Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (5) (2023), 1396 – 1407
DOI : 10.15672/hujms.1180888

Research Article

On the nth-Order subfractional Brownian motion

Mohamed El Omari*, Mohamed Mabdaoui

Chouaïb Doukkali University, Polydisciplinary Faculty of Sidi Bennour, B.P. 299, Jabrane Khalil Jabrane Street, 24000 El Jadida, Morocco

Abstract

In the present work, we introduce the nth-Order subfractional Brownian motion $S_H^n = \{S_H^n(t),\ t \geq 0\}$ with Hurst index $H \in (n-1, n)$ and order $n \geq 1$; then we examine some of its basic properties: self-similarity, long-range dependence, non Markovian nature and semimartingale property. A local law of iterated logarithm for S_H^n is also established.

Mathematics Subject Classification (2020). 60G05, 60G15, 60G17, 60G18, 60G22

Keywords. Gaussian self-similar process, non Markovian process, subfractional Brownian motion, semimartingale property, local law of the iterated logarithm

1. Introduction

The self-similarity and long range-dependence have become two important aspects of stochastic models. The first one means that as scale is changing, the process looks like identical. For this reason, it has been applied in image processing for modeling texture having multiscale patterns such as natural scenes [13, 19], bone texture radiographs [16], or rough surfaces [28]. The long range-dependence is strongly related to long memory phenomena arising in a variety of different scientific fields, including hydrology [18], biology [5], medicine [14], economics [11] or traffic network [25]. The fractional Brownian motion (fBm) is the best known and most widely used self-similar process that exhibits the long-range dependence. Thus, it is not surprising that a large number of publications are devoted to the study of fBm and its generalizations (see, e.g., [7–10] and references therein). The two-sided fBm with Hurst index $H \in (0, 1)$ is formally defined as a centered Gaussian process $B_H = \{B_H(t),\ t \in \mathbb{R}\}$ having the covariance function

$$\mathbb{E}\left(B_H(t)B_H(s)\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad \text{for all } t, s \in \mathbb{R}.$$

The fBm is of stationary increments and reduces to the standard Brownian motion (Bm) in the case $H = 1/2$. Compared to the extensive studies on fBm, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments. As an extension of Brownian motion, the authors in [2] introduced and studied a rather special class of self-similar Gaussian processes which they call sub-fractional Brownian motions (sub-fBm). As mentioned by the authors, these

*Corresponding Author.
Email addresses: elomari.m@ucd.ac.ma (M. El Omari), mabdaoui.m@ucd.ac.ma (M. Mabdaoui)
Received: 27.09.2022; Accepted: 30.05.2023

processes have properties analogous to those of fBm, and they are intermediate between Bm and fBm in the sense that their increments on nonoverlapping intervals are more weakly correlated and their covariance decays faster than for fBm. The sub-fBm is formally defined as a zero mean Gaussian process S_H^1, $H \in (0, 2)$ with covariance

$$\mathcal{R}_H^1(t,s) = (2-2H)\left(t^{2H} + s^{2H} - \frac{1}{2}\left[(t+s)^{2H} + |t-s|^{2H}\right]\right), \quad \text{for all } t,s \geq 0.$$

The existence of S_H^1 for all $H \in (0,1)$ follows from the equality in distribution $(S_H^1(t))_{t\geq 0} \triangleq c_H\,(B_H(t) + B_H(-t))_{t\geq 0}$, where B_H is a two-sided fBm and c_H is some nonnegative constant. It is important to note that S_H^1 is semimartingale if $H = 1/2$ or $H \in (1,2)$. This kind of processes arises from occupation time fluctuations of branching particle systems with Poisson initial condition. More works on sub-fBm can be found in [2–4, 22–24]. In [20], the authors introduced nth-order fBms as extensions of the standard fBm. Such extensions are very smooth as the order n increases and they exhibit long-range dependence; while the stationarity of increments is achieved at the order n. One of the main features of nth-order fBm's is their ability to describe a wide class of $1/f^{\alpha}$-nonstationary signals with the range $\alpha \in (1,\infty)$. It is shown in [9] that nth order fBm's are semimartingales whenever $n \geq 2$. Some extensions of them can be found in [8, 9]. Motivated by this kind of processes we introduce the nth-order sub-fBm S_H^n and establish some of its basic properties. In comparison with the fBm, S_H^n extendes the usual sub-fBm and share many properties with the nth-order fBm. Especially, the semimartingale property required for modeling fluctