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Hyper-Dual Leonardo Numbers

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Abstract

In the present paper, the hyper-dual Leonardo numbers will be introduced with the use of Leonardo numbers. Some algebraic properties of these numbers such as recurrence relation, generating function, Catalan's and Cassini's identity, Binet's formula, sum formulas will also be obtained.

Keywords: *Fibonacci numbers; Hyper-dual numbers; Leonardo numbers.*
2010 Mathematics Subject Classification: *11B37; 11B39; 11R52; 05A15.*

1. Introduction

The algebra of dual numbers was introduced by W. Clifford in 1873 [6]. An extension of dual numbers called hyper-dual numbers have been introduced by Fike to accomplish second derivative problem in the complex-step derivative approximation [9]. For the calculations first and second derivatives, Fike and Alonso have used hyper-dual numbers in [10]. In [7, 8], an expression for a hyper-dual numbers in terms of two dual numbers have been obtained by Cohen and Shoham, also, hyper-dual vectors have been examined in the sense of Study and Kotelnikov. It is important to study hyper dual numbers since its arithmetic allows us to calculate derivatives without truncation and subtractive cancellation errors and it is applicable to arbitrarily complex software [8, 10].
There are several fascinating sequences of integers. The foremost wide studied sequence of numbers is Fibonacci sequence. In the existing literature, one can find many papers on Fibonacci and Lucas numbers. (See [16, 13, 12]). Moreover, they have been examined on different number systems, for example, quaternions and hybrid numbers [11, 18, 2, 14, 15, 22]. It is well-known that Fibonacci and Lucas sequences are defined as follows: for $n \geq 0$,

$$F_{n+2} = F_{n+1} + F_n,$$

$$L_{n+2} = L_{n+1} + L_n,$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$, respectively. The Binet's formulas for F_n and L_n as

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi},$$

$$L_n = \phi^n + \psi^n, \tag{1.1}$$

where ϕ and ψ are the roots of the characteristic equation $x^2 - x - 1 = 0$.
During the paper, we consider Leonardo sequence which has similar properties with Fibonacci sequence and denoted the nth Leonardo numbers by L_{e_n}. Some properties of Leonardo numbers have been given by Catarino and Borges in [4] and it is noteworthy to recall that Leonardo sequence is defined by the following recurrence relation: for $n \geq 2$,

$$L_{e_n} = L_{e_{n-1}} + L_{e_{n-2}} + 1, \tag{1.2}$$

with the initial conditions $L_{e_0} = L_{e_1} = 1$. One can find large number of sequences indexed in *The Online Encyclopedia of Integer Sequences*, being in this case $\{L_{e_n}\} : A001595$ in [21].

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Also, the following holds for Leonardo numbers for $n \geq 2$,

$$L_{e_{n+1}} = 2L_{e_n} - L_{e_{n-2}}. \tag{1.3}$$

The Binet's formula for Leonardo numbers is

$$L_{e_n} = \frac{2\phi^{n+1} - 2\psi^{n+1} - \phi + \psi}{\phi - \psi}. \tag{1.4}$$

Here ϕ and ψ are roots of characteristic equation $x^3 - 2x^2 + 1 = 0$.
From the Binet formula, the relationship between Leonardo and Fibonacci numbers is

$$L_{e_n} = 2F_{n+1} - 1, \tag{1.5}$$

where F_n is nth Fibonacci number.
In [4], Cassini's identity, Catalan's identity and d'Ocagne's identity have been obtained for Leonardo numbers by Catarino and et.al. Moreover they have presented the two-dimensional recurrences relations and matrix representation of Leonardo numbers. In [20], Shannon have defined generalized Leonardo numbers which are considered Asveld's extension and Horadam's generalized sequence.
Now, we are ready to recall some identities involving Fibonacci, Lucas and Leonardo numbers as follows. For more details related to them, please refer [1, 4, 16, 22].

$$F_n + L_n = 2F_{n+1}, \tag{1.6}$$

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_iF_j, \tag{1.7}$$

$$L_{e_{n+m}} + (-1)^m L_{e_{n-m}} = L_m(L_{e_n} + 1) - 1 - (-1)^m, \tag{1.8}$$

$$L_{e_{n+m}} - (-1)^m L_{e_{n-m}} = L_{n+1}(L_{e_{m-1}} + 1) - 1 + (-1)^m, \tag{1.9}$$

