New Exact Soliton Solutions and Multistability for the Modified Zakharov-Kuznetsov Equation with Higher Order Dispersion

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Abstract

The aim of the present paper is to obtain and analyze new exact travelling wave solutions and bifurcation behavior of modified Zakharov-Kuznetsov (mZK) equation with higher-order dispersion term. For this purpose, the first and second simplest methods are used to build soliton solutions of travelling wave solutions. Furthermore, the bifurcation behavior of traveling waves including new types of quasiperiodic and multi-periodic traveling wave motions have been examined depending on the physical parameters. Multistability for the nonlinear mZK equation has been investigated depending on fixed values of physical parameters with various initial conditions. The suggested methods for the analytical solutions are powerful and beneficial tools to obtain the exact travelling wave solutions of nonlinear evolution equations (NLEEs). Two and three-dimensional plots are also provided to illustrate the new solutions. Bifurcation and multistability behaviors of traveling wave solution of the nonlinear mZK equation with higher-order dispersion will add some value to the literature of mathematical and plasma physics.

Keywords: Bifurcation; First simplest method; Modified Zakharov-Kuznetsov equation; Second simplest method; Quasiperiodic motion.

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1. Introduction

NLEEs involve nonlinear complex physical phenomena and play an outstanding role while characterizing complicated phenomena rooting in different branches of science for example fluid flow, wave propagations, fluid mechanics, nonlinear optics, optical fibres, chemical kinematics, chemical physics, plasma physics, solid-state physics, hydrodynamic, nonlinear transmission lines, plasma physics, geochemistry, biology and soil consolidations. Therefore, obtaining exact solutions of such nonlinear equations are a rich area of research for the scientists because

the resulting solutions can describe physical behaviour of concerned problems in the best way [1–3]. These solutions define various phenomena in nature, such as vibrations, solitons and propagation with a finite speed [4]. In the recent past, many researcher developed a wide range of methods and still trying to construct new methods to establish analytical and solitary traveling wave solutions of the NLEEs. Some of these methods are: inverse scattering method [5, 6], Backlund transformation method [7], modified simplest equation method [8], homogeneous balance method [9], direct algebraic method [10, 11], Hirota bilinear transformation method [12], tanh–sech method [13, 14], extended tanh method [15–17], Jacobi elliptic function expansion method [18–20] , generalized Riccati equation method [21], sine–cosine method [22], F-expansion method [23–25], homogeneous balance method [26], Exp function method [27–30], Cole-Hopf transformation method [31], Adomian decomposition method [32], homotopy analysis method [33, 34], homotopy perturbation method [35], first and second simplest method [36, 37], bifurcation method [38, 39] and first integral method [40].

The nonlinear Zakharov-Kuznetsov (NZK) equation is an another alternative version of nonlinear model describing (2+1)-dimensional modulation of a KdV soliton equation in fluid mechanics [41–43]. In two-and three-dimensional spaces, the NZK equation is given by

$$u_t + auu_x + bu_{xxx} + cu_{xyy} = 0, \tag{1.1}$$

and

$$u_t + auu_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) = 0, \tag{1.2}$$

respectively, where a is known as the coefficient of nonlinear term and b, c are called the coefficients of dispersion terms. Here x, y, z are space variables, t is time and u is acoustic wave potential. This equation was first derived to model the propagation of weakly nonlinear ion-acoustic waves in plasma, which involves cold ions and hot-isothermal electrons in a medium with a uniform magnetic field. The equation is also used to define different types of acoustic waves in magnetized plasmas [44]. It has been shown the equation is not integrable by means of the inverse scattering transform method. It was found that the solitary-wave solutions of the ZK equation are inelastic. Hesam et al. [45] developed differential transform method for Zakharov equation. The nonlinear ZK equation with higher order dispersion term is given by

$$u_t + auu_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) + du_{xxxxx} = 0, \tag{1.3}$$

where a, b, c are same as equation (1.2) and d is the coefficient of fifth order dispersion. With an appropriately modified form of the electron number density given in [46], Munro and Parkes [41] demonstrated that reductive perturbation can induce following modified Zakharov–Kuznetsov (mZK) equation

$$16(u_t - ku_x) + 30u^{1/2}u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{1.4}$$

where k is a positive constant. The mZK equation have solutions that symbolize plane-periodic and solitary traveling waves propagating. It is noted that the mZK equation is a high dimensional nonlinear evolution equation and, thus, the study of its reduction problem is of theoretical interest [47]. Park et al. [48] applied modified Khater method to equation. The extended mapping method is developed to study the traveling wave solution for a mZK equation by Peng [49]. In our manuscript, we study the following nonlinear modified ZK equation with higher order dispersion term as

$$u_t + au^2u_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) + du_{xxxxx} = 0. \tag{1.5}$$

Multistability alludes to an interesting phenomenon where a dynamical system provides more than one numerical solution for a fixed values of the parameters at various initial conditions [50, 51]. Arecchi et al. [52] performed experimental observation of multistability behavior in a Q-switch laser system. Natiq et al. [53] experienced coexisting features involving chaotic and quasi-periodic phenomena and the coexistence of symmetric Hopf bifurcations. Morfu et al. [54] reported multistability in Cellular Nonlinear Network in image processing. Rahim et al. [55] investigated multistability behavior in a hyperchaotic system. Li and Sprott [56] studied multistability phenomenon in the famous Lorenz system in a special parametric range space. In various fields of plasmas, multistability behavior also known as coexisting features were extensively investigated in discharge plasmas [57], plasma diodes [58], solar wind plasma [59], electron-ion plasma [60], and in various quantum plasmas [61, 62].

The aims of this study are twofold and will take place for the first time in the literature. Firstly, we introduce the soliton solutions of the mZK equation with higher order dispersion term using different typies of two simplest methods:

· First simplest method was suggested by Nikolay A. Kudryashov [63], its applications have also been shown in [36, 64] and

· Second simplest method was suggested by Khalid K. Ali [37].

Secondly, we examined the bifurcation behavior of traveling waves including quasiperiodic, multi-periodic, and multistability motion for the mZK equation with higher-order dispersion depending on the physical parameters. Thus, we construct new exact and travelling wave solutions in soliton.

The remnant rest of this paper is systematized as follows: An introduction is given in Section 1. The main steps of the first and second simplest methods are specified in Section 2. In the next section, in Section 3, we apply these methods in detail with finding the exact travelling wave solutions of the mZK equation. In Section 4, some figures are presented in two and three-dimensional to display the solutions given in Section 3. Bifurcation behavior of travelling wave solution containing: the Quasiperiodic, multi-periodic, and multistability wave motion of the mZK equation is investigated in Section 5. A discussion about the equation is given in Section 6. Finally, the paper ends with a conclusion in Section 7.

2. Overview of the methods

2.1 First simplest method [63]

Let's consider the

$$F(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xx}, u_{yy}, u_{zz}, ...) = 0, \tag{2.1}$$

nonlinear partial differential equation where $u = u(x, y, z, t)$ is the unknown function.

Step 1: Use the following wave transformation:

$$u(x, y, z, t) = u(\xi), \ \ \xi = lx + my + nz - vt, \tag{2.2}$$

where l, m, n are constants and v is velocity of the traveling wave.

$$P(u', u'', u''', ...) = 0, u' = \frac{du}{d\xi}. \tag{2.3}$$

By using above terms, equation (2.1) is reduced to a non-linear ordinary differential equation.

Step 2: Assume solution of (2.3) takes form of a finite series

$$u(\xi) = \sum_{i=0}^{N} (A_i(Q(\xi))^i, \tag{2.4}$$

$A_i(i = 0, 1, 2, ..., N), A_N \neq 0$, are unknowns with $(A_i \neq 0)$ to be calculated. N is a positive integer and will be computed by homogeneous balance algorithm.

Step 3: The function $Q(\xi)$ satisfies auxiliary differential equation:

$$(Q'(\xi))^2 = \alpha^2 Q(\xi)^2 (1 - \Omega Q(\xi)^2), \tag{2.5}$$

(2.5) gives the following solution:

$$Q(\xi) = \frac{4\sigma \exp(-\alpha\xi)}{4\sigma^2 + \Omega \exp(-2\alpha\xi)}. \tag{2.6}$$

Step 4: By substituting (2.4) and (2.5) into (2.3) and collecting all terms with the same power of $Q(\xi)$ together, (2.3) turn into a polynomial, taking each coefficient equal to zero, a system of algebraic equations are obtained.

Step 5: By using the Mathematica 11 program, we can obtain the exact solution of (2.3).

2.2 Second simplest method [37]

We illustrate modified Kudryashov method in this section as follows:

Step 1: Assume a solution of (2.3) given in a series form:

$$u(\xi) = \sum_{i=0}^{N} (A_i (Q(\xi))^i, \tag{2.7}$$

where A_i is the same as in First simple method.

Step 2: Function $Q(\xi)$ fulfills the differential equation:

$$(Q'(\xi))^2 = \alpha^2 (\log(C))^2 Q(\xi)^2 (1 - \Omega Q(\xi)^2), \tag{2.8}$$

the solution of (2.8) is introduced by:

$$Q(\xi) = \frac{4\sigma C^{(-\alpha\xi)}}{4\sigma^2 + \Omega C^{(-2\alpha\xi)}}. \tag{2.9}$$

Step 3: Putting (2.7) and (2.8) into (2.3), we procure a polynomial of $Q(\xi)$. Setting all the coefficients of the like powers of $Q(\xi)$ to zero, a system of algebraic equations are obtained.

Step 4: System of equations are solved by Mathematica 11 program. Consequently, we can obtain exact solution of (2.3).

3. Implementations of the methods

We employ the transformation (2.2) with $l^2 + m^2 + n^2 = 1$. Then, the equation (1.5) becomes

$$-(v - alu^2)u_\xi + (bl^3 + clm^2 + cln^2)u_{\xi\xi\xi} + dl^5 u_{\xi\xi\xi\xi\xi} = 0. \tag{3.1}$$

Integrating equation (3.1) according to ξ,

$$-(v - \frac{alu^2}{3})u + (bl^3 + clm^2 + cln^2)u_{\xi\xi} + dl^5 u_{\xi\xi\xi\xi} = c_1, \tag{3.2}$$

is obtained and here c_1 is an integrating constant. Applying the boundary conditions $u \to 0$, $u_\xi \to 0$, $u_{\xi\xi} \to 0$, $u_{\xi\xi\xi} \to 0$, $u_{\xi\xi\xi\xi} \to 0$ as $\xi \to \pm\infty$ in equation (3.2), one can obtain $c_1 = 0$. Then equation (3.2) becomes

$$-(v - \frac{alu^2}{3})u + (bl^3 + cl(1 - l^2))u_{\xi\xi} + dl^5 u_{\xi\xi\xi\xi} = 0. \tag{3.3}$$

Balancing u^3 with $u_{\xi\xi\xi\xi}$ in (3.3), following relation is obtained:

$$3N = N + 4 \Rightarrow N = 2. \tag{3.4}$$

3.1 First simplest method

From (2.4) and (3.4), the solution of (3.3) is written in the form:

$$u(\xi) = A_0 + A_1 Q(\xi) + A_2 Q^2(\xi). \tag{3.5}$$

By setting above solution in equation (3.3) and equating factors of each power of $Q(\xi)$ in resulting equation to zero, we reach following nonlinear algebraic system:

$$\frac{1}{3}aA_0^3 l - A_0 v = 0,$$

$$aA_0^2 A_1 l + \alpha^2 A_1 l \left(bl^2 - cl^2 + c + \alpha^2 dl^4\right) - A_1 v = 0,$$

$$aA_0 A_1^2 l + aA_0^2 A_2 l + 4\alpha^2 A_2 l \left(bl^2 - cl^2 + c + 4\alpha^2 dl^4\right) - A_2 v = 0,$$

$$\frac{1}{3}A_1 l\left(aA_1^2 - 6\alpha^2\Omega\left(bl^2 - cl^2 + c + 10\alpha^2 dl^4\right)\right) + 2aA_0A_1A_2 l = 0,$$

$$aA_0A_2^2 l + aA_1^2 A_2 l - 6\alpha^2 A_2 bl^3\Omega + 6\alpha^2 A_2 cl^3\Omega - 6\alpha^2 A_2 cl\Omega - 120\alpha^4 A_2 dl^5\Omega = 0,$$

$$aA_1 A_2^2 l + 24\alpha^4 A_1 dl^5\Omega^2 = 0,$$

$$\frac{1}{3}aA_2^3 l + 120\alpha^4 A_2 dl^5\Omega^2 = 0.$$

Solving the previous system, we obtain the following solutions:

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = \mp\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}},$$

$$d = -\frac{v}{64\alpha^4 l^5}, \quad b = -\frac{c}{l^2} + c + \frac{5v}{16\alpha^2 l^3}. \tag{3.6}$$

Substituting (3.6) in (3.5) with (2.6) and (2.2), we get the following solutions of (1.5):

$$u_{1,2}(x,y,z,t) = \mp\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}}\left(\frac{4\sigma\exp(-\alpha(lx+my+nz-vt))}{4\sigma^2 + \Omega\exp(-2\alpha(lx+my+nz-vt))}\right)^2. \tag{3.7}$$

3.2 Second simplest method

From (2.7) and (3.4), the solution of (3.3) is written in the form:

$$u(\xi) = A_0 + A_1 Q(\xi) + A_2 Q^2(\xi). \tag{3.8}$$

By setting above solution (3.8) in (3.3) and equating coefficients of like powers of $Q(\xi)$, we obtain following set of non-linear algebraic equations:

$$\frac{1}{3}aA_0^3 l - A_0 v = 0,$$

$$aA_0^2 A_1 l + \alpha^2 A_1 l \log^2(C)\left(bl^2 - cl^2 + c + \alpha^2 dl^4 \log^2(C)\right) - A_1 v = 0,$$

$$aA_0 A_1^2 l + aA_0^2 A_2 l + 4\alpha^2 A_2 l \log^2(C)\left(bl^2 - cl^2 + c + 4\alpha^2 dl^4 \log^2(C)\right) - A_2 v = 0,$$

$$\frac{1}{3}aA_1^3 l + 2aA_0 A_1 A_2 l + 2\alpha^2 A_1 l^3\Omega(c-b)\log^2(C)$$

$$-2\alpha^2 A_1 cl\Omega\log^2(C) - 20\alpha^4 A_1 dl^5\Omega\log^4(C) = 0,$$

$$aA_0 A_2^2 l + aA_1^2 A_2 l - 6\alpha^2 A_2 bl^3\Omega\log^2(C) + 6\alpha^2 A_2 cl^3\Omega\log^2(C)$$

$$-6\alpha^2 A_2 cl\Omega\log^2(C) - 120\alpha^4 A_2 dl^5\Omega\log^4(C) = 0,$$

$$aA_1 A_2^2 l + 24\alpha^4 A_1 dl^5\Omega^2 \log^4(C) = 0,$$

$$\frac{1}{3}aA_2^3 l + 120\alpha^4 A_2 dl^5\Omega^2 \log^4(C) = 0.$$

Now, the following new exact solutions for (1.5) will be produced:

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = -\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}},$$

$$d = -\frac{v}{64\alpha^4 l^5 \log^4(C)}, \quad b = \frac{16\alpha^2 cl^3 \log^2(C) - 16\alpha^2 cl \log^2(C) + 5v}{16\alpha^2 l^3 \log^2(C)}. \tag{3.9}$$

Substituting (3.9) in (3.8) with (2.9) and (2.2), we get the following solutions of (1.5):

$$u_{1,2}(x,y,z,t) = -\frac{3\sqrt{\frac{5}{2}}\sqrt{v}\Omega}{2\sqrt{a}\sqrt{l}}\left(\frac{4\sigma C^{(-\alpha(lx+my+nz-vt))}}{4\sigma^2 + \Omega C^{(-2\alpha(lx+my+nz-vt))}}\right)^2. \tag{3.10}$$

