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On The Solutions of Nonlinear Fractional Klein-Gordon Equation by Means of Local Fractional Derivative Operators

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Abstract. In this paper, an application of the local fractional decomposition method (LFDM) is analyzed to search for an approximate analytical solution of nonlinear fractional Klein-Gordon equation. The fractional derivatives are described in Jumarie's modified Riemann-Liouville sense. A new application of the local fractional decomposition method (LFDM) is extended to derive the approximate solutions in series form for this model problem. Solutions have been plotted for different values of the fractional order. It is concluded that the solutions for nonlinear partial equations with Riemann- Liouville derivative obtained with LFDM are useful, reliable and efficient.

2010 AMS Classification: 26A33

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1. Introduction

The local fractional calculus (LFC) was applied to model and process the non-differentiable phenomena in several fractal physical phenomena [3, 5, 7–10, 28, 29, 32, 39, 41–45]. Some local fractional models are the nonhomogeneous local fractional Volterra equation (NLFVE) [44], the local fractional Mechanics of Elastic Materials (LFMEM) [9], the Local fractional variational iteration method (LFVIM) [45], wave equations on the Cantor sets (WECSs) [3], local fractional Laplace equation (LFLE) [42] and Newtonian mechanics (NM) on fractals subset of real-line [17]. As known, linear and nonlinear Klein-Gordon equations have many applications in pattern formation in solitons, condensed matter physics, classical and quantum mechanics [6, 13, 33, 34, 38, 48].

The Adomian decomposition method [1, 2, 37], homotopy analysis method (HAM) [11], the variational iteration method (VIM) [19–22] and the homotopy perturbation method (HPM) [14, 16, 23, 36] were successfully applied to autonomous ordinary, partial, integral and fractional differential equations. Recently, a new modified Riemann-Liouville left derivative is suggested by Jumarie [24–28].

More recently, Golmankhaneh et al. [18] used the homotopy perturbation method (HPM), Local Fractional Series Expansion Method [47], the homotopy analysis method (HAM) [31] and the fractional variational method FVIM [33] for solving the nonlinear fractional Klein-Gordon equation.

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In this paper, we extend the application of the LFDM to derive the analytical approximate solutions of the nonlinear fractional Klein-Gordon problem

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) + bu(x,t)^2 + cu(x,t)^3,$$
$$0 < x \le 1,\ 0 < \alpha \le 1,\ t > 0,$$
$$u(x,0) = h(x),\ 0 < x \le 1,$$

where a, b and c are real constants. The aim of this paper is to extend the application of the local fractional PDEs (LF-PDEs) within local fractional derivative (LFD) to solve the fractional Klein-Gordon equations with modified Riemann-Liouville derivative.

The paper is organized as follows: In section 2, we briefly review the definitions related to the local fractional calculus theory. Section 3 deals with the solution procedure of the local fractional decomposition method method to show the inefficiency of this method. We present the application of the fractional Klein-Gordon equations with modified Riemann-Liouville derivative using LFDM and the numerical results in Section 4. The conclusions are given in the last section.

2. Basic Definitions

Here, some basic definitions and properties of the local fractional contunity (LFC), local fractional derivative (LFD) and local fractional integral (LFI) of non-differential functions, which can be found in [3,4,12,15,17,30,35,40,42,43,45,46], are given.

Definition 2.1. If there exists the relation [4,15,39,40,46]

$$|f(u) - f(u_0)| < \epsilon^\alpha \tag{2.1}$$

with $|u - u_0| < \delta$ for $\epsilon, \delta > 0$ and $\epsilon, \delta \in \mathbb{R}$. Now, $f(u)$ is called local fractional continuous at $u = u_0$ and is denoted by $\lim_{u \to u_0} f(u) = f(u_0)$. Also, $f(u)$ is called local fractional continuous on the interval (a, b) when denoted by

$$f(u) \in C_\alpha(a,b).$$

Definition 2.2. The function $f(x)$ is called a non-differentiable function of exponent $0 < \alpha \le 1$ which satisfies the Hölder function of exponent α such that for $u, v \in X$ [4,15,39,40,46]

$$|f(u) - f(v)| < C|u - v|^\alpha. \tag{2.2}$$

Definition 2.3. The function $f(x)$ is siad to be continuous of order α, $0 < \alpha \le 1$, or shortly α continuous, if the following relations [4,15,39,40,46] are satisfied; $|f(u) - f(u_0)| < \epsilon^\alpha$,

$$f(u) - f(u_0) = o((u - u_0)^\alpha). \tag{2.3}$$

Compared with (2.3), (2.1) is standard definition of local fractional continuity (LFC) and (2.2) is unified local fractional continuity (ULFC).

