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Arithmetic statistically convergent on neutrosophic normed spaces

Nötrosofik normlu uzaylarda aritmetik istatistiksel yakınsaklık

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Abstract

This work is concerned with several important different types of convergence that will be described on neutrosophic normed spaces. In the study, arithmetic convergence was combined with different types of statistical convergence and then integrated into the structure of neutrosophic spaces established through the membership function. For this purpose, in the neutrosophic normed space, firstly, the concepts of arithmetic convergence and arithmetic statistical convergence are given, then some important definitions that can be established with lacunary sequences and ideal structures and some relationships between convergent sequences in this sense are examined. Furthermore, new convergence definitions were established by evaluating lambda sequences together with arithmetic convergence and statistical convergence with the help of neutrosophic normed space structure properties. Finally, with the help of the definition of graduated convergence, the inclusion relation between the two set is given.

Keywords: Arithmetic statistically convergence, Ideals, Neutrosophic normed spaces.

Öz

Bu çalışma, nötrosofik normlu uzaylar üzerinde tanımlanacak olan önemli birkaç yakınsaklık türü ile ilgilidir. Çalışmada, aritmetik yakınsaklık, farklı istatistiksel yakınsaklık türleri ile birleştirilmiş ve daha sonra üyelik fonksiyonu yardımıyla kurulan nötrosofik uzayların yapısına dahil edilmiştir. Bu amaçla nötrosofik normlu uzayda öncelikle aritmetik yakınsaklık ve aritmetik istatistiksel yakınsaklık kavramları verilmiş, ardından lacunary diziler ve ideal yapılar ile kurulabilecek bazı önemli tanımlar ve yakınsak diziler arasında bu anlamdaki bazı ilişkiler incelenmiştir. Ayrıca nötrosofik normlu uzay yapısı özellikleri yardımıyla lambda diziler; aritmetik yakınsaklık ve istatistiksel yakınsaklıkla birlikte değerlendirilerek yeni yakınsaklık tanımları kurulmuştur. Son olarak dereceli yakınsaklık tanımı yardımıyla iki küme arasında kapsama bağıntısı verilmiştir.

Anahtar kelimeler: *Aritmetik istatistiksel yakınsaklık, İdealler, Nötrosofik normlu uzaylar.*

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1. Introduction

It is not easy to explain many uncertainties that arise in our daily lives with classical methods. In order to eliminate these ambiguities, first fuzzy, then intuitionistic fuzzy, and then neutrosophic concepts emerged. The fuzzy set theory, which is constructed with the help of the membership function that converts the elements of a set to the interval [0,1], paved the way for the development of the intuitionistic fuzzy set theory with the addition of the non-member function that converts the elements of the set to the interval [0,1]. In this theory, the sum of the values of non-membership and membership function for all element of set remained in range of [0,1].

Then, with the emergence of the neutrosophic sets, a more realistic decision-making process came to the fore with the concept that the accuracy, inaccuracy and indefinite membership values are independent of each other, and thus the sum of the three can take values in the range of [0,3] (Smarandache, 1998). The neutrosophic concept structure has been widely used in the solution of many problems in the fields of basic sciences, medicine and engineering. Many studies have been carried out in the field of mathematics related to this concept in order to be presented to the use of other fields. Kirisci & Simsek (2020) defined new concept called as neutrosophic normed space. By evaluating the new development area of analysis studies such as statistical convergence and the concept of neutrosophic together, Kisi (2020) investigated the concept of lacunary statistical convergence on this space. Some notable results in this regard can be reviewed here: (Khan et al., 2021; Gonul Bilgin, 2022a, 2022b; Kisi, 2021a).

On the other hand, Ruckle, (2012) established the concept of arithmetic convergence. It has been started to be studied by different authors by associating it with arithmetic convergence, statistical convergence, metric spaces and invariance. Arithmetic convergence is examined in (Yaying & Hazarika, 2018, 2020; Kisi, 2021b, 2022).

The concept of ideal convergence is presented in (Kostyrko et al., 2000). After then, this concept and its various generalizations have been studied from many different perspectives in fuzzy, intuitionistic fuzzy and neutrosophic normed spaces.e.g (Kisi, 2021a). Moreover, using some special sequences, such as lacunary and fibonacci sequences, convergence and statistical versions in this sense are important study areas of the last ten years.

Another concept combined with statistical convergence is lambda sequences. $\lambda -$sequence is introduced by (Mursaleen, 2000). With the help of these sequences with certain properties, many different studies have been carried out in the spaces given above, where uncertainty is investigated similar to the method of convergence of lacunary sequences e.g (Esi & Hazarika, 2013).

After a summary of the relevant literature, what has been done in this study can be briefly summarized as follows. The main motivation in the preparation of this study is to establish the equivalent of the concept of arithmetic convergence, on which important studies have been made, in rapidly developing neutrosophic structures with a new perspective brought to daily life problems. Since the concept of arithmetic convergence has not yet been transferred to spaces with neutrosophic norms, this study was created by taking into account the gap in the literature. In addition, the fact that there are very few examples of arithmetic convergence enabled the study to be shaped in this direction. The concepts that will be transferred to the Neutrosophic normed space will be introduced in order and connections will be made between them. For this purpose, the terms of arithmetic, statistical convergence and lambda, ideal and lacunary convergence are based on and the general structure of the neutrosophic normed space is integrated with their appropriate combinations. Finally, the degree of convergence definitions of the defined concepts were given and the inclusion status was examined.

Now some definitions necessary for the study will be reminded.

Definition 1.1 (Kirisci & Simsek, 2020) Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces.$(\mathfrak{s}_n)$ is called to be convergent to $\mathfrak{s}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1)$ and $\mathfrak{p} > 0$, there exists $n^* \in \mathbb{N}$, such that for each $n \geq n^*$,

$$\mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}, \mathfrak{p}) > 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}, \mathfrak{p}) < \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}, \mathfrak{p}) < \varepsilon^{\mathfrak{N}}. \quad (1)$$

Then, it is denoted $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}$. Here, (1) is also expressed by

$$\lim_{n\to\infty} \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}, \mathfrak{p}) = 1, \lim_{n\to\infty} \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}, \mathfrak{p}) = 0, \lim_{n\to\infty} \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}, \mathfrak{p}) = 0. \quad (2)$$

Definition 1.2 (Kisi, 2020) Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces, $\theta^{\mathfrak{N}} = (\mathscr{k}_n)$ be lacunary sequence. It is named to be lacunary statistically convergent to $\delta^{\mathfrak{N}}$ in $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1)$ and $\mathfrak{p} > 0$,

$$\lim_{n\to\infty} \frac{1}{\mathfrak{h}_n} \left| \left\{ n \in \mathcal{I}_n : \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \delta^{\mathfrak{N}}, \mathfrak{p}\right) \le 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \delta^{\mathfrak{N}}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \delta^{\mathfrak{N}}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}} \right\} \right| = 0. \quad (3)$$

Here, $\left(\theta^{\mathfrak{N}}\right)$ is an increasing integer sequence, $\mathscr{k}_0 = 0$, $\mathfrak{h}_n = \mathscr{k}_n - \mathscr{k}_{n-1} \to \infty$ as $n \to \infty$ and $\mathcal{I}_n = (\mathscr{k}_{n-1}, \mathscr{k}_n]$.

Definition 1.3 (Yaying & Hazarika, 2020) $(\mathfrak{s}_n)$ is called to be arithmetic statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$ if for all $\varepsilon^{\mathfrak{N}} > 0$, there is an integer r so that,

$$\lim_{v\to\infty} \frac{1}{v} \left| \left\{ n \le v : \left| \mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle} \right| \ge \varepsilon \right\} \right| = 0. \quad (4)$$

In this case, it is denoted with $\lim \mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle} (aSt)$. Here, for two integers n, r the greatest common divisor of n, r denoted by $\langle n, r\rangle$ is the largest number that divides both n and r.

Definition 1.4 (Yaying & Hazarika, 2020) $(\mathfrak{s}_n)$ is named to be lacunary arithmetic statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$ if for a lacunary sequence $\theta = (\mathscr{k}_v)$ and for all $\varepsilon > 0$, there is an integer r so that,

$$\lim_{v\to\infty} \frac{1}{\mathfrak{h}_v} \left| \left\{ n \in \mathcal{I}_v : \left| \mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle} \right| \ge \varepsilon \right\} \right| = 0. \quad (5)$$

In this case, it is denoted with $\lim \mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle} (aSt_\theta)$.

After the necessary definitions are given, different convergence situations can be constructed with the help of new concepts in next section.

2. Material and method

Firstly, definition of arithmetic convergence on neutrosophic normed space will be given.

Definition 2.1 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces, $(\mathfrak{s}_n)$ is called to be a arithmetic convergent to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1)$, $\mathfrak{p} > 0$ and there is an integer r and $n^* \in \mathbb{N}$, such that for each $n \ge n^*$,

$$\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) > 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) < \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) < \varepsilon^{\mathfrak{N}}. \quad (6)$$

Then, it is denoted $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}$. Here, (6) means

$$\lim_{n\to\infty} \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) = 1, \lim_{n\to\infty} \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) = 0, \lim_{n\to\infty} \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) = 0. \quad (7)$$

Lemma 2.1 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces. then the next properties are hold.

1) $(\mathfrak{s}_n)$ is arithmetic convergent on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ then it is unique.
2) Let $\lim_{n\to\infty} \mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}$ and $\lim_{n\to\infty} \breve{\mathfrak{s}}_n = \breve{\mathfrak{s}}_{\langle n,r\rangle}$. Then $\lim_{n\to\infty} \mathfrak{s}_n + \breve{\mathfrak{s}}_n = \mathfrak{s}_{\langle n,r\rangle} + \breve{\mathfrak{s}}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$.
3) Let $\lim_{n\to\infty} \mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ and $\mathfrak{c} \neq 0$, then $\lim_{n\to\infty} \mathfrak{c}\,\mathfrak{s}_n = \mathfrak{c}\mathfrak{s}_{\langle n,r\rangle}$.

Proof. The proof is easily obtained from the (6) or (7).

The definition of arithmetic convergence on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ can be modified with the concept of statistical convergence as below.

Definition 2.2 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be neutrosophic normed spaces. $(\mathfrak{s}_n)$ is called to be a arithmetic statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1), \mathfrak{p} > 0$ and there is an integer r so that,

$$\lim_{v\to\infty} \frac{1}{v} \left|\left\{n \le v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \le 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}\right\}\right| = 0 \quad (8)$$

Then, it is denoted $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt)$.

This definition given by the equation (8) can be used in the following equivalent statement instead.

Lemma 2.2 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces. Then the next properties are equal.