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r} F_r^2, \tag{1.10}$$

$$L_{r+s} - (-1)^s L_{r-s} = 5F_rF_s, \tag{1.11}$$

$$\sum_{k=1}^{n} (-1)^{k-1} F_{k+1} = (-1)^{n-1} F_n. \tag{1.12}$$

Recently, many researchers considered two dimensional number systems. One of them is hyper-dual numbers that are an extension of dual numbers. The hyper-dual numbers are denoted by HD and are defined as

$$HD = \left\{ \begin{array}{l} x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_1\varepsilon_2 : x_0, x_1, x_2, x_3 \in \mathbb{R}, \\ \varepsilon_1^2 = \varepsilon_2^2 = 0, \varepsilon_1 \neq 0, \varepsilon_2 \neq 0,\ \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \neq 0 \end{array} \right\}.$$

Let $x = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_1\varepsilon_2$ and $y = y_0 + y_1\varepsilon_1 + y_2\varepsilon_2 + y_3\varepsilon_1\varepsilon_2$ be any two hyper-dual numbers. Then the main operations on hyper-dual numbers are defined as follows:
$x = y$ if and only if $x_0 = y_0$, $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$.
$x + y = (x_0 + y_0) + (x_1 + y_1)\varepsilon_1 + (x_2 + y_2)\varepsilon_2 + (x_3 + y_3)\varepsilon_1\varepsilon_2$.
$kx = kx_0 + kx_1\varepsilon_1 + kx_2\varepsilon_2 + kx_3\varepsilon_1\varepsilon_2$ where $k \in \mathbb{R}$.
The product of any two hyper-dual numbers is
$x \cdot y = (x_0y_0) + (x_0y_1 + x_1y_0)\varepsilon_1 + (x_0y_2 + x_2y_0)\varepsilon_2 + (x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1)\varepsilon_1\varepsilon_2$.
There are many generalizations of Fibonacci and Lucas numbers in the existing literature. One of these generalizations is the hyper-dual generalized Fibonacci and Lucas numbers and they have been defined by Omur and some properties have been presented in [19]. They have defined hyper-dual generalized Fibonacci and Lucas numbers as follows:

$$\widetilde{U_{k_r}} = U_{k_r} + U_{k_{(r+1)}}\varepsilon_1 + U_{k_{(r+2)}}\varepsilon_2 + U_{k_{(r+3)}}\varepsilon_1\varepsilon_2, \tag{1.13}$$

and

$$\widetilde{V_{k_r}} = V_{k_r} + V_{k_{(r+1)}}\varepsilon_1 + V_{k_{(r+2)}}\varepsilon_2 + V_{k_{(r+3)}}\varepsilon_1\varepsilon_2, \tag{1.14}$$

where $\{U_{k_r}\}$ and $\{V_{k_r}\}$ are generalized Fibonacci sequence and generalized Lucas sequences, respectively. In [17] the authors have investigated Leonardo Pisano polynomials and hybrinomials with the use of the Leonardo Pisano numbers and hybrid numbers. They also describe the basic algebraic properties and some identities of the Leonardo Pisano polynomials and hybrinomials.
With the motivation of there mentioned papers, here, we introduce a hyper-dual numbers with Leonardo number components. We also aim to obtain generating function, Binet's formula, recurrence relation, summation formula, Catalan's, Cassini's and other identities.
For more details about Leonardo numbers, see [3, 4, 5, 17].

2. Hyper-Dual Leonardo Numbers

In this section, we define the hyper-dual Leonardo numbers. Then, we obtain generating function, Binet's formula, summation formulas, Catalan's identity, Cassini's identity and other identities.
For $n \geqslant 1$, the nth hyper-dual Leonardo numbers are defined by

$$HDL_{e_n} = L_{e_n} + L_{e_{n+1}}\varepsilon_1 + L_{e_{n+2}}\varepsilon_2 + L_{e_{n+3}}\varepsilon_1\varepsilon_2. \tag{2.1}$$