Definition 2.4. For $f(u) \in C_\alpha(a,b)$, the LFD of $f(x)$ of order α at $u = u_0$ is defined as [4,15,39,40,46]:

$$f^\alpha(u_0) = \left.\frac{d^\alpha f(u)}{du^\alpha}\right|_{u=u_0} = \lim_{u \to u_0} \frac{\Delta^\alpha (f(u) - f(u_0))}{(u - u_0)^\alpha},\ 0 < \alpha \le 1,$$

where $\Delta^\alpha (f(u) - f(u_0)) \simeq \Gamma(1+\alpha)\Delta(f(u) - f(u_0))$. For any $u \in (a,b)$, $f^\alpha(u) = D_u^\alpha f(u)$ exists and is denoted by $f(u) \in D_u^\alpha(a,b)$. LFD of higher order can be expressed as follows:

$$f^{(k\alpha)}(u) = \underbrace{D_u^\alpha \cdots D_u^\alpha f(u)}_{k\ times},$$

and local fractional partial derivative (LFPD) of higher order is shown as:

$$\frac{\partial^{(k\alpha)} f(u)}{\partial u^{k\alpha}} = \underbrace{\frac{\partial^\alpha}{\partial u^\alpha} \cdots \frac{\partial^\alpha}{\partial u^\alpha} f(u)}_{k\ times}.$$

Definition 2.5. For $f(u) \in C_\alpha(a,b)$, the local fractional integral (LFI) of $f(u)$ of order α in the interval $[a,b]$ is defined as [4, 12, 15, 30, 35, 40, 46]:

$$ {}_aI_b^\alpha f(u) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, 0 < \alpha \leq 1, $$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = max\{\Delta t_1, \Delta t_2, \Delta t_3, \cdots\}$ and $[\Delta t_j, \Delta t_{j+1}]$, $j = 0, 1, \cdots, N-1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a,b]$. For any $u \in (a,b)$, ${}_aI_b^\alpha f(u)$ exists and is denoted by $f(u) \in I_u^\alpha(a,b)$. If $f(u) = D_u^\alpha f(a,b)$, or $I_u^{(\alpha)}(a,b)$, then $f(u) \in C_\alpha(a,b)$. For any $f(u) \in C_\alpha(a,b), 0 < \alpha \leq 1$, we have local fractional multiple integrals (LFMIs)

$$ {}_{u_0}I_u^{(k\alpha)} f(u) = \overbrace{{}_{u_0}I_u^\alpha \cdots {}_{u_0}I_u^\alpha f(u)}^{k\ times}, $$

For any $0 < \alpha \leq 1$, if $f^{(k\alpha)}(u) \in C_\alpha^k(a,b)$, then we have

$$ ({}_{u_0}I_u^{(k\alpha)} f(u))^{(k\alpha)} = f(u), $$

where ${}_{u_0}I_u^{(k\alpha)} f(u) = \underbrace{{}_{u_0}I_u^{(\alpha)} \cdots {}_{u_0}I_u^{(\alpha)} f(u)}_{k\ times}$ and $f^{(k\alpha)}(u) = \underbrace{D_u^\alpha \cdots D_u^\alpha f(u)}_{k\ times}$.

Definition 2.6. The Mittag-Leffler function (MLF) in fractional space is defined by (see [4, 12, 15, 30, 35, 39, 40, 46])

$$ E_\alpha(u^\alpha) = \sum_{n=0}^{\infty} \frac{u^{(n\alpha)}}{\Gamma(1+n\alpha)}, 0 < \alpha \leq 1. $$

Some useful formulas of LFD can be given as follows [4, 12, 15, 30, 35, 39, 40, 46]:

$$ \frac{d^\alpha u^{n\alpha}}{du^\alpha} = \frac{\Gamma(1+n\alpha)u^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}, $$

$$ \frac{d^\alpha E_\alpha(u^\alpha)}{du^\alpha} = E_\alpha(u^\alpha), $$

$$ \frac{d^\alpha E_\alpha(nu^\alpha)}{du^\alpha} = nE_\alpha(u^\alpha), $$

$$ \frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(u^\alpha)(du)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha), $$

$$ \frac{1}{\Gamma(1+\alpha)} \int_a^b u^{n\alpha}(du)^\alpha = \frac{\Gamma(1+n\alpha)(b^{(n+1)\alpha} - a^{(n+1)\alpha})}{\Gamma(1+(n+1)\alpha)}. $$

3. Local Fractional Decomposition Method

To describe the solution procedure of local fractional decomposition method, we consider the following fractional differential equation [4, 18, 33]

$$ \begin{aligned} &\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) + bu(x,t)^2 + cu(x,t)^3, \\ &0 < x \leq 1,\ 0 < \alpha \leq 1,\ t > 0, \end{aligned} \tag{3.1} $$

According to the LFDM, we can build a local fractional differential operator form for (3.1) as

$$ L_t^{(\alpha)} u(x,t) = u_{xx}(x,t) + au(x,t) + bu(x,t)^2 + cu(x,t)^3, \tag{3.2} $$

where $0 < \alpha \leq 1$, and $u(x,t)$ is a local fractional continuous function. Applying the inverse operator $L_t^{(-\alpha)}$ to both sides of (3.2) yields

$$ \begin{aligned} &u_{n+1}(x,t) = L_t^{(-\alpha)} \left[\frac{\partial^2 u_n(x,t)}{\partial x^2} + a_n u(x,t) + b_n u(x,t)^2 + c_n u(x,t)^3 \right], \quad n \geq 0 \\ &u_0(x,t) = u(x,0). \end{aligned} $$

The successive approximations $u_{n+1}(x,t), n \geq 0$ of the solution $u(x,t)$ will be obtained upon using any function u_0. The initial values are usually used for choosing the zeroth approximation u_0. Consequently, the exact solution may be procured by using

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Thus, we can find that the following condition

$$|f(x) - f(x_0)| < \epsilon^{\alpha}$$

where fractional dimension $f(x)$ is equal to α for any $x \in (a,b)$.

4. Applications

In this section, we present the solution of nonlinear fractional partial differential equations as the applications of local fractional decomposition method (LFDM).

Example 4.1. Consider the nonlinear fractional Klein-Gordon equation for $0 < \alpha \leq 1, \alpha = 1$, and $0 < x \leq 1, t > 0$. We get

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) \tag{4.1}$$

with the initial condition [4, 18, 33]

$$u(x,0) = 1 + \sin(x). \tag{4.2}$$

Using (4.2), the recurrence relation reads as $u_0(x,t) = u(x,0)$ and thus

$$u_{n+1}(x,t) = L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_n(x,t) + u_n(x,t) \right], n \geq 0 \tag{4.3}$$

Applying the recursive relation (4.3) and the condition (4.2), we get the followings results:

$$\begin{aligned}
u_0(x,t) &= 1 + \sin(x), \\
u_1(x,t) &= L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_0(x,t) + u_0(x,t) \right] = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \\
u_2(x,t) &= L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_1(x,t) + u_1(x,t) \right] = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\
u_3(x,t) &= L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_2(x,t) + u_2(x,t) \right] = \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\
&\cdots \\
u_n(x,t) &= L_t^{(-\alpha)} \left[\frac{\partial^2}{\partial x^2} u_{n-1}(x,t) + u_{n-1}(x,t) \right] = \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}
\end{aligned}$$

Then, the approximate solution is given in a series form as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = sin(x) + \left(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \right) = \sin x + \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}$$

which has the exact solution

$$u(x,t) = \sin x + E_{\alpha}(t^{\alpha}),$$

For the special case $\alpha = 1$,

$$u(x,t) = \sin x + \sum_{n=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = \sin x + e^t$$

which is an exact solution to the nonlinear Klein-Gordon equation. Fig. 1 shows the approximate solutions for the nonlinear Klein-Gordon equation (4.1) obtained through local fractional decomposition method (LFDM). Fig. 2 shows the approximate solution of (4.1) for $\alpha = 0.9, 0.8, 0.7, 0.6$. Fig. 3 and 4 are given to show the influence of α on the function $u(x,t)$.

TABLE 1. Numerical values of $u(x,t)$ for $\alpha = 0.5, 0.75, 1$ obtained with the LFDM.

t	x	u_{6LFDM}			u_{Exact}	u_{Error}
		$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 1$	
0.2	0	1.798626054	1.404675299	1.221402756	1.221402758	0.2 e-8
	0.25	2.046030013	1.652079258	1.468806715	1.468806717	0.2 e-8
	0.50	2.278051593	1.884100838	1.700828295	1.700828297	0.2 e-8
	0.75	2.480264814	2.086314059	1.903041516	1.903041518	0.2 e-8
	1	2.640097039	2.246146284	2.062873741	2.062873743	0.2 e-8
0.4	0	2.425071937	1.800405083	1.491824356	1.491824698	0.342 e-6
	0.25	2.672475896	2.047809042	1.739228315	1.739228657	0.342 e-6
	0.50	2.904497476	2.279830622	1.971249895	1.971250237	0.342 e-6
	0.75	3.106710697	2.482043843	2.173463116	2.173463458	0.342 e-6
	1	3.266542922	2.641876068	2.333295341	2.333295683	0.342 e-6
0.6	0	3.123561961	2.262467262	1.822112800	1.822118800	0.6 e-5
	0.25	3.370965920	2.509871221	2.069516759	2.069522759	0.6 e-5
	0.50	3.602987500	2.741892801	2.301538339	2.301544339	0.6 e-5
	0.75	3.805200721	2.944106022	2.503751560	2.503757560	0.6 e-5
	1	3.965032946	3.103938247	2.663583785	2.663589785	0.6 e-5

(4.1) is solved in [33] using fractional variational iteration method (FVIM) and HPM [18]. It is shown that the present algorithm for Local fractional decomposition method (LFDM) performs with considerable efficiency, simplicity and reliability. The results obtained from LFDM are fully compatible with FVIM and HPM.