1) $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt)$.
2) $\lim_{v\to\infty} \frac{1}{v} \left|\left\{n \le v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \le 1 - \varepsilon^{\mathfrak{N}}\right\}\right| = \lim_{v\to\infty} \frac{1}{v} \left|\left\{n \le v: \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}\right\}\right|$
$= \lim_{v\to\infty} \frac{1}{v} \left|\left\{n \le v: \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}\right\}\right| = 0.$
3) Let $\lim_{n\to\infty} \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) = 1(aSt)$, $\lim_{n\to\infty} \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) = 0(aSt)$
and $\lim_{n\to\infty} \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) = 0(aSt)$.

Proof. The proof is easily obtained from the definition.

The relationship between the two definitions above is given in the following lemma.

Lemma 2.3 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces and $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}$. In this case, $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt)$.

Proof. Let $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}$, so for all $\varepsilon^{\mathfrak{N}} \in (0,1)$, $\mathfrak{p} > 0$ and there is an integer r and $n^* \in \mathbb{N}$, such that for each $n \ge n^*$,
$\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) > 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) < \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) < \varepsilon^{\mathfrak{N}}$.
Hence, the density of $\left\{n \le v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \le 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}\right\}$ is zero. So,
$$\lim_{v\to\infty} \frac{1}{v} \left|\left\{n \le v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \le 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \ge \varepsilon^{\mathfrak{N}}\right\}\right| = 0\,.$$
That is, $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt)$.

The following hybrid definitions integrates the concepts of arithmetic and lacunary convergence.

Definition 2.3 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be neutrosophic normed spaces, $\theta = (\mathscr{k}_v)$ be lacunary sequence. $(\mathfrak{s}_n)$ is named to be a arithmetic strong lacunary convergent to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1)$, $\mathfrak{p} > 0$, there is an integer r and $n^* \in \mathbb{N}$, such that for each $n \ge n^*$,

$$\frac{1}{\mathfrak{h}_v}\sum_{n\in\mathfrak{I}_v} \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) > 1 - \varepsilon^{\mathfrak{N}}, \frac{1}{\mathfrak{h}_v}\sum_{n\in\mathfrak{I}_v} \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) < \varepsilon^{\mathfrak{N}} and \frac{1}{\mathfrak{h}_v}\sum_{n\in\mathfrak{I}_v} \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) < \varepsilon^{\mathfrak{N}}.$$

Then, it is denoted with $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (a\theta^s)$.

Theorem 2.1 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces, $\theta = (\mathscr{k}_v)$ be a lacunary sequence. $(\mathfrak{s}_n)$ is arithmetic strong lacunary convergent on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$, in this case, this limit is unique.

Proof. Let's accept that $\mathfrak{N}-lim\mathfrak{s}_n=\mathfrak{s}_{\langle n,r\rangle}\,(a\theta^s)$ and $\mathfrak{N}-lim\mathfrak{s}_n=\breve{\mathfrak{s}}_{\langle n,r\rangle}\,(a\theta^s)$. For given $\varepsilon^{\mathfrak{N}}\in(0,1)$, choosing $\mathfrak{x}\in(0,1)$ so that $(1-\mathfrak{x})\otimes(1-\mathfrak{x})>1-\varepsilon^{\mathfrak{N}}$ and $\mathfrak{x}\boxtimes\mathfrak{x}<\varepsilon^{\mathfrak{N}}$. For all $\mathfrak{p}>0$, there is an integer r and $n^*\in\mathbb{N}$, such that for each $n\geq n^*$,

$$\frac{1}{\mathfrak{h}_v}\sum_{n\in\mathcal{I}_v}\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)>1-\varepsilon^{\mathfrak{N}},\frac{1}{\mathfrak{h}_v}\sum_{n\in\mathcal{I}_v}\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)<\varepsilon^{\mathfrak{N}}\ and\ \frac{1}{\mathfrak{h}_v}\sum_{n\in\mathcal{I}_v}\mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)<\varepsilon^{\mathfrak{N}}$$

At the same time, there is $n^{**}\in\mathbb{N}$ such that, for each $n\geq n^{**}$,

$$\frac{1}{\mathfrak{h}_v}\sum_{n\in\mathcal{I}_v}\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\breve{\mathfrak{s}}_{\langle n,r\rangle},\mathfrak{p}\right)>1-\varepsilon^{\mathfrak{N}},\frac{1}{\mathfrak{h}_v}\sum_{n\in\mathcal{I}_v}\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\breve{\mathfrak{s}}_{\langle n,r\rangle},\mathfrak{p}\right)<\varepsilon^{\mathfrak{N}}\ and\ \frac{1}{\mathfrak{h}_v}\sum_{n\in\mathcal{I}_v}\mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\breve{\mathfrak{s}}_{\langle n,r\rangle},\mathfrak{p}\right)<\varepsilon^{\mathfrak{N}}$$

If it is choosing $n^{\wedge}=max\{n^*,n^{**}\}$. Hence, for $n\geq n^{\wedge}$, taking a $\mathfrak{k}\in\mathbb{N}$ such that

$$\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_{\mathfrak{k}}-\mathfrak{s}_{\langle n,r\rangle},\frac{\mathfrak{p}}{2}\right)>\frac{1}{\mathfrak{h}_v}\sum_{n\in\mathcal{I}_v}\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\frac{\mathfrak{p}}{2}\right)>1-\mathfrak{x}$$

So,

$$\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_{\langle n,r\rangle}-\breve{\mathfrak{s}}_{\langle n,r\rangle},\mathfrak{p}\right)>\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_{\mathfrak{k}}-\mathfrak{s}_{\langle n,r\rangle},\frac{\mathfrak{p}}{2}\right)\otimes\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_{\mathfrak{k}}-\breve{\mathfrak{s}}_{\langle n,r\rangle},\frac{\mathfrak{p}}{2}\right)>(1-\mathfrak{x})\otimes(1-\mathfrak{x})>1-\varepsilon^{\mathfrak{N}}.$$

Using $\varepsilon^{\mathfrak{N}}>0$ is arbitrary, for each $\mathfrak{p}>0$, it is written $\lim_{n\to\infty}\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_{\langle n,r\rangle}-\breve{\mathfrak{s}}_{\langle n,r\rangle},\mathfrak{p}\right)=1$, by doing the same with others then $\mathfrak{s}_{\langle n,r\rangle}=\breve{\mathfrak{s}}_{\langle n,r\rangle}$.

Definition 2.4 Let $(\Omega,\mathfrak{N},\otimes,\boxtimes)$ be a neutrosophic normed spaces, $\theta=(\mathcal{k}_v)$ be a lacunary sequence. $(\mathfrak{s}_n)$ is called to be a arithmetic lacunary statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega,\mathfrak{N},\ \otimes,\ \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}}\in(0,1)$, $\mathfrak{p}>0$ and there is an integer r so that,

$$\lim_{v\to\infty}\frac{1}{\mathfrak{h}_v}\left|\left\{n\in\mathcal{I}_v:\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)\leq1-\varepsilon^{\mathfrak{N}},\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)\geq\varepsilon^{\mathfrak{N}},\mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)\geq\varepsilon^{\mathfrak{N}}\right\}\right|=0\quad(9)$$

Then, it is denoted $\mathfrak{N}-lim\mathfrak{s}_n=\mathfrak{s}_{\langle n,r\rangle}\,(aSt_\theta)$.

Equivalent expressions that can be used instead of (9) are given in the lemma below.

Lemma 2.4 Let $(\Omega,\mathfrak{N},\otimes,\boxtimes)$ be a neutrosophic normed spaces and $\theta=(\mathcal{k}_v)$ be a lacunary sequence. Then the next properties are equivalent.

1) $\mathfrak{N}-lim\mathfrak{s}_n=\mathfrak{s}_{\langle n,r\rangle}\,(aSt_\theta)$.
2) $\lim_{v\to\infty}\frac{1}{\mathfrak{h}_v}\left|\left\{n\in\mathcal{I}_v:\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)\leq1-\varepsilon^{\mathfrak{N}}\right\}\right|=\lim_{v\to\infty}\frac{1}{\mathfrak{h}_v}\left|\left\{n\in\mathcal{I}_v:\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)\geq\varepsilon^{\mathfrak{N}}\right\}\right|$ $=\lim_{v\to\infty}\frac{1}{\mathfrak{h}_v}\left|\left\{n\in\mathcal{I}_v:\mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)\geq\varepsilon^{\mathfrak{N}}\right\}\right|=0.$
3) $\lim_{v\to\infty}\frac{1}{\mathfrak{h}_v}\left|\left\{n\in\mathcal{I}_v:\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)>1-\varepsilon^{\mathfrak{N}}\right\}\right|=\lim_{v\to\infty}\frac{1}{\mathfrak{h}_v}\left|\left\{n\in\mathcal{I}_v:\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)<\varepsilon^{\mathfrak{N}}\right\}\right|$ $=\lim_{v\to\infty}\frac{1}{\mathfrak{h}_v}\left|\left\{n\in\mathcal{I}_v:\mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)<\varepsilon^{\mathfrak{N}}\right\}\right|=1.$
4) Let $\mathfrak{N}-lim\,\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)=1(aSt_\theta)$, $\mathfrak{N}-lim\,\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)=0(aSt_\theta)$ and $\mathfrak{N}-lim\,\mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)=0(aSt_\theta)$.

3. Results

In this section, the relations of the definitions established above with $\lambda-$sequences and ideals will be presented.

Definition 3.1 Let $(\Omega,\mathfrak{N},\otimes,\boxtimes)$ be a neutrosophic normed spaces. $(\mathfrak{s}_n)$ is called to be a arithmetic $\lambda-$statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega,\mathfrak{N},\ \otimes,\ \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}}\in(0,1)$, $\mathfrak{p}>0$ and there is an integer r so that,

$$\lim_{v\to\infty}\frac{1}{\lambda_v}\left|\left\{n\in\mathfrak{I}_v:\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\le 1-\varepsilon^{\mathfrak{N}},\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\ge\varepsilon^{\mathfrak{N}},\mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\ge\varepsilon^{\mathfrak{N}}\right\}\right|=0 \quad (10)$$

Then, it is denoted $\mathfrak{N}-lim\mathfrak{s}_n=\mathfrak{s}_{\langle n,r\rangle}\,(aSt_\lambda)$. Here, $\lambda=(\lambda_v)$, be a non-decreasing sequence of positive numbers, $\lambda_1=1$, $\lim_{v\to\infty}\lambda_v=\infty$, $\lambda_{v+1}\le\lambda_v+1$ and $\mathfrak{I}_v=[v-\lambda_v+1,v]$ well known in the literature.

Let us consider the following norms, which are frequently used in studies in the field of neutrosophic normed space, see (Kirisci & Simsek, 2020) and then let's choose the sequence to which we will apply the concept of arithmetic convergence.