Note that, throughout this paper, nth hyper-dual Leonardo numbers are denoted by HDL_{e_n}.
From the recurrence relation (1.2) and the definition of hyper-dual Leonardo numbers (2.1), for $n \geq 2$ we get

$$\begin{aligned} HDL_{e_n} &= (L_{e_{n-1}} + L_{e_{n-2}} + 1) + (L_{e_n} + L_{e_{n-1}} + 1)\varepsilon_1 \\ &\quad + (L_{e_{n+1}} + L_{e_n} + 1)\varepsilon_2 + (L_{e_{n+2}} + L_{e_{n+1}} + 1)\varepsilon_1\varepsilon_2, \\ &= HDL_{e_{n-1}} + HDL_{e_{n-2}} + A. \end{aligned}$$

Here $A = 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2$ and this notation will be used in the whole paper. Also initial conditions are $HDL_{e_0} = 1 + \varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_1\varepsilon_2$ and $HDL_{e_1} = 1 + 3\varepsilon_1 + 5\varepsilon_2 + 9\varepsilon_1\varepsilon_2$.
Now, let us give another recurrence relation of hyper-dual Leonardo numbers. That is

$$HDL_{e_{n+1}} = 2HDL_{e_n} - HDL_{e_{n-2}}. \tag{2.2}$$

Using the definition of hyper-dual Leonardo numbers (2.1) and the recurrence relation of Leonardo numbers (1.3), for $n \geqslant 2$ we get

$$\begin{aligned} HDL_{e_{n+1}} &= 2L_{e_n} - L_{e_{n-2}} + (2L_{e_{n+1}} - L_{e_{n-1}})\varepsilon_1 \\ &\quad + (2L_{e_{n+2}} - L_{e_n})\varepsilon_2 + (2L_{e_{n+3}} - L_{e_{n+1}})\varepsilon_1\varepsilon_2, \\ &= L_{e_{n+1}} + L_{e_{n+2}}\varepsilon_1 + L_{e_{n+3}}\varepsilon_2 + L_{e_{n+4}}\varepsilon_1\varepsilon_2, \\ &= 2HDL_{e_n} - HDL_{e_{n-2}}, \end{aligned}$$

with the initial values $HDL_{e_0} = 1 + \varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_1\varepsilon_2$ and $HDL_{e_1} = 1 + 3\varepsilon_1 + 5\varepsilon_2 + 9\varepsilon_1\varepsilon_2$.

Theorem 2.1. *The generation function for the hyper-dual Leonardo numbers, denoted by $gHDL_{e_n}(t)$, is*

$$gHDL_{e_n}(t) = \frac{HDL_{e_0} + t(-1 + \varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2) + t^2(1 - \varepsilon_1 - \varepsilon_2 - 3\varepsilon_1\varepsilon_2)}{1 - 2t + t^3}.$$

Proof. Let the formal power series representation of the generating function for $\{HDL_{e_n}\}_{n=0}^{\infty}$ be as

$$gHDL_{e_n}(t) = \sum_{n=0}^{\infty} HDL_{e_n} t^n. \tag{2.3}$$

That is,

$$gHDL_{e_n}(t) = HDL_{e_0} + HDL_{e_1}t + HDL_{e_2}t^2 + \ldots + HDL_{e_k}t^k + \ldots$$

Then we have

$$\begin{aligned} \left(1 - 2t + t^3\right) gHDL_{e_n}(t) &= \left(1 - 2t + t^3\right) \begin{pmatrix} HDL_{e_0} + HDL_{e_1}t \\ +HDL_{e_2}t^2 + \ldots + HDL_{e_k}t^k + \ldots \end{pmatrix} \\ \left(1 - 2t + t^3\right) gHDL_{e_n}(t) &= HDL_{e_0} + HDL_{e_1}t + HDL_{e_2}t^2 + \ldots + \\ &\quad -2HDL_{e_0}t - 2HDL_{e_1}t^2 - 2HDL_{e_2}t^3 - \ldots \\ &\quad + HDL_{e_0}t^3 + HDL_{e_1}t^4 + HDL_{e_2}t^5 + \ldots \\ &= HDL_{e_0} + t\left(HDL_{e_1} - 2HDL_{e_0}\right) + t^2\left(HDL_{e_2} - 2HDL_{e_1}\right) \\ &\quad + t^3\left(HDL_{e_3} - 2HDL_{e_2} + HDL_{e_0}\right) + \ldots \\ &\quad + t^{k+1}\left(HDL_{e_{k+1}} - 2HDL_{e_k} + HDL_{e_{k-2}}\right) + \ldots \end{aligned}$$