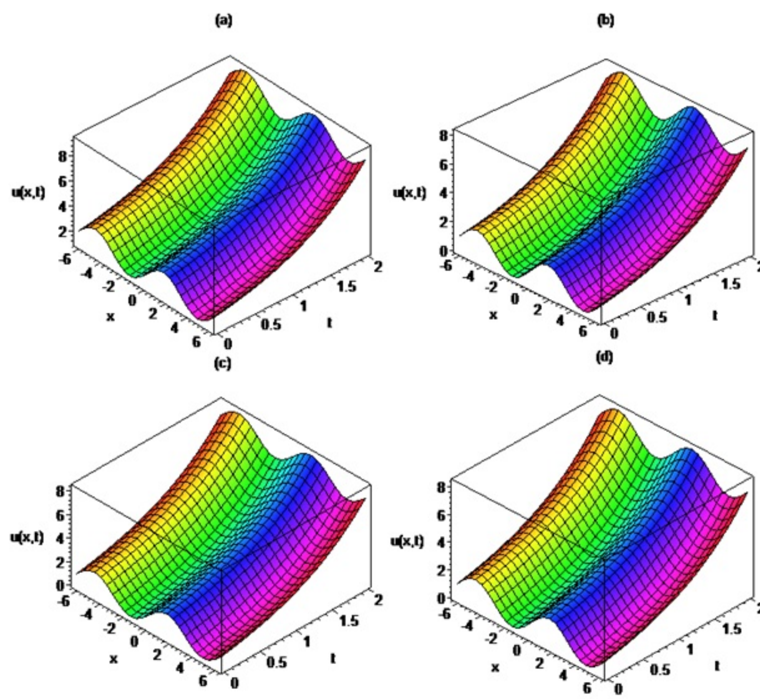

FIGURE 1. The surface indicates the solution $u(x,t)$ of (4.1) for $\alpha = 1$. (a) Exact solution (b) 5-iterate LFD approximate solution, (c) 6-iterate LFD approximate solution and (d) 7-iterate LFD approximate solution.

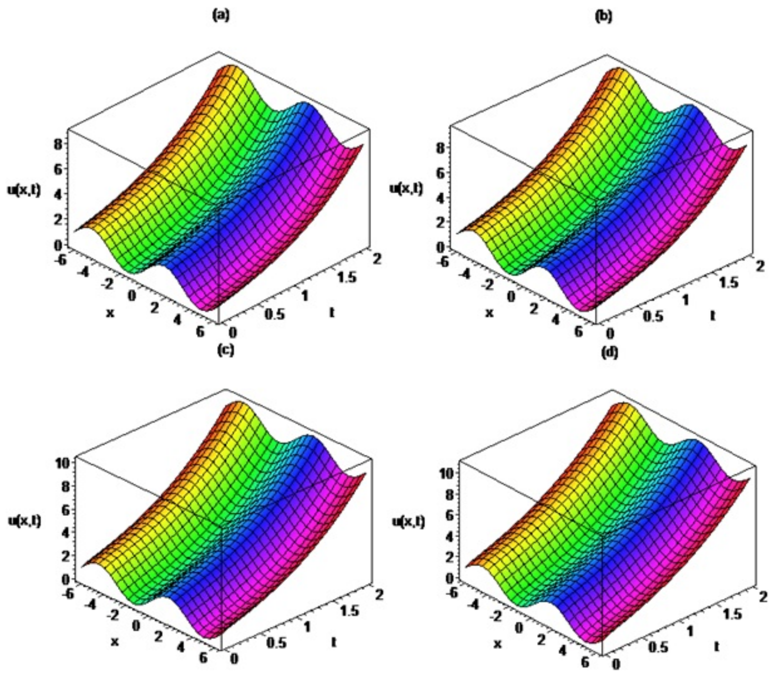

FIGURE 2. The surface indicates the solution $u(x,t)$ of (4.1) (a) 5-iterate LFD approximate solution for $\alpha = 0.9$ (b) 5-iterate LFD approximate solution for $\alpha = 0.8$ (c) 5-iterate LFD approximate solution for $\alpha = 0.7$ and (d) 5-iterate LFD approximate solution for $\alpha = 0.6$.