Example 3.1
Let $(\Omega,\mathfrak{N},\otimes,\boxtimes)$ be a neutrosophic normed spaces and $\mathfrak{p}>\|\mathfrak{s}_n\|$. For all $\varepsilon^{\mathfrak{N}}\in(0,1),\mathfrak{p}>0$ there is an integer r such that $\mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n,\mathfrak{p})=\frac{\mathfrak{p}}{\|\mathfrak{s}_n\|+\mathfrak{p}}$, $\mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n,\mathfrak{p})=\frac{\|\mathfrak{s}_n\|}{\|\mathfrak{s}_n\|+\mathfrak{p}}$, $\mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n,\mathfrak{p})=\frac{\|\mathfrak{s}_n\|}{\mathfrak{p}}$. If we choose $(\lambda_v)=v$, then $\mathfrak{I}_v=[1,v]$. For $\mathfrak{s}_n=\begin{cases}1, & n=k^2\\ 0, & n\ne k^2\end{cases}$ and let's take $r=10$. Let's calculate a few $\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}$ values to see the structure of the sequences.

$\mathfrak{s}_1-\mathfrak{s}_{\langle 1,10\rangle}=1-1=0,\mathfrak{s}_2-\mathfrak{s}_{\langle 2,10\rangle}=\mathfrak{s}_2-\mathfrak{s}_2=0,\mathfrak{s}_3-\mathfrak{s}_{\langle 3,10\rangle}=\mathfrak{s}_3-\mathfrak{s}_1=-1,\mathfrak{s}_4-\mathfrak{s}_{\langle 4,10\rangle}=\mathfrak{s}_4-\mathfrak{s}_2=1$
$\mathfrak{s}_5-\mathfrak{s}_{\langle 5,10\rangle}=\mathfrak{s}_5-\mathfrak{s}_5=0,\mathfrak{s}_6-\mathfrak{s}_{\langle 6,10\rangle}=\mathfrak{s}_6-\mathfrak{s}_2=0,\mathfrak{s}_7-\mathfrak{s}_{\langle 7,10\rangle}=0-\mathfrak{s}_1=-1,\mathfrak{s}_8-\mathfrak{s}_{\langle 8,10\rangle}=\mathfrak{s}_8-\mathfrak{s}_2=0$
$\mathfrak{s}_9-\mathfrak{s}_{\langle 9,10\rangle}=\mathfrak{s}_9-\mathfrak{s}_1=0,\mathfrak{s}_{10}-\mathfrak{s}_{\langle 10,10\rangle}=\mathfrak{s}_{10}-\mathfrak{s}_{10}=0,\ldots,\mathfrak{s}_{15}-\mathfrak{s}_{\langle 15,10\rangle}=\mathfrak{s}_{15}-\mathfrak{s}_5=0,\ldots,$
$\mathfrak{s}_{20}-\mathfrak{s}_{\langle 20,10\rangle}=\mathfrak{s}_{20}-\mathfrak{s}_{10}=0,\ldots,\mathfrak{s}_{100}-\mathfrak{s}_{\langle 100,10\rangle}=\mathfrak{s}_{100}-\mathfrak{s}_{10}=1\ldots$

$$\left\{n\le v:\frac{\mathfrak{p}}{\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|+\mathfrak{p}}\le 1-\varepsilon^{\mathfrak{N}},\frac{\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|}{\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|+\mathfrak{p}}\ge\varepsilon^{\mathfrak{N}},\frac{\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|}{\mathfrak{p}}\ge\varepsilon^{\mathfrak{N}}\right\}$$
$$=\left\{n\le v:\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|\ge\frac{\mathfrak{p}\varepsilon^{\mathfrak{N}}}{1-\varepsilon^{\mathfrak{N}}},\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|\ge\mathfrak{p}\varepsilon^{\mathfrak{N}}\right\}$$
$$=\left\{n\le v:\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}=1\right\}=\{n\le v:n=k^2,k\in\mathbb{N}\}.$$

So,

$$\frac{1}{\lambda_v}\left|\{n\le v:n=k^2,k\in\mathbb{N}\}\right|=\frac{1}{v}\left|\{n\in\mathfrak{I}_v:n=k^2,k\in\mathbb{N}\}\right|\le\frac{\sqrt{v}}{v}.$$

Here, if we take sufficiently large, $\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)=\frac{\mathfrak{p}}{\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|+\mathfrak{p}}$ becomes less than $1-\varepsilon^{\mathfrak{N}}$ and similarly $\mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)=\frac{\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|}{\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|+\mathfrak{p}}$, $\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle},\mathfrak{p}\right)=\frac{\|\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle}\|}{\mathfrak{p}}$ becomes larger than $\varepsilon^{\mathfrak{N}}$.

Definition 3.2 Let $(\Omega,\mathfrak{N},\otimes,\boxtimes)$ be a neutrosophic normed spaces. $(\mathfrak{s}_n)$ is called to be a arithmetic $\mathcal{J}^*$ −statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega,\mathfrak{N},\otimes,\boxtimes)$ if for all $\varepsilon^{\mathfrak{N}}\in(0,1)$, $\gamma^{\mathfrak{N}}>0$ and $\mathfrak{p}>0$ there is an integer r so that,

$$\left\{v\in\mathbb{N}:\frac{1}{v}\left|\left\{n\le v:\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\le 1-\varepsilon^{\mathfrak{N}},\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\ge\varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\ge\varepsilon^{\mathfrak{N}}\right\}\right|\ge\gamma^{\mathfrak{N}}\right\}\in\mathcal{J}^* \quad (11)$$

Then, it is denoted $\mathfrak{N}-lim\mathfrak{s}_n=\mathfrak{s}_{\langle n,r\rangle}\,(aSt_{\mathcal{J}^*})$.

In the following definition, the statistical convergence-based concept established by evaluating the (10) and (11) together will be given.

Definition 3.3 Let $(\Omega,\mathfrak{N},\otimes,\boxtimes)$ be a neutrosophic normed spaces. $(\mathfrak{s}_n)$ is called to be a arithmetic $\lambda\mathcal{J}^*$ −statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega,\mathfrak{N},\otimes,\boxtimes)$ if for all $\varepsilon^{\mathfrak{N}}\in(0,1),\gamma^{\mathfrak{N}}>0$ and $\mathfrak{p}>0$ there is an integer r so that,

$$\left\{v\in\mathbb{N}:\frac{1}{\lambda_v}\left|\left\{n\in\mathfrak{I}_v:\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\le 1-\varepsilon^{\mathfrak{N}},\mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\ge\varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n-\mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\ge\varepsilon^{\mathfrak{N}}\right\}\right|\ge\gamma^{\mathfrak{N}}\right\}\in\mathcal{J}^* (12)$$

Then, the situation given in (12) is denoted by $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt_{\lambda\mathcal{J}^*})$.

Now, let's give the arithmetic $\lambda\mathcal{J}^*$ −statistically summability definition with the help of the sum, which is known as the sum of the Valée-Pousin in the literature by $t_n(\mathfrak{s}) = \frac{1}{\lambda_n}\sum_{v\in\mathfrak{I}_n}\mathfrak{s}_v$.

Definition 3.4 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces. $(\mathfrak{s}_n)$ is called to be a arithmetic $\lambda\mathcal{J}^*$ −statistically summable to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1), \gamma^{\mathfrak{N}} > 0$ and $\mathfrak{p} > 0$ there is an integer r so that,

$$\left\{v \in \mathbb{N}: \frac{1}{\lambda_v}\left|\left\{n \in \mathfrak{I}_v: \mathcal{M}^{\mathfrak{t}}\left(t_n(\mathfrak{s}) - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(t_n(\mathfrak{s}) - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(t_n(\mathfrak{s}) - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right| \geq \gamma^{\mathfrak{N}}\right\} \in \mathcal{J}^*$$

Then, it is denoted $\mathfrak{N} - [\lambda\mathcal{J}^*] - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt_{\lambda\mathcal{J}^*})$.

Let's give the following theorem explaining the relationship between arithmetic $\lambda\mathcal{J}^*$ −statistically summable and arithmetic $\lambda\mathcal{J}^*$ − statistically convergent.

Theorem 3.1 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces. If $(\mathfrak{s}_n)$ is a arithmetic $\lambda\mathcal{J}^*$ −statistically summable to $\mathfrak{s}_{\langle n,r\rangle}$ then $(\mathfrak{s}_n)$ is arithmetic $\lambda\mathcal{J}^*$ −statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$.

Proof. Let $(\mathfrak{s}_n)$ is a arithmetic $\lambda\mathcal{J}^*$ −statistically summable to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$. Then

$$\sum_{n\in\mathfrak{I}_v}\left(\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right), \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) or \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\right)$$

$$\geq \sum_{\substack{n\in\mathfrak{I}_v,\ \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}) < 1-\varepsilon^{\mathfrak{N}} \\ ,\mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}) > \varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}) > \varepsilon^{\mathfrak{N}}}}\left(\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right), \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) or\ \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right)\right)$$

$$\geq \varepsilon^{\mathfrak{N}}\left|\ n \in \mathfrak{I}_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right|.$$

So,

$$\frac{1}{\lambda_v}\left|\left\{n \in \mathfrak{I}_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right| \geq \gamma^{\mathfrak{N}}.$$

Hence,

$$\frac{1}{\lambda_v}\sum_{v\in\mathfrak{I}_n}\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \leq \left(1 - \varepsilon^{\mathfrak{N}}\right)\gamma^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \gamma^{\mathfrak{N}}\varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \gamma^{\mathfrak{N}}\varepsilon^{\mathfrak{N}}.$$

$$\left\{v \in \mathbb{N}: \frac{1}{\lambda_v}\left|\left\{n \in \mathfrak{I}_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right| \geq \gamma^{\mathfrak{N}}\right\}$$

$$\subset \left\{v \in \mathbb{N}: \frac{1}{\lambda_v}\sum_{n\in\mathfrak{I}_v}\mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \leq \left(1 - \varepsilon^{\mathfrak{N}}\right), \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \gamma^{\mathfrak{N}}\varepsilon^{\mathfrak{N}}\right\}.$$

Thus, using that $(\mathfrak{s}_n)$ is a arithmetic $\lambda\mathcal{J}^*$ −statistically summable to $\mathfrak{s}_{\langle n,r\rangle}$ then it is arithmetic $\lambda\mathcal{J}^*$ −statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$.

In the following theorem, the inclusion case between arithmetic $\mathcal{J}^*$ −statistically convergent and arithmetic $\lambda\mathcal{J}^*$ −statistically convergent is given.

Theorem 3.2 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces, arithmetic $\mathcal{J}^*$ −statistically convergent sequences and arithmetic $\lambda\mathcal{J}^*$ −statistically convergent sequences spaces are denoted with $S(aSt_{\mathcal{J}^*})$ and $S(aSt_{\lambda\mathcal{J}^*})$, respectively. If $\lim_{v\to\infty}\frac{\lambda_v}{v} > 0$, then $S(aSt_{\mathcal{J}^*}) \subset S(aSt_{\lambda\mathcal{J}^*})$.

Proof. Given $\varepsilon^{\mathfrak{N}} > 0$,

$$\frac{1}{v}\left|\left\{n \leq v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle,}\mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|$$

$$\geq \frac{1}{v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} or \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|$$

$$= \frac{\lambda_v}{v}\frac{1}{\lambda_v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|.$$

Here, using $\lim_{v\to\infty} \frac{\lambda_v}{v} = d > 0$, then $\left\{n \in \mathbb{N}: \frac{\lambda_v}{v} < \frac{d}{2}\right\}$ is finite. So, for $\gamma^{\mathfrak{N}} > 0$,

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right| \geq \gamma^{\mathfrak{N}}\right\}$$
$$\subset \left\{n \in \mathbb{N}: \frac{1}{v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right| \geq \frac{d}{2}\gamma^{\mathfrak{N}}\right\}$$
$$\cup \left\{n \in \mathbb{N}: \frac{\lambda_v}{v} < \frac{d}{2}\right\}.$$

The proof is complete because of the set in right-hand side belongs to $\mathcal{J}^*$.

Theorem 3.3 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces. If $\lim_{v\to\infty} \frac{\lambda_v}{v} = 1$, then $S(aSt_{\lambda\mathcal{J}^*}) \subset S(aSt_{\mathcal{J}^*})$.

Proof. Given for $\gamma^{\mathfrak{N}} > 0$, let $\lim_{v\to\infty} \frac{\lambda_v}{v} = 1$. Then, for all $v \geq \breve{v}$, there exists a $\breve{v} \in \mathbb{N}$ such that $\left|\frac{\lambda_v}{v} - 1\right| < \gamma^{\mathfrak{N}}$. For all $\varepsilon^{\mathfrak{N}} > 0$ and $v \geq \breve{v}$,

$$\frac{1}{v}\left|\left\{n \leq v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|$$
$$= \frac{1}{v}\left|\left\{n \leq v - \lambda_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|$$
$$+ \frac{1}{v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|$$
$$\leq \frac{v - \lambda_v}{v} + \frac{1}{v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|$$
$$\leq 1 - \left(1 - \frac{\gamma^{\mathfrak{N}}}{2}\right) + \frac{1}{v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|$$

$$= \frac{\gamma^{\mathfrak{N}}}{2} + \frac{1}{v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right|.$$

Then,

$$\left\{n \in \mathbb{N}: \frac{1}{v}\left|\left\{n \leq v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right| \geq \gamma^{\mathfrak{N}}\right\}$$

$$\subset \left\{n \in \mathbb{N}: \frac{1}{v}\left|\left\{n \in \Im_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}} \text{ or } \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right| \geq \frac{\gamma^{\mathfrak{N}}}{2}\right\}$$

$$\cup \{1,2,\ldots,\breve{v}\}.$$

If, $(\mathfrak{s}_n)$, $\lambda\mathcal{J}^*$ −statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$ in this case, the set in right-hand side belongs to $\mathcal{J}^*$. Therefore, $(\mathfrak{s}_n)$ arithmetic $\mathcal{J}^*$ −statistically convergent to $\mathfrak{s}_{\langle n,r\rangle}$.

Some definitions of degree of convergence are given below.

Definition 3.5 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces, $\theta = (\hbar_v)$ be lacunary sequence, $\alpha \in (0,1]$. $(\mathfrak{s}_n)$ is named to be a arithmetic lacunary statistically convergent of order α to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1)$, $\mathfrak{p} > 0$ and there is an integer r so that,

$$\lim_{v\to\infty} \frac{1}{\mathfrak{h}_v{}^{\alpha}}\left|\left\{n \in \mathcal{J}_v: \mathcal{M}^{\mathfrak{t}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \leq 1-\varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{w}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}\left(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}\right) \geq \varepsilon^{\mathfrak{N}}\right\}\right| = 0 \quad (13)$$

In this case, it is demonstrated $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}$ $(aSt_{\theta}{}^{\alpha})$.

Definition 3.6 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be neutrosophic normed spaces, $\alpha \in (0,1]$ and $\theta = (\mathscr{k}_v)$ be lacunary sequence. $(\mathfrak{s}_n)$ is named to be a arithmetic $\mathcal{J}^*$ −lacunary statistically convergent of order α to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1), \gamma^{\mathfrak{N}} > 0$ and $\mathfrak{p} > 0$ there is an integer r so that,

$$\lim_{v\to\infty} \frac{1}{\mathfrak{h}_v{}^{\alpha}} \{ |\{n \in \mathcal{J}_v : \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}\} | \geq \gamma^{\mathfrak{N}} \} = 0 \quad (14)$$

Then, it is denoted $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt_{\mathcal{J}^*\theta}{}^{\alpha})$.

Definition 3.7 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces, $\alpha \in (0,1]$. $(\mathfrak{s}_n)$ is called to be a arithmetic λ −statistically convergent of order α to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1)$, $\mathfrak{p} > 0$ and there is an integer r so that,

$$\lim_{v\to\infty} \frac{1}{\lambda_v{}^{\alpha}} |\{n \in \mathfrak{I}_v : \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}\} | = 0 \quad (15)$$

Then, it is denoted $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt_{\lambda}{}^{\alpha})$.