4. Graphical illustrations

Now, some figures in two and three dimensional have been drawn to exemplify solutions given above. The graph of (3.7) using the first simplest method at $c = 0.2, \sigma = 5, a = 4, \Omega = 6, v = 0.5, l = 0.55, m = 0.35, n = 0.1, y = z = 2$ is introduced in Fig. (1). Finally, we shown the graph of (3.10) using the second simplest method at $c = 0.2, a = 4, \sigma = 5, k = 0.001, \Omega = 6, v = 0.5, l = 0.55, m = 0.35, n = 0.1, C = 0.4, y = z = 2$ in Fig. (2).
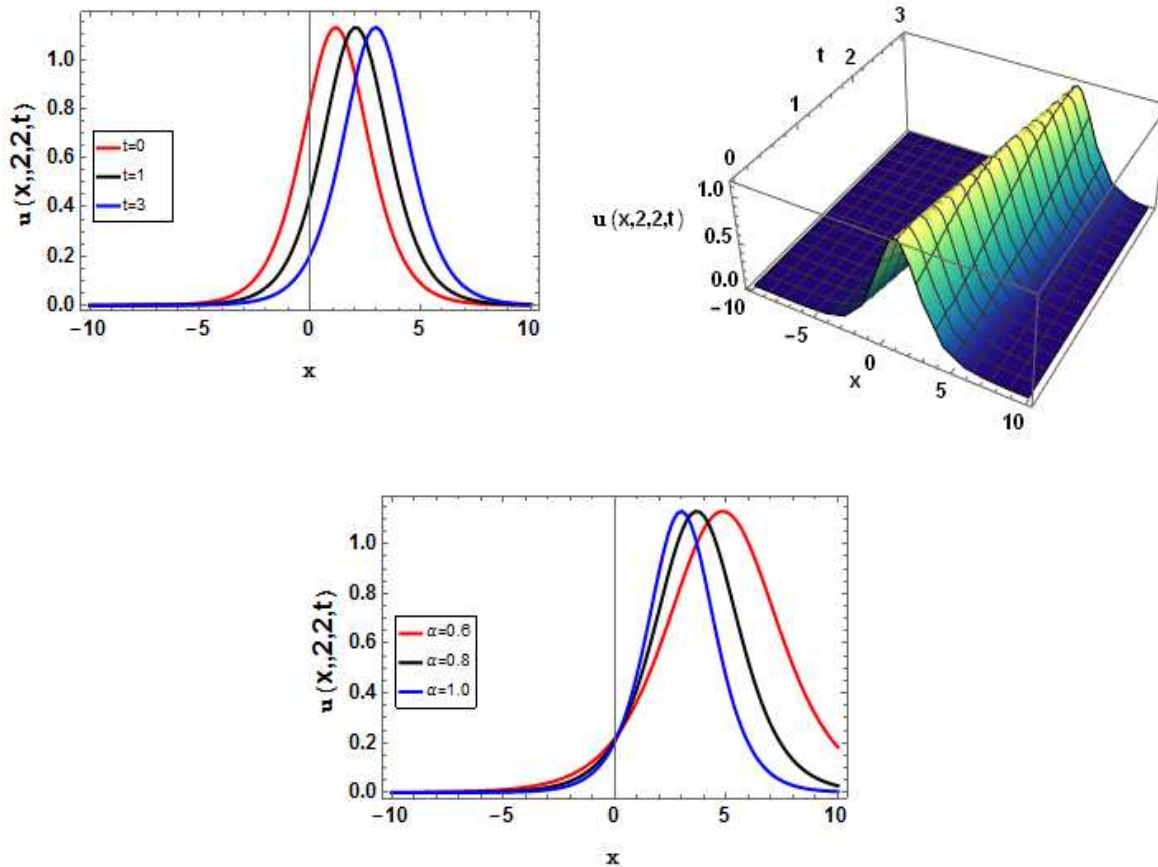
Figure 1. Profile of (3.7) using the first simplest method at c=0.2, $\sigma = 5$, a=4,$\Omega = 6$, v=0.5, l=0.55, m=0.35, n=0.1.
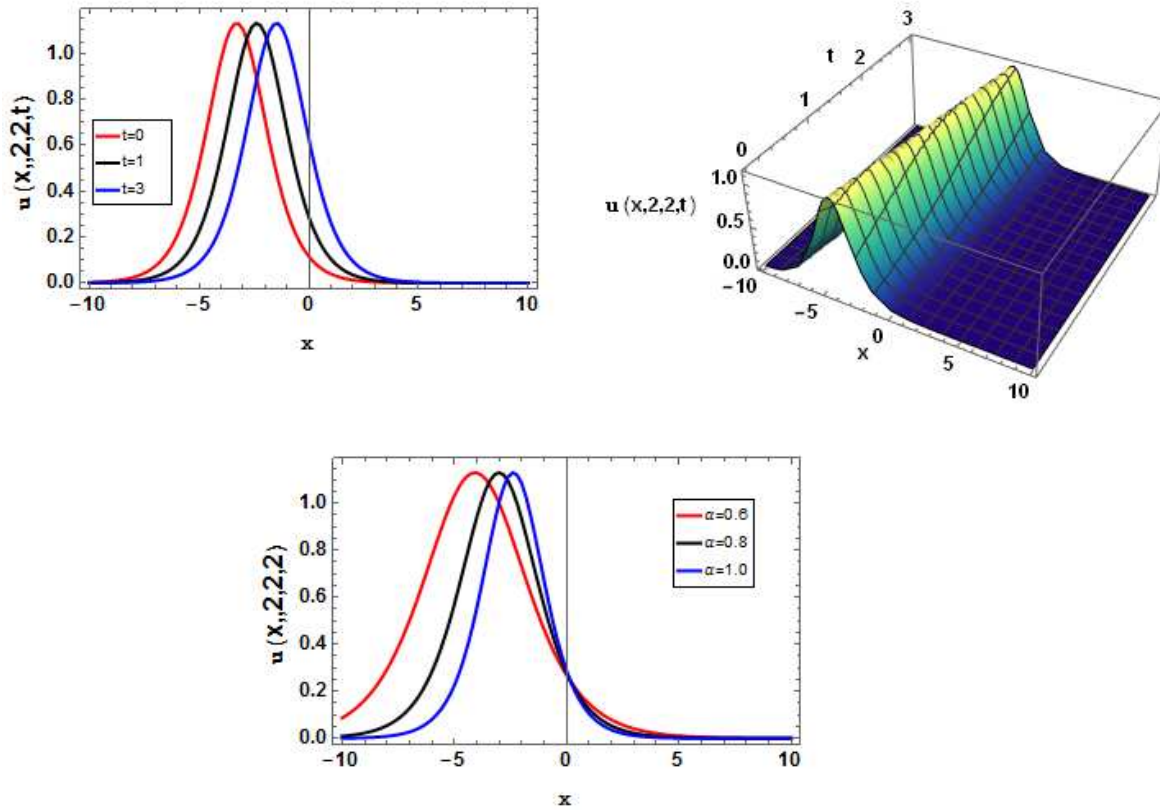
Figure 2. Profile of (3.10) using the second simplest method at c=0.2, a=4, $\sigma = 5$, k=0.001, $\Omega = 6$, v=0.5, l=0.55, m=0.35, n=0.1, C=0.4.

5. Bifurcation analysis

We investigate bifurcation behavior of traveling wave solution of the nonlinear modified ZK equation with higher order dispersion (1.5) for the first time in the literature. To discover all possible traveling wave solutions of nonlinear modified ZK equation (1.5), we form the following dynamical system [65–70] (with parameters a, b, c, d, l and v) from equation (3.3):

$$\begin{cases} u_\xi = X, \\ X_\xi = Y, \\ Y_\xi = Z, \\ Z_\xi = \left(v - \frac{al}{3}u^2\right)\frac{u}{dl^5} - \frac{(bl^2 + c(1-l^2))Y}{dl^4}. \end{cases} \tag{5.1}$$

Let $\overrightarrow{F} = \left(X,\ Y,\ Z,\ \left(v - \frac{al}{3}u^2\right)\frac{u}{dl^5} - \frac{(bl^2+c(1-l^2))Y}{dl^4}\right)$. Then divergence of $\overrightarrow{F}$ is:

$$\overrightarrow{\nabla} \cdot \overrightarrow{F} = \frac{\partial u_\xi}{\partial u} + \frac{\partial X_\xi}{\partial X} + \frac{\partial Y_\xi}{\partial Y} + \frac{\partial Z_\xi}{\partial Z} = 0. \tag{5.2}$$

Thus one can make a conclusion on the conservativeness of the dynamical system (5.1). The singular points of the system (5.1) are given by solutions of the following set of equations:

$$\begin{cases} X = 0, \\ Y = 0, \\ Z = 0, \\ \left(v - \frac{al}{3}u^2\right)\frac{u}{dl^5} - \frac{(bl^2+c(1-l^2))Y}{dl^4} = 0. \end{cases} \tag{5.3}$$

The dynamical system (5.1) has three equilibrium points at $P_1(u_1, 0, 0, 0)$, $P_2(u_2, 0, 0, 0)$ and $P_3(u_3, 0, 0, 0)$, where $u_1 = 0$, $u_2 = \sqrt{\frac{3v}{al}}$, and $u_3 = -\sqrt{\frac{3v}{al}}$.

The stability of the singular point based on the character of eigenvalues of the Jacobian matrix J_P. After making linearisation of the dynamical system (5.1) at the singular point $P(u, X, Y, Z)$, the Jacobian matrix J_P can be written as:

$$J_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{dl^5}(v - alu^2) & 0 & -\frac{1}{dl^4}(bl^2 + c(1 - l^2)) & 0 \end{pmatrix}. \tag{5.4}$$

One can acquire eigenvalues of the system (5.1) at $P(u, X, Y, Z)$ by making solution of the following equation:

$$|\lambda I - J_P| = 0. \tag{5.5}$$

Then one can obtain the following characteristic equation as:

$$\lambda^4 + M_1\lambda^3 + M_2\lambda^2 + M_3\lambda + M_4 = 0, \tag{5.6}$$

where

$$\begin{cases} M_1 = 0, \\ M_2 = -\frac{1}{dl^4}(bl^2 + c(1 - l^2)), \\ M_3 = 0, \\ M_4 = \frac{1}{dl^5}(alu^2 - v). \end{cases}$$

The singular point $P(u, X, Y, Z)$ is considered as stable if all possible solutions of equation (5.6) having real parts less than zero for the singular point or it will be considered as unstable.

5.1 Quasiperiodic and multi-periodic traveling wave motions

Some new types of quasiperiodic and multi-periodic motions for the travelling wave solutions of the modified ZK equation (1.5) are investigated through the conservative dynamical system (5.1) based on suitable values of the parameters a, b, c, d, l, and v in Figures (3-7).

In Figure (3), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (1.5) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.3$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two leafs faced to each other. In Figure (4), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (1.5) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.4$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space looks like a heart-shape. In Figure (5), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (1.5) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.48$ and $v = 6$ with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two leafs faced to each other with multi-bends. In Figure (6), we present phase space and variation of wave profile u for a quasiperiodic motion of the modified ZK equation (1.5) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.494$ and $v = 6$. with initial condition $(3.1, 1.1, -0.1, -0.2)$. In this case, the phase space forms a torus connected with two sets of multi-torus structures faced to each other.
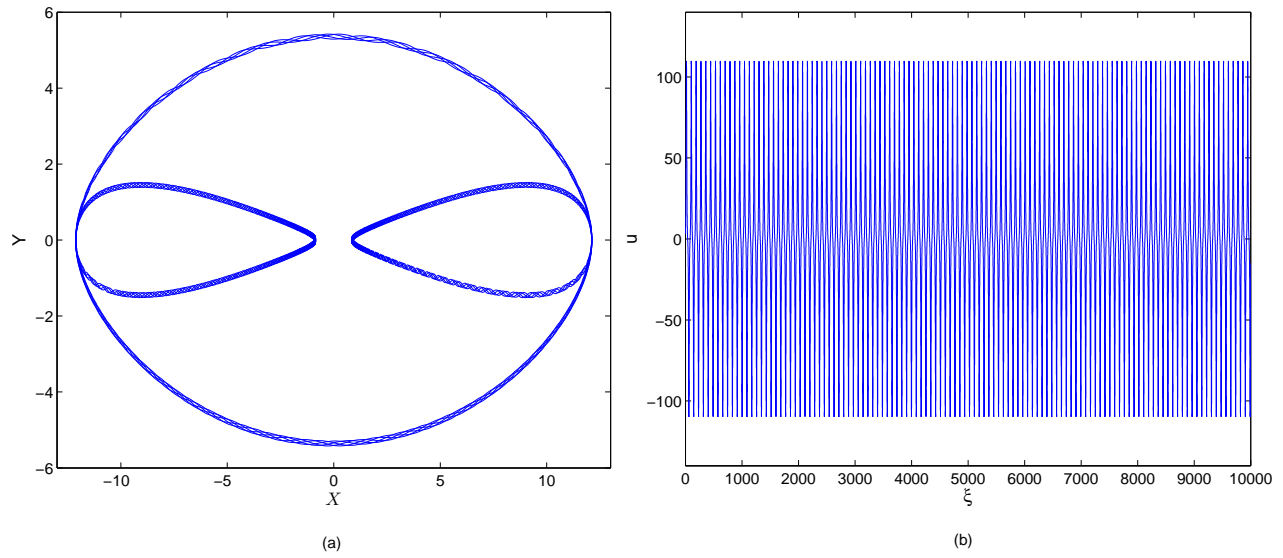
Figure 3. Phase space of the system (5.1) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.3$ and $v = 6$.
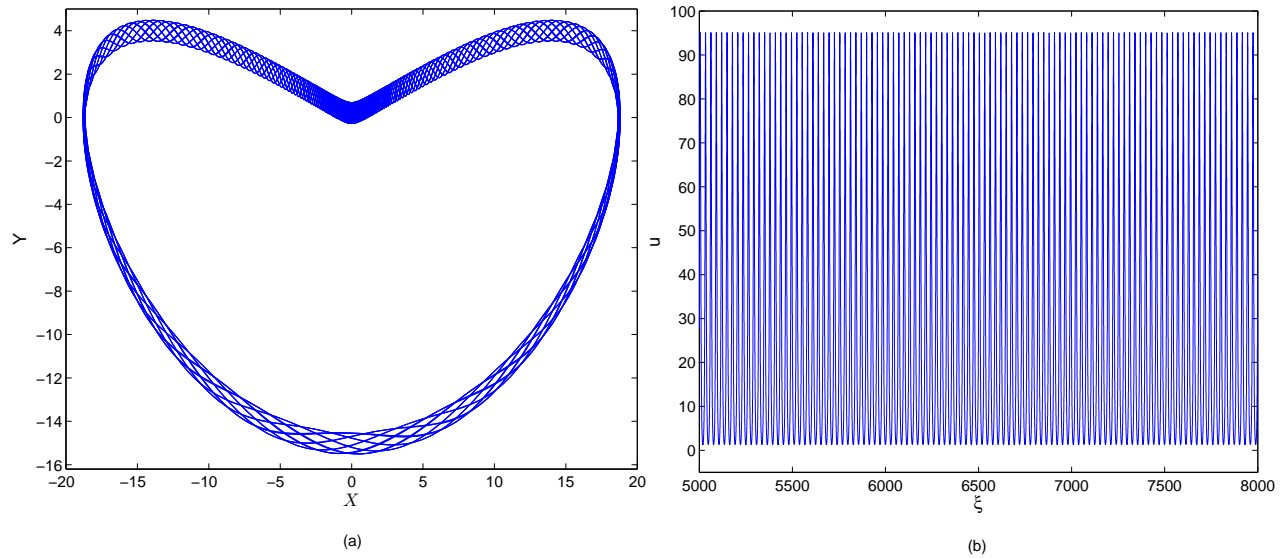

Figure 4. Phase space of the system (5.1) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.4$ and $v = 6$.
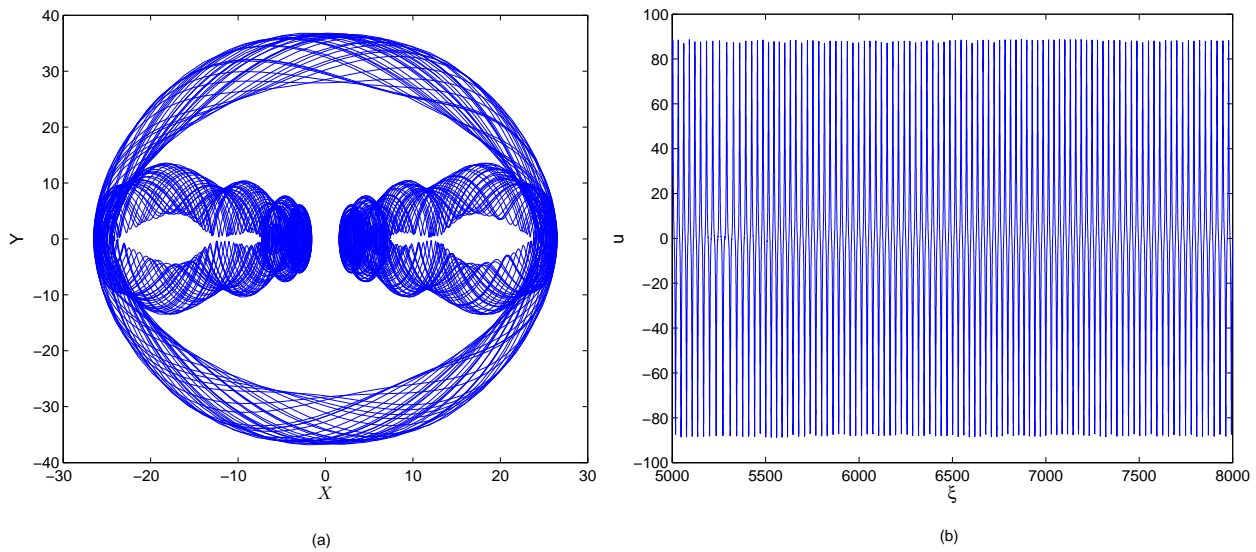

Figure 5. Phase space of the system (5.1) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.48$ and $v = 6$.
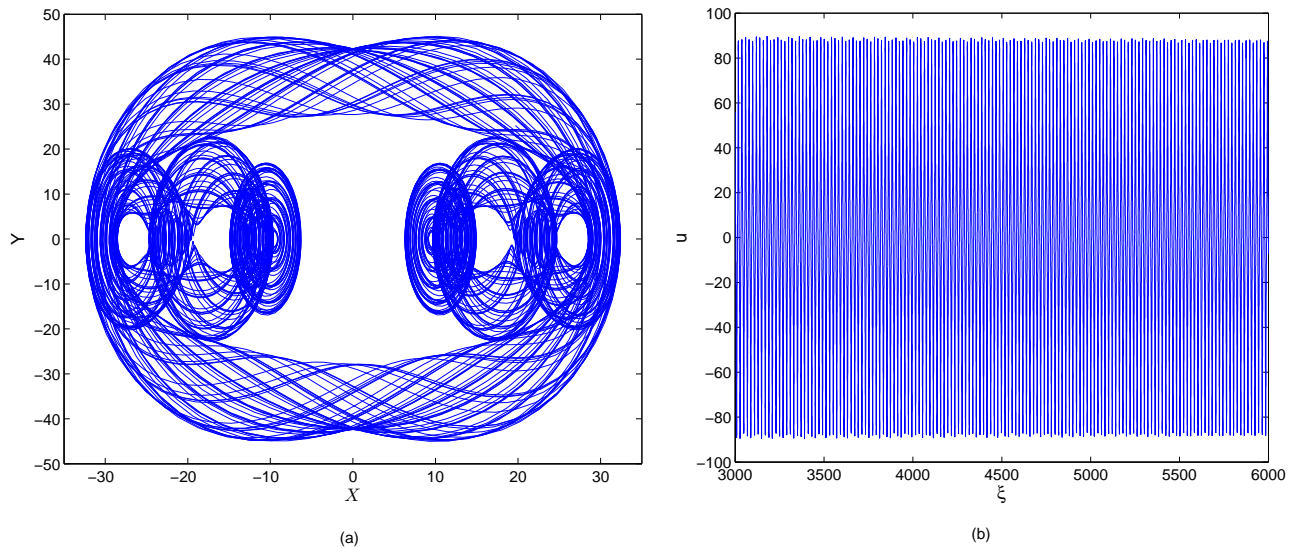
Figure 6. Phase space of the system (5.1) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.494$ and $v = 6$.

There exists a period-9 motion of the dynamical system (5.1) and corresponding phase portrait is shown in Figure (7) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.5$ and $v = 6$. with initial condition $(3.1, 1.1, -0.1, -0.2)$. It is important to note that all phase spaces, presented in Figures (3-7), are symmetric with respect to Y-axis. Such phase spaces, presented in Figures (3-7), are observed for the first time in the literature of nonlinear modified ZK equation (1.5) with higher order dispersion term.
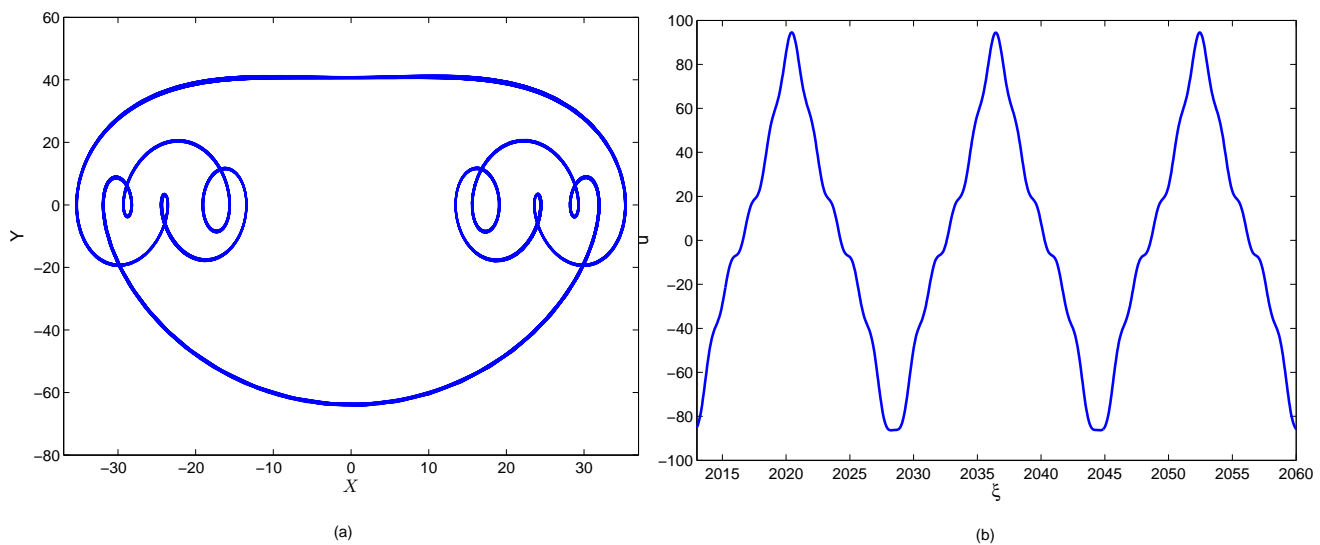

Figure 7. Phase space of the system (5.1) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.5$ and $v = 6$.