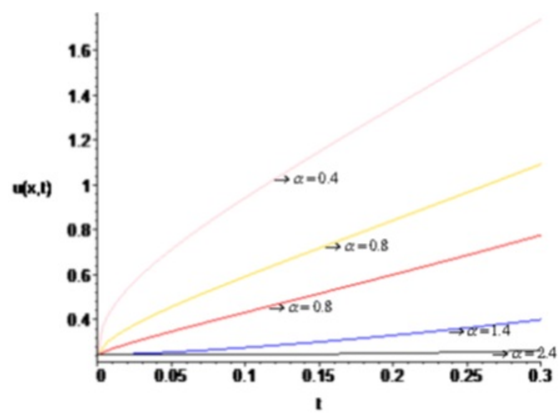

FIGURE 3. 8-iterate LFD approximate solution at $x = 4$ for different values of α.

Example 4.2. Next, consider the non-homogenous nonlinear fractional Klein-Gordon equation where $0 < \alpha \leq 1,\ 0 < x \leq 1, t > 0$,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t)^2 \tag{4.4}$$

with the initial condition [4, 18, 33]

$$u(x,0) = 1 + \sin x. \tag{4.5}$$

According to LFDM, the recurrence relation reads as $u_0(x,t) = u(x,0)$ and thus

$$u_{n+1}(x,t) = L_t^{(-\alpha)}\left[\frac{\partial^2}{\partial x^2} u_n(x,t) - u_n(x,t)^2\right], n \geq 0 \tag{4.6}$$

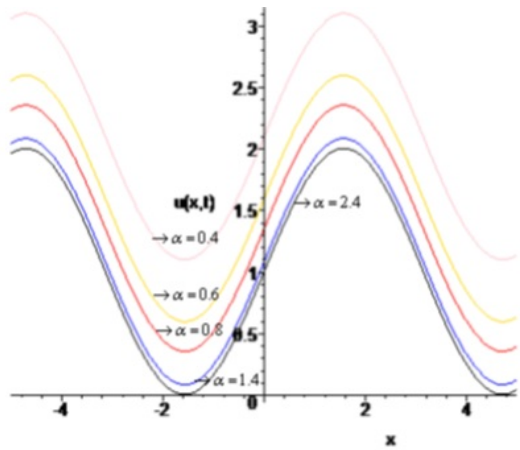

Figure 4. 8-iterate LFD approximate solution at $t = 0.2$ for different values of α.

Applying the recursive relation (4.6) and the condition in (4.5), we get the followings results:

$$u_0(x,t) = 1 + \sin x,$$

$$u_1(x,t) = L_t^{(-\alpha)}\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - u_0(x,t)^2\right] = \frac{-(1+3\sin x+\sin^2 x)t^{\alpha}}{\Gamma(1+\alpha)},$$

$$u_2(x,t) = L_t^{(-\alpha)}\left[\frac{\partial^2}{\partial x^2}u_1(x,t) - u_1(x,t)^2\right] = \frac{(3\sin x - 2\cos 2x)t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{(6\sin x + 11\sin^2 x + 6\sin^3 x + \sin^4 x)\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(3\alpha+1)},$$

$$\begin{aligned} u_3(x,t) =& L_t^{(-\alpha)}\left[\frac{\partial^2}{\partial x^2}u_2(x,t) - u_2(x,t)^2\right] = \frac{(-3\sin x + 8\cos 2x)t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &- \frac{(22\cos 2x - 6\sin x + 36\sin x\cos^2 x - 18\sin^3 x + 3\sin^2 2x - 4\sin^4 x)\Gamma(2\alpha+1)^2 t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(3\alpha+1)^2}, \\ &\cdots \end{aligned}$$

$$\begin{aligned} u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) =& 1 + \sin x - \frac{(1+3\sin x+\sin^2 x)t^{\alpha}}{\Gamma(1+\alpha)} + \frac{(3\sin x - 2\cos 2x)t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &- \frac{(6\sin x + 11\sin^2 x + 6\sin^3 x + \sin^4 x)\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(3\alpha+1)} + \frac{(-3\sin x + 8\cos 2x)t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &- \frac{(22\cos 2x - 6\sin x + 36\sin x\cos^2 x - 18\sin^3 x + 3\sin^2 2x - 4\sin^4 x)\Gamma(2\alpha+1)^2 t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(3\alpha+1)^2} + \cdots \end{aligned}$$

For the special case $\alpha = 1$ [33]

$$\begin{aligned} u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) =& 1 + \sin x - (1+3\sin x+\sin^2 x)t + \frac{(3\sin x - 2\cos 2x)t^2}{2} \\ &- \frac{(6\sin x + 11\sin^2 x + 6\sin^3 x + \sin^4 x)t^3}{3} + \frac{(-3\sin x + 8\cos 2x)t^3}{3} \\ &- \frac{(22\cos 2x - 6\sin x + 36\sin x\cos^2 x - 18\sin^3 x + 3\sin^2 2x - 4\sin^4 x)t^3}{9} + \cdots \end{aligned}$$

Finally, the solution surfaces of the non-homogenous nonlinear fractional Klein-Gordon equation are depicted in Fig. 5 for different values of α. From the graphical results in Figs. 5 and 6, it can be seen the influence of α on the function $u(x,t)$. It is clearly seen that $u(x,t)$ increases with the increases in t for $\alpha = 1, 0.9, 0.8, 0.7, 0.6$.

Eq. (4.4) is solved in [33] using the FVIM, HPM [18] and the results in Fig. 5 compare well with those obtained from the Adomian decomposition method and HPM.

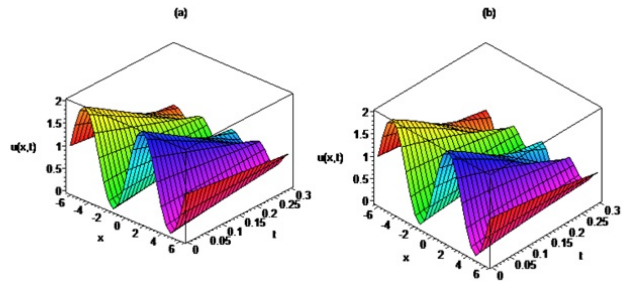

FIGURE 5. The surface indicates the solution $u(x,t)$ of (4.4). (a) 1-iterate LFD approximate solution, (b) 2-iterate LFD approximate solution and for $\alpha = 1$.

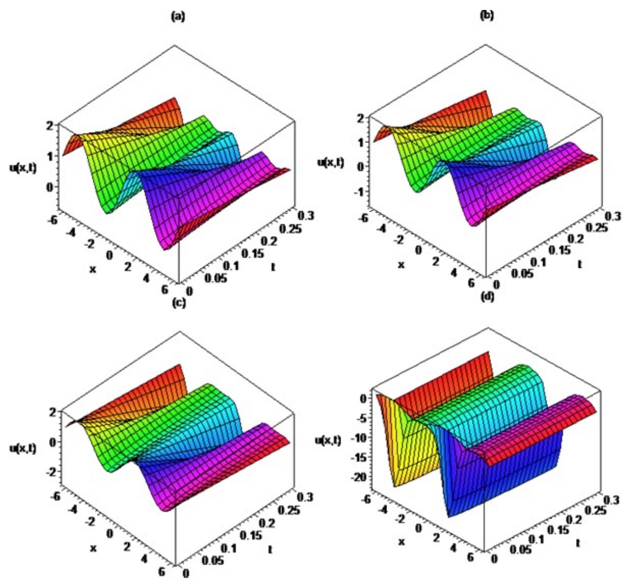

FIGURE 6. The surface indicates the solution $u(x,t)$ of (4.4) (a) 2-iterate LFD approximate solution for $\alpha = 0.9$ (b) 2-iterate LFD approximate solution for $\alpha = 0.8$ (c) 2-iterate LFD approximate solution for $\alpha = 0.7$ and (d) 2-iterate LFD approximate solution for $\alpha = 0.01$.

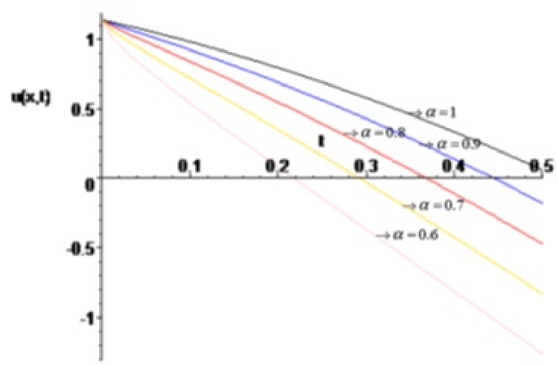

FIGURE 7. 2-iterate LFD approximate solution at $x = 3$ for different values of α.

