-3}	5.0×10^{-3}	2.6×10^{-3}	2.9×10^{-16}
	0.002	2.0×10^{-2}	1.3×10^{-2}	8.5×10^{-3}	4.6×10^{-3}	9.4×10^{-15}
	0.003	2.6×10^{-2}	1.8×10^{-2}	1.1×10^{-2}	6.3×10^{-3}	7.1×10^{-14}
	0.004	3.1×10^{-2}	2.2×10^{-2}	1.4×10^{-2}	7.9×10^{-3}	3.0×10^{-13}
	0.005	3.5×10^{-2}	2.5×10^{-2}	1.6×10^{-2}	9.4×10^{-3}	9.1×10^{-13}
0.2	0.001	1.4×10^{-2}	9.3×10^{-2}	5.5×10^{-3}	2.9×10^{-3}	3.2×10^{-16}
	0.002	2.2×10^{-2}	1.5×10^{-2}	9.4×10^{-3}	5.1×10^{-3}	1.0×10^{-14}
	0.003	2.9×10^{-2}	2.0×10^{-2}	1.2×10^{-2}	7.0×10^{-3}	7.9×10^{-14}
	0.004	3.4×10^{-2}	2.4×10^{-2}	1.5×10^{-2}	8.8×10^{-3}	3.3×10^{-13}
	0.005	3.9×10^{-2}	2.8×10^{-2}	1.8×10^{-2}	1.0×10^{-2}	1.0×10^{-12}
0.3	0.001	1.6×10^{-2}	1.0×10^{-2}	0.6×10^{-3}	3.2×10^{-3}	3.5×10^{-16}
	0.002	2.5×10^{-2}	1.6×10^{-2}	1.0×10^{-2}	5.6×10^{-3}	1.1×10^{-14}
	0.003	3.2×10^{-2}	2.2×10^{-2}	1.4×10^{-2}	7.8×10^{-3}	8.7×10^{-14}
	0.004	3.8×10^{-2}	2.7×10^{-2}	1.7×10^{-2}	9.7×10^{-3}	3.6×10^{-13}
	0.005	4.3×10^{-2}	3.1×10^{-2}	2.0×10^{-2}	1.1×10^{-2}	1.1×10^{-12}
0.4	0.001	1.7×10^{-2}	1.1×10^{-2}	6.7×10^{-3}	3.5×10^{-3}	3.9×10^{-16}
	0.002	2.7×10^{-2}	1.8×10^{-2}	1.1×10^{-2}	6.2×10^{-3}	1.2×10^{-14}
	0.003	3.5×10^{-2}	2.4×10^{-2}	1.5×10^{-2}	8.6×10^{-3}	9.6×10^{-14}
	0.004	4.2×10^{-2}	3.0×10^{-2}	1.9×10^{-2}	1.0×10^{-2}	4.0×10^{-13}
	0.005	4.8×10^{-2}	3.4×10^{-2}	2.2×10^{-2}	1.2×10^{-2}	1.2×10^{-12}
0.5	0.001	1.9×10^{-2}	1.2×10^{-2}	7.5×10^{-3}	3.9×10^{-3}	4.3×10^{-16}
	0.002	3.0×10^{-2}	2.0×10^{-2}	1.2×10^{-2}	6.9×10^{-3}	1.4×10^{-14}
	0.003	3.9×10^{-2}	2.7×10^{-2}	1.7×10^{-2}	9.5×10^{-3}	1.0×10^{-13}
	0.004	4.7×10^{-2}	3.3×10^{-2}	2.1×10^{-2}	1.1×10^{-2}	4.4×10^{-13}
	0.005	5.3×10^{-2}	3.8×10^{-2}	2.4×10^{-2}	1.4×10^{-2}	1.3×10^{-12}

6. Discussion

In Figure 1, the behaviours of the exact solution, CFEDM solution and absolute error for Ex. 5.1 are plotted. Therefore, we observe that CFEDM solution is close to the exact solution. The numerical solutions for different μ values are evaluated in Table 1. From Table 1, it has been observed that the solutions get closer to the exact solution, when μ gets closer to 1. Also, especially for $\mu = 1$, it is concluded that the absolute error is extremely small in Table1. Similarly, the behaviours of the exact solution, CFEDM solution and absolute error for Ex. 5.2 are plotted in Figure 3. Therefore, we observe that CFEDM solution is close to the exact solution. The numerical solutions for different μ values are evaluated in Table 2. From Table 2, it has been observed that the solutions get closer to the exact solution, when μ gets closer to 1. Also, especially for $\mu = 1$, it is concluded that the absolute error is extremely small in Table 2. Additionally, 2D graphs of solutions of Ex. 5.1 and Ex. 5.2 for distinct μ values illustrate the behavior of CFEDM in Figures 2 and 5.

7. uld be given briefly. Besides, forward-looking suggestions and opinions related to the study results can be stated.

In the present framework, we profitably applied a new hybrid method, namely CFEDM to solve the conformable time-fractional linear telegraph equations. During the investigation, the obtained solutions are illustrated in terms of plots and tables with diverse values of space and time variables. We have observed that CFEDM is powerful, fast and efficient method to solve the conformable time-fractional linear telegraph equations.

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