5.2 Multistability of traveling wave motion

Multistability behaviors for the travelling wave solutions of modified ZK equation (1.5) are examined through conservative dynamical system (5.1) based on fixed values of the parameters $a, b, c, d, l,$ and v in Figure (8) with different initial conditions: (a) $(0.1, 1.1, -0.1, -0.2)$, (b) $(0.5, 1.1, -0.1, -0.2)$, (c) $(0.1, 0.1, -0.1, -0.2)$, (d) $(0.1, 0.1, 0.1, -0.2)$, (e) $(0.1, 0.1, 0.1, 0.2)$, and (f) $(0.1, 1.1, 0.9, 0.2)$. All these phase spaces are qualitatively different from each other. It is important to note that all phase spaces of Figure (8) are symmetric in nature with respect to Y-axis. This kind of multistability behaviors for the travelling wave solutions of the modified ZK equation (1.5) with higher order dispersion term are reported for the first time in the literature.
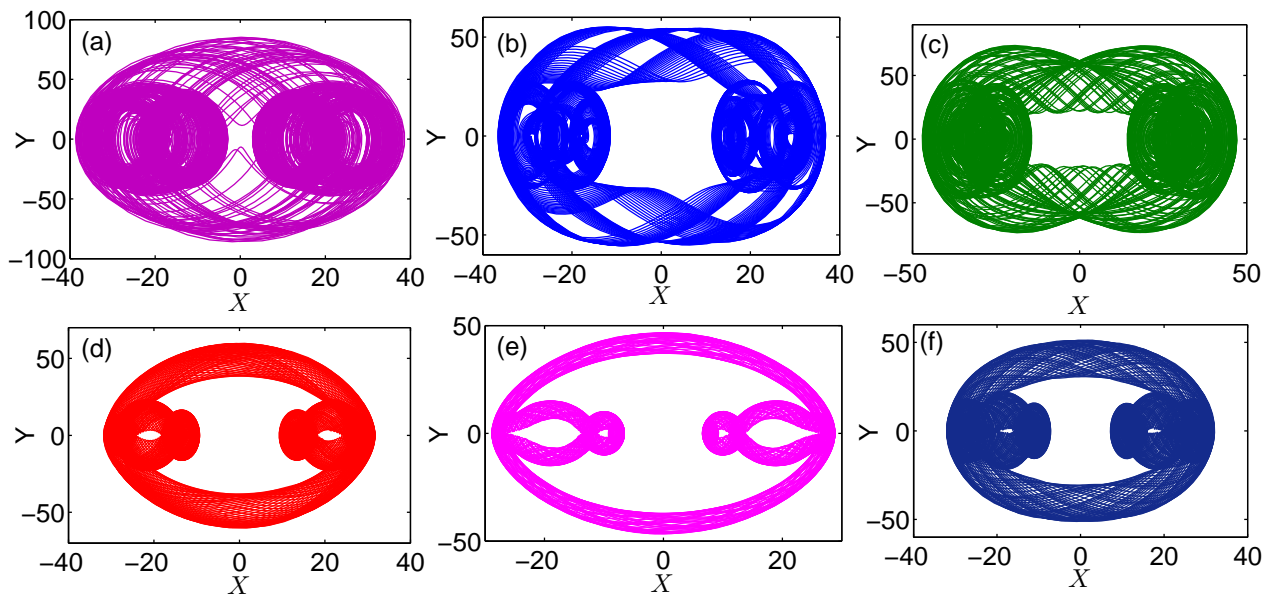
Figure 8. Phase spaces of the system (5.1) for $a = 0.01$, $b = 1$, $c = 1$, $d = 1$, $l = 0.494$ and $v = 6$. with different initial conditions: (a) $(0.1, 1.1, -0.1, -0.2)$, (b) $(0.5, 1.1, -0.1, -0.2)$, (c) $(0.1, 0.1, -0.1, -0.2)$, (d) $(0.1, 0.1, 0.1, -0.2)$, (e) $(0.1, 0.1, 0.1, 0.2)$, and (f) $(0.1, 1.1, 0.9, 0.2)$.

6. Discussion

The (3+1)-dimensional modified Zakharov-Kuznetsov equation is a mathematical model that describes the propagation of nonlinear waves in a medium. It is an extension of the Zakharov-Kuznetsov equation, which is a well-known model for the propagation of ion-acoustic waves in plasma. The modified Zakharov-Kuznetsov equation takes into account the effects of both dispersion and nonlinearity, which can lead to the formation of solitary waves. The physical meaning of the equation is that it describes the evolution of these solitary waves as they propagate through the medium. Overall, the (3+1)-dimensional modified Zakharov-Kuznetsov equation is a useful tool for studying the behavior of waves in a variety of physical systems, including plasmas, fluids, and optical fibers. Its solutions can help researchers better understand the dynamics of these systems and predict how they will behave under different conditions.

7. Conclusions

In this paper, by successfully implementing the two simplest methods, traveling wave solutions for the nonlinear (3+1) dimensional mZK equation have been obtained. New soliton solutions are derived. For a clear understanding, solutions are illustrated with details in 2D and 3D. These solutions have many applications and can supply a beneficial contribution for researchers to examine and discover wave features in several areas of physics and applied sciences. The bifurcation behavior of travelling wave solutions of the mZK equation was also analyzed. A collection of new types of quasiperiodic motions was reported for the first time in the literature on the mZK equation with higher-order dispersion terms. Considering fixed values of the parameters, the multistability behavior of the mZK equation was shown at different initial conditions. In a conclusion, it can be easily seen that the methods used in this paper may further be improved to solve and analyze the qualitative behavior of nonlinear traveling wave solutions for other NLEEs in mathematical and plasma physics.

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