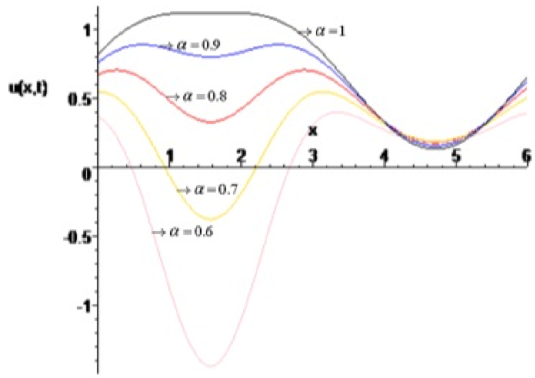

FIGURE 8. 2-iterate LFD approximate solution at $t = 0.15$ for different values of α.

Example 4.3. We next consider the non-homogenous nonlinear fractional Klein-Gordon equation for $0 < \alpha \leq 1,\ 0 < x \leq 1, t > 0$,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + u(x,t)^3 \tag{4.7}$$

with initial condition [4, 18, 33]

$$u(x,0) = -sechx \tag{4.8}$$

According to LFDM, the recurrence relation reads as $u_0(x,t) = u(x,0)$ and thus

$$u_{n+1}(x,t) = L_t^{(-\alpha)}\left[\frac{\partial^2}{\partial x^2}u_n(x,t) - u_n(x,t) + u_n(x,t)^3\right], n \geq 0 \tag{4.9}$$

Applying the recursive relation (4.9) and the condition in (4.8), we get the followings results:

$$u_0(x,t) = -sechx$$

$$u_1(x,t) = L_t^{(-\alpha)}\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - u_0(x,t) + u_0(x,t)^3\right] = \frac{sech^3 x t^\alpha}{\Gamma(1+\alpha)},$$

$$\begin{aligned} u_2(x,t) =& L_t^{(-\alpha)}\left[\frac{\partial^2}{\partial x^2}u_1(x,t) - u_1(x,t) + u_1(x,t)^3\right] \\ =& \frac{(9sech^3 x tanh^2 x)t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{(5sech^5 x)t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{(sech^3 x)t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ \frac{sech^9 x\Gamma(3\alpha+1)t^{4\alpha}}{\Gamma(1+\alpha)^3\Gamma(4\alpha+1)}, \end{aligned}$$

$$\begin{aligned} u_3(x,t) =& L_t^{(-\alpha)}\left[\frac{\partial^2}{\partial x^2}u_2(x,t) - u_2(x,t) + u_2(x,t)^3\right] \\ =& \frac{(81sech^3 x tanh^4 x)t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{(18sech^3 x tanh^2 x)t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &- \frac{(246sech^5 x tanh^2 x)t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{(33sech^7 x)t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{(6sech^5 x)t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ \frac{(81sech^9 x tanh^2 x)\Gamma(1+3\alpha)t^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+5\alpha)} + \cdots \end{aligned}$$

$$\cdots$$

Then, the approximate solution in a series form is

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\
&= -\, sechx + \frac{sech^3 x t^{\alpha}}{\Gamma(1+\alpha)} + \frac{(9 sech^3 x tanh^2 x) t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{(5 sech^5 x) t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad - \frac{(sech^3 x) t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{sech^9 x \Gamma(3\alpha+1) t^{4\alpha}}{\Gamma(1+\alpha)^3 \Gamma(4\alpha+1)} + \frac{(81 sech^3 x tanh^4 x) t^{3\alpha}}{\Gamma(1+3\alpha)} \\
&\quad - \frac{(18 sech^3 x tanh^2 x) t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{(246 sech^5 x tanh^2 x) t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{(33 sech^7 x) t^{3\alpha}}{\Gamma(1+3\alpha)} \\
&\quad + \frac{(6 sech^5 x) t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{(81 sech^9 x tanh^2 x) \Gamma(1+3\alpha) t^{5\alpha}}{\Gamma(1+\alpha)^3 \Gamma(1+5\alpha)} + \cdots
\end{aligned}$$

For the special case $\alpha = 1$ [33]

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\
&= -\, sechx + t sech^3 x + \frac{(9 sech^3 x tanh^2 x) t^2}{2} - \frac{(5 sech^5 x) t^2}{2} - \frac{(sech^3 x) t^2}{2} \\
&\quad + \frac{sech^9 x t^4}{4} + \frac{(81 sech^3 x tanh^4 x) t^3}{6} - \frac{(18 sech^3 x tanh^2 x) t^3}{6} \\
&\quad - \frac{(246 sech^5 x tanh^2 x) t^3}{6} + \frac{(33 sech^7 x) t^3}{6} + \frac{(6 sech^5 x) t^3}{6} \\
&\quad + \frac{(81 sech^9 x tanh^2 x) t^5}{20} + \cdots
\end{aligned}$$

Finally, the solution surfaces of the non-homogenous nonlinear fractional Klein-Gordon equation are depicted in Fig. 9 for different values of α. From the graphical results in Figs. 9 and 10, it can be seen the influence of α on the function $u(x,t)$. It is clearly seen that $u(x,t)$ increases with the increases in t for $\alpha = 1, 0.9, 0.8, 0.7, 0.6$.