Since the recurrence relation of hyper-dual numbers (2.2) and also by using initial conditions, we get

$$\begin{aligned} gHDL_{e_n}(t)(1 - 2t + t^3) &= (1 + \varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_1\varepsilon_2) \\ &\quad + t(-1 + \varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2) + t^2(1 - \varepsilon_1 - \varepsilon_2 - 3\varepsilon_1\varepsilon_2). \end{aligned}$$

Therefore we get the the generating function for $\{HDL_{e_n}\}_{n=0}^{\infty}$ as

$$\sum_{n=0}^{\infty} HDL_{e_n} t^n = \frac{HDL_{e_0} + t(-1 + \varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2) + t^2(1 - \varepsilon_1 - \varepsilon_2 - 3\varepsilon_1\varepsilon_2)}{1 - 2t + t^3}.$$

□

Theorem 2.2.

$$HDL_{e_n} = 2\widetilde{U}_{n+1} - A, \tag{2.4}$$

holds for any integer $n \geqslant 0$. Here $\widetilde{U}_n$ is nth hyper-dual Fibonacci number.

Proof. Using the definition of hyper-dual Leonardo numbers (2.1) and the recurrence relation between Leonardo and Fibonacci numbers (1.5) we get

$$\begin{aligned} HDL_{e_n} &= L_{e_n} + L_{e_{n+1}}\varepsilon_1 + L_{e_{n+2}}\varepsilon_2 + L_{e_{n+3}}\varepsilon_1\varepsilon_2, \\ &= (2F_{n+1}-1) + (2F_{n+2}-1)\varepsilon_1 \\ &\quad +(2F_{n+3}-1)\varepsilon_2 + (2F_{n+4}-1)\varepsilon_1\varepsilon_2, \\ &= 2(F_{n+1} + F_{n+2}\varepsilon_1 + F_{n+3}\varepsilon_2 + F_{n+4}\varepsilon_1\varepsilon_2) - A, \\ &= 2\widetilde{U}_{n+1} - A. \end{aligned}$$

□

Theorem 2.3. *The Binet's formula for the hyper-dual Leonardo numbers HDL_{e_n}*

$$HDL_{e_n} = 2\left(\frac{\underline{\Phi}\phi^{n+1} - \underline{\Psi}\psi^{n+1}}{\phi - \psi}\right) - A, \tag{2.5}$$

holds for any integer $n \geqslant 0$. Here ϕ and ψ are roots of characteristic equation $x^3 - 2x^2 + 1 = 0$, $\underline{\Phi} = 1 + \phi\varepsilon_1 + \phi^2\varepsilon_2 + \phi^3\varepsilon_1\varepsilon_2$ and $\underline{\Psi} = 1 + \psi\varepsilon_1 + \psi^2\varepsilon_2 + \psi^3\varepsilon_1\varepsilon_2$.

Proof. By using the definition of hyper-dual Leonardo numbers (2.1) and the Binet's formula of Leonardo numbers (1.4), we get

$$HDL_{e_n} = 2\left(\begin{array}{c} \frac{\phi^{n+1}-\psi^{n+1}}{\phi-\psi} + \frac{\phi^{n+2}-\psi^{n+2}}{\phi-\psi}\varepsilon_1 \\ + \frac{\phi^{n+3}-\psi^{n+3}}{\phi-\psi}\varepsilon_2 + \frac{\phi^{n+4}-\psi^{n+4}}{\phi-\psi}\varepsilon_1\varepsilon_2 \end{array} \right) - (1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2).$$

If the expression $\underline{\Phi} = 1 + \phi\varepsilon_1 + \phi^2\varepsilon_2 + \phi^3\varepsilon_1\varepsilon_2$, $\underline{\Psi} = 1 + \psi\varepsilon_1 + \psi^2\varepsilon_2 + \psi^3\varepsilon_1\varepsilon_2$ are used in the last equation, we can easily obtained the result. □

Theorem 2.4. *For $n \geq 0$ summation formulas of hyper-dual Leonardo numbers are*