Definition 3.8 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces and $\alpha \in (0,1]$. $(\mathfrak{s}_n)$ is called to be a arithmetic $\lambda\mathcal{J}^*$ −statistically convergent of order α to $\mathfrak{s}_{\langle n,r\rangle}$ on $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ if for all $\varepsilon^{\mathfrak{N}} \in (0,1), \gamma^{\mathfrak{N}} > 0$ and $\mathfrak{p} > 0$ there is an integer r so that,

$$\lim_{v\to\infty} \frac{1}{\lambda_v{}^{\alpha}} \{ |\{n \in \mathfrak{I}_v : \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}\}| \geq \gamma^{\mathfrak{N}} \} = 0 \quad (16)$$

Then, it is denoted $\mathfrak{N} - lim\mathfrak{s}_n = \mathfrak{s}_{\langle n,r\rangle}\ (aSt_{\lambda\mathcal{J}^*}{}^{\alpha})$.

Let all arithmetic $\mathcal{J}^*$ − statistically convergent sequences spaces is showed by $S(aSt_{\mathcal{J}^*}{}^{\alpha})$. Then the following inclusion relation is valid for $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$.

Theorem 3.4 Let $(\Omega, \mathfrak{N}, \otimes, \boxtimes)$ be a neutrosophic normed spaces $\alpha, \mu \in (0,1]$ and $\alpha \leq \mu$. Then, $S(aSt_{\mathcal{J}^*\theta}{}^{\alpha}) \subset S(aSt_{\mathcal{J}^*\theta}{}^{\mu})$.

Proof. Let $\alpha, \mu \in (0,1]$ and $\alpha \leq \mu$. In this case,

$$\frac{1}{\mathfrak{h}_v{}^{\mu}} |\{n \in \mathcal{J}_v : \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}\} |$$

$$\leq \frac{1}{\mathfrak{h}_v{}^{\alpha}} |\{n \in \mathcal{J}_v : \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}\} |.$$

So, for all $\gamma^{\mathfrak{N}} > 0$,

$$\left\{ v \in \mathbb{N} : \frac{1}{\mathfrak{h}_v{}^{\mu}} |\{n \in \mathcal{J}_v : \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}\} | \geq \gamma^{\mathfrak{N}} \right\} \subset$$
$$\left\{ v \in \mathbb{N} : \frac{1}{\mathfrak{h}_v{}^{\alpha}} |\{n \in \mathcal{J}_v : \mathcal{M}^{\mathfrak{t}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \leq 1 - \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{u}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}, \mathcal{M}^{\mathfrak{f}}(\mathfrak{s}_n - \mathfrak{s}_{\langle n,r\rangle}, \mathfrak{p}) \geq \varepsilon^{\mathfrak{N}}\} | \geq \gamma^{\mathfrak{N}} \right\}$$

Hence, belongs to $\mathcal{J}^*$ in the left-hand set, which is covered by the right-hand side of this expression belonging to $\mathcal{J}^*$.

4. Discussion and conclusions

In this study, which is one of the most important research subjects for researchers working in functional analysis, a neutrosophic approach was used to conduct decision -making processes more realistic in daily life problems.The concepts of ideals, lacunary convergence, statistical convergence on neutrosophic normed space have been studied by different researchers. However, this research was conducted since arithmetic convergence has not been studied in this sense yet. The equivalents of λ convergence in fuzzy, intuitionistic fuzzy and neutrosophic normed spaces were examined and it was thought that arithmetic convergence concept could be evaluated together with λ convergence in this sense. This concept, whose different properties are studied in all three spaces used for decision processes with the help of ideals, has been integrated with the concept of arithmetic convergence. This paper; it has been prepared according to the neutrosophic approach, which argues that uncertainty exist in right and wrong in the decision-making process, and since it is obtained by combining two different convergences in the normed space structure, it has the advantages of both convergence types. This study, which was established to determine the properties of sequences that should be evaluated in terms of arithmetic and statistical convergence in neutrosophic spaces, is a resource for researchers whose convergence type is important in the space they work. When the studies available in the literature are examined, it is seen that the insufficiency in the fuzzy structure has not been overcome yet or the convergences related to this convergence type are examined in the classical sense, which does not include uncertainty and indecision situations. The equivalent of these studies in the literature in the neutrosophic space has not been examined yet. Therefore, the prepared paper is a pioneering study that will fill this gap. A few definitions have been given for the types of degree of convergence, and in future studies, detailed analyzes can be made from different perspectives for the relations of these concepts with each other. For example, using the convergence methods in this study, studies can be made in topological and metric spaces with the technique in (Riaz, 2022a, 2022b) and this convergence structure can be created for function sequences. Concepts that can be evaluated together with topological spaces such as fermatean and pythagorean neutrosophic structures can be blended with arithmetic convergence and their application to daily life problems may follow this study similar to (Gonul Bilgin et al, 2022). In addition, arithmetic convergence properties on the triple sequences used in the studies of (Kisi & Gurdal, 2022a, 2022b) and the relationship of the convergence type used in (Kisi, 2021) with arithmetic convergence can be investigated.

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Declaration of ethical code

The authors of this article declare that the materials and methods used in this study do not require ethical committee approval and/or legal-specific permission.

Conflicts of interest

The author declares that there is no conflict of interest.

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