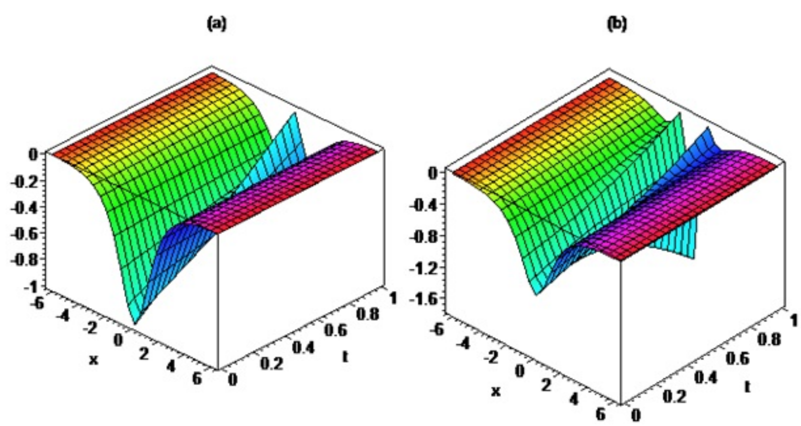

FIGURE 9. The surface indicates the solution $u(x,t)$ of (4.7). (a) 1-iterate LFD approximate solution, (b) 2-iterate LFD approximate solution and for $\alpha = 1$.

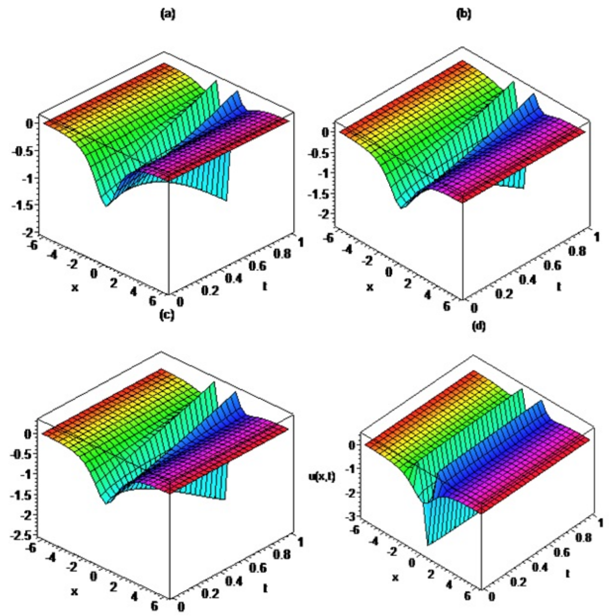

FIGURE 10. The surface indicates the solution $u(x, t)$ of (4.7) (a) 2-iterate LFD approximate solution for $\alpha = 0.9$ (b) 2-iterate LFD approximate solution for $\alpha = 0.8$ (c) 2-iterate LFD approximate solution for $\alpha = 0.7$ and (d) 2-iterate LFD approximate solution for $\alpha = 0.01$.

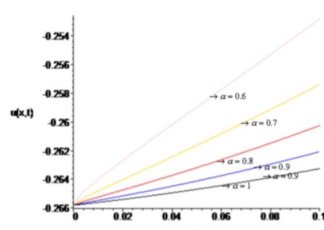

FIGURE 11. 2-iterate LFD approximate solution at $x = 2$ for different values of α.

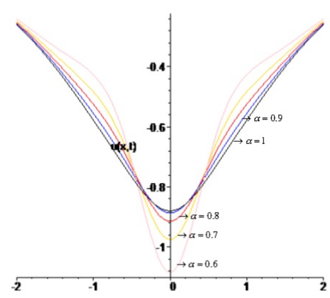

FIGURE 12. 2-iterate LFD approximate solution at $t = 0.2$ for different values of α.

5. Conclusions

Local fractional decomposition method (LFDM) may be used in solving nonlinear problems and ordinary, partial, fractional, integral equations. In this paper, we have taken Local fractional decomposition method having integral w. r. t. $(d\tau)^{\alpha}$ used for the first time by Jumarie. The obtained results show that this method since, for $\alpha = 1$, is powerful and meaningful for solving the nonlinear fractional differential equations. Three examples indicate that the results of local fractional decomposition method having integral w. r. t. $(d\tau)^{\alpha}$ are in accordance with those obtained by classical MVIM, HAM, HPM and Local Fractional Series Expansion Method which are available in the literature.

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