1) $\sum\limits_{r=0}^{n} HDL_{e_r} = HDL_{e_{n+2}} - (n+2)A - (2\varepsilon_1 + 4\varepsilon_2 + 8\varepsilon_1\varepsilon_2)$,
2) $\sum\limits_{r=0}^{n} HDL_{e_{2r}} = HDL_{e_{2n+1}} - nA - (2\varepsilon_1 + 2\varepsilon_2 + 4\varepsilon_1\varepsilon_2)$,
3) $\sum\limits_{r=0}^{n} HDL_{e_{2r+1}} = HDL_{e_{2n+1}} - (n+2)A - (2\varepsilon_2 + 4\varepsilon_1\varepsilon_2)$,
also for $n \geq 1$
4) $\sum\limits_{r=0}^{n} (-1)^{r-1} HDL_{e_r} = \begin{cases} -\left(HDL_{e_{n-1}} + 2 + \varepsilon_2 + \varepsilon_1\varepsilon_2\right), & n \text{ is even} \\ HDL_{e_{n-1}} - 1 + \varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2, & n \text{ is odd} \end{cases}$.

Proof. With the use of the sums and products of terms of the Leonardo sequence in [4] and also the definition of hyper-dual Leonardo numbers, the proof of (1), (2) and (3) follows easily.
Finally to prove (4) we use definition of the hyper-dual Leonardo numbers, we have

$$\begin{aligned} \sum_{r=0}^{n} (-1)^{r-1} HDL_{e_r} &= \sum_{r=0}^{n} (-1)^{r-1} L_{e_r} + \varepsilon_1 \sum_{r=0}^{n} (-1)^{r-1} L_{e_{r+1}} \\ &\quad + \varepsilon_2 \sum_{r=0}^{n} (-1)^{r-1} L_{e_{r+2}} + \varepsilon_1\varepsilon_2 \sum_{r=0}^{n} (-1)^{r-1} L_{e_{r+3}}. \end{aligned}$$

By using (1.2), (1.5) and (1.12) we get

$$\sum_{r=0}^{n} (-1)^{r-1} HDL_{e_r} = \begin{cases} -2\widetilde{U}_n - 1 + \varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2, & n \text{ is even} \\ 2\widetilde{U}_n - 2 - 2\varepsilon_2 - 2\varepsilon_1\varepsilon_2, & n \text{ is odd} \end{cases}$$

where $\widetilde{U}_n$ is *nth* hyper-dual Fibonacci number. Then from (2.4) we obtain the desired result. □

Now we present the following interesting identities in accordance with the Binet's formula (2.5) for the sequence $\{L_{e_n}\}$.

Theorem 2.5. ***(Catalan's Identity)*** *For positive integers n and r with $n \geq r$, we have*

$$\begin{aligned} & HDL_{e_n}^2 - HDL_{e_{n-r}} HDL_{e_{n+r}} \\ = \; & (HDL_{e_{n-r}} + HDL_{e_{n+r}} - 2HDL_{e_n})A \\ & +4(-1)^{n-r+1}(2(\varepsilon_2 + \varepsilon_1\varepsilon_2) + A)F_r^2, \end{aligned} \tag{2.6}$$

where F_n is nth Fibonacci number.

Proof. First use the Binet's formula (2.5) to left hand side (LHS), then one can get that

$$\begin{aligned} LHS &= \left(2\left(\frac{\underline{\Phi}\phi^{n+1}-\underline{\Psi}\psi^{n+1}}{\phi-\psi}\right)-A\right)\left(2\left(\frac{\underline{\Phi}\phi^{n+1}-\underline{\Psi}\psi^{n+1}}{\phi-\psi}\right)-A\right) \\ &\quad -\left(2\left(\frac{\underline{\Phi}\phi^{n-r+1}-\underline{\Psi}\psi^{n-r+1}}{\phi-\psi}\right)-A\right)\left(2\left(\frac{\underline{\Phi}\phi^{n+r+1}-\underline{\Psi}\psi^{n+r+1}}{\phi-\psi}\right)-A\right). \end{aligned} \tag{2.7}$$

By considering ϕ, ψ, $\underline{\Phi}=1+\phi\varepsilon_1+\phi^2\varepsilon_2+\phi^3\varepsilon_1\varepsilon_2$ and $\underline{\Psi}=1+\psi\varepsilon_1+\psi^2\varepsilon_2+\psi^3\varepsilon_1\varepsilon_2$, one can also have

$$\underline{\Phi\Psi}=1+\varepsilon_1+3\varepsilon_2+3\varepsilon_1\varepsilon_2. \tag{2.8}$$

By taking into account (1.10) and (2.8) in (LHS),

$$\begin{aligned} LHS &= (HDL_{e_{n-r}}+HDL_{e_{n+r}}-2HDL_{e_n})A \\ &\quad +4(-1)^{n-r+1}(2(\varepsilon_2+\varepsilon_1\varepsilon_2)+A)F_r^2. \end{aligned}$$

This completes the proof. □

Remark that in the case $r=1$ in (2.6), it reduces to Cassini's identity of the hyper-dual Leonardo numbers.

Corollary 2.6. ***(Cassini's Identity)*** *For $n\geq 1$, we have*

$$\begin{aligned} HDL_{e_n}^2-HDL_{e_{n-1}}HDL_{e_{n+1}} &= (HDL_{e_{n-1}}-HDL_{e_n-2})A \\ &\quad +4(-1)^n(2(\varepsilon_2+\varepsilon_1\varepsilon_2)+A). \end{aligned}$$

Theorem 2.7. *Let k, m, and s be positive integers. Then the following holds between the Fibonacci numbers and hyper-dual Leonardo numbers*

$$\begin{aligned} & HDL_{e_{k+m}}HDL_{e_{k+s}}-HDL_{e_k}HDL_{e_{k+m+s}} \\ =\ & (HDL_{e_k}-HDL_{e_{k+m}}+HDL_{e_{k+m+s}}-HDL_{e_{k+s}})A \\ & +4(-1)^{k+1}F_mF_s\left(2\left(\varepsilon_2+\varepsilon_1\varepsilon_2\right)+A\right), \end{aligned}$$

where F_n is nth Fibonacci number.

Proof. By using the Binet's formula (2.5) to left hand side (LHS), we get

$$\begin{aligned} LHS &= \left(\frac{2\underline{\Phi}\phi^{k+m+1}-2\underline{\Psi}\psi^{k+m+1}}{\phi-\psi}-A\right)\left(\frac{2\underline{\Phi}\phi^{k+s+1}-2\underline{\Psi}\psi^{k+s+1}}{\phi-\psi}-A\right) \\ &\quad -\left(\frac{2\underline{\Phi}\phi^{k+1}-2\underline{\Psi}\psi^{k+1}}{\phi-\psi}-A\right)\left(\frac{2\underline{\Phi}\phi^{k+m+s+1}-2\underline{\Psi}\psi^{k+m+s+1}}{\phi-\psi}-A\right), \\ &= (HDL_{e_k}-HDL_{e_{k+m}}+HDL_{e_{k+m+s}}-HDL_{e_{k+s}})A \\ &\quad +\frac{4\underline{\Phi\Psi}}{(\phi-\psi)^2}\left(\phi^{k+1}\psi^{k+1}\left(-\phi^m\psi^s-\phi^s\psi^m+\psi^{m+s}+\phi^{m+s}\right)\right). \end{aligned}$$

Then by using the Vajda's identity for Fibonacci numbers (1.7) and (2.8), we have

$$\begin{aligned} LHS &= (HDL_{e_k}-HDL_{e_{k+m}}+HDL_{e_{k+m+s}}-HDL_{e_{k+s}})A \\ &\quad +4(-1)^{k+1}F_mF_s\left(2\left(\varepsilon_2+\varepsilon_1\varepsilon_2\right)+A\right). \end{aligned}$$

□

Theorem 2.8. *For positive integers n and m, with $n\geq m$, then the following identities between the Lucas, Leonardo, hyper-dual Lucas and hyper-dual Leonardo numbers are provided:*

$$HDL_{e_{n+m}}+(-1)^mHDL_{e_{n-m}} = L_mHDL_{e_n}+(L_m-(-1)^m-1)A, \tag{2.9}$$

$$HDL_{e_{n+m}}-(-1)^mHDL_{e_{n-m}} = (L_{e_{m-1}}+1)\widetilde{V}_{n+1}+((-1)^m-1)A, \tag{2.10}$$

where L_n is nth Lucas number, L_{e_n} is nth Leonardo number, $\widetilde{V}_n$ is nth hyper-dual Lucas number.

Proof. For the proof (2.9), using the definition of hyper-dual Leonardo numbers to left hand side (LHS), we get

$$\begin{aligned} LHS &= (L_{e_{n+m}}+(-1)^mL_{e_{n-m}})+(L_{e_{n+m+1}}+(-1)^mL_{e_{n-m+1}})\varepsilon_1 \\ &\quad +(L_{e_{n+m+2}}+(-1)^mL_{e_{n-m+2}})\varepsilon_2+(L_{e_{n+m+3}}+(-1)^mL_{e_{n-m+3}})\varepsilon_1\varepsilon_2. \end{aligned}$$

Then by using (1.8), we obtain

$$LHS=L_mHDL_{e_n}+A(L_m-(-1)^m-1).$$

The proof of (2.10) can be made by the same way. □

Theorem 2.9. *For positive integers m, r and s with $m \geq r$ and $m \geq s$, then the following holds between the Fibonacci and hyper-dual Leonardo numbers*

$$\begin{aligned} & HDL_{e_{m+r}} HDL_{e_{m-r}} - HDL_{e_{m+s}} HDL_{e_{m-s}} \\ = & (HDL_{e_{m+s}} - HDL_{e_{m+r}} + HDL_{e_{m-s}} - HDL_{e_{m-r}})A \\ & +4\left(A+2\left(\varepsilon_2+\varepsilon_1\varepsilon_2\right)\right)\left((-1)^{m-r}F_r^2-(-1)^{m-s}F_s^2\right), \end{aligned}$$

where F_n is nth Fibonacci number.

Proof. By using the Binet's formula of the hyper-dual Leonardo numbers (2.5) to left hand side (LHS), we have

$$\begin{aligned} LHS &= \left(2\frac{\underline{\Phi}\phi^{m+r+1}-\underline{\Psi}\psi^{m+r+1}}{\phi-\psi}-A\right)\left(2\frac{\underline{\Phi}\phi^{m-r+1}-\underline{\Psi}\psi^{m-r+1}}{\phi-\psi}-A\right) \\ & \quad -\left(2\frac{\underline{\Phi}\phi^{m+s+1}-\underline{\Psi}\psi^{m+s+1}}{\phi-\psi}-A\right)\left(2\frac{\underline{\Phi}\phi^{m-s+1}-\underline{\Psi}\psi^{m-s+1}}{\phi-\psi}-A\right). \\ &= (HDL_{e_{m+s}} - HDL_{e_{m+r}} + HDL_{e_{m-s}} - HDL_{e_{m-r}})A \\ & \quad +\frac{4\underline{\Phi}\underline{\Psi}}{(\phi-\psi)^2}\left(\phi^m\psi^m\left(\phi^r\psi^{-r}+\psi^r\phi^{-r}-\phi^s\psi^{-s}-\psi^s\phi^{-s}\right)\right). \end{aligned}$$

Also by using (1.10) and (2.8) in (LHS), we get

$$\begin{aligned} LHS &= (HDL_{e_{m+s}} - HDL_{e_{m+r}} + HDL_{e_{m-s}} - HDL_{e_{m-r}})A \\ & \quad +4\left(2\left(\varepsilon_2+\varepsilon_1\varepsilon_2\right)+A\right)\left((-1)^{m-r}F_r^2-(-1)^{m-s}F_s^2\right). \end{aligned}$$

□

Theorem 2.10. *Let be positive integers $n, m, s,$ and r with $n \geq m$, $s \geq r$. For the conditions $n+m = s+r$, then the following identity between the Lucas and hyper-dual Leonardo numbers is provided:*

$$\begin{aligned} & HDL_{e_n} HDL_{e_m} - HDL_{e_s} HDL_{e_r} \\ = & (HDL_{e_s} - HDL_{e_n} + HDL_{e_r} - HDL_{e_m})A \\ & +\frac{4}{5}\left(2(\varepsilon_2+\varepsilon_1\varepsilon_2)+A\right)\left((-1)^m L_{n-m}-(-1)^r L_{s-r}\right), \end{aligned}$$

where L_n is nth Lucas number.

Proof. By using the Binet's formula of hyper-dual Leonardo numbers (2.5) to left hand side (LHS), we get

$$\begin{aligned} LHS &= \left(2\frac{\underline{\Phi}\phi^{n+1}-\underline{\Psi}\psi^{n+1}}{\phi-\psi}-A\right)\left(2\frac{\underline{\Phi}\phi^{m+1}-\underline{\Psi}\psi^{m+1}}{\phi-\psi}-A\right) \\ & \quad -\left(2\frac{\underline{\Phi}\phi^{s+1}-\underline{\Psi}\psi^{s+1}}{\phi-\psi}-A\right)\left(2\frac{\underline{\Phi}\phi^{r+1}-\underline{\Psi}\psi^{r+1}}{\phi-\psi}-A\right). \\ &= (HDL_{e_s} - HDL_{e_n} + HDL_{e_r} - HDL_{e_m})A \\ & \quad +\frac{4\underline{\Phi}\underline{\Psi}}{(\phi-\psi)^2}\left(\phi^n\psi^m+\phi^m\psi^n-\phi^s\psi^r-\phi^r\psi^s\right). \end{aligned}$$

From (1.11) and (2.8), we get

$$\begin{aligned} LHS &= (HDL_{e_s} - HDL_{e_n} + HDL_{e_r} - HDL_{e_m})A \\ & \quad +\frac{4}{5}\left(2(\varepsilon_2+\varepsilon_1\varepsilon_2)+A\right)\left((-1)^m L_{n-m}-(-1)^r L_{s-r}\right). \end{aligned}$$

□

Theorem 2.11. *For r and s positive integers with $r \geq 1$, $s \geq 1$, we have*

$$\begin{aligned} & HDL_{e_{s+1}} HDL_{e_{r+1}} - HDL_{e_{s-1}} HDL_{e_{r-1}} \\ = & -(HDL_{e_s} + HDL_{e_r})A - 2A^2 \\ & +8\widetilde{U}_{s+r+2} - 4F_{s+r+2} + 8\varepsilon_1\varepsilon_2 F_{s+r+5}, \end{aligned}$$

where F_n and $\widetilde{U}_n$ are nth Fibonacci and nth hyper-dual Fibonacci numbers, respectively.

Proof. By using the Binet's formula of hyper-dual Leonardo numbers (2.5) to left hand side (LHS), we get

$$\begin{aligned} LHS &= \left(2\frac{\underline{\Phi}\phi^{s+2}-\underline{\Psi}\psi^{s+2}}{\phi-\psi}-A\right)\left(2\frac{\underline{\Phi}\phi^{r+2}-\underline{\Psi}\psi^{r+2}}{\phi-\psi}-A\right) \\ &\quad -\left(2\frac{\underline{\Phi}\phi^{s}-\underline{\Psi}\psi^{s}}{\phi-\psi}-A\right)\left(2\frac{\underline{\Phi}\phi^{r}-\underline{\Psi}\psi^{r}}{\phi-\psi}-A\right). \\ &= -(HDL_{e_s}+HDL_{e_r})A-2A^2 \\ &\quad +\frac{4}{(\phi-\psi)^2}\left(\underline{\Phi}^2\phi^{s+r}\left(\phi^4-1\right)+\underline{\Psi}^2\psi^{s+r}\left(\psi^4-1\right)\right). \end{aligned}$$

Also by using the Binet's formula for the Lucas numbers (1.1), (1.11) and (1.13), we get

$$\begin{aligned} LHS &= -(HDL_{e_s}+HDL_{e_r})A-2A^2 \\ &\quad +8\widetilde{U}_{s+r+2}-4F_{s+r+2}+8\varepsilon_1\varepsilon_2F_{s+r+5}. \end{aligned}$$

□

3. Conclusion

In the present paper, hyper-dual Leonardo numbers with coefficients of basis of Leonardo numbers have been introduces. First of all the recurrence relation and generating function for these numbers have been obtained. Then summation formulas for these numbers have been provided. Furthermore, Catalan's and Cassini's identities, and some interesting properties have been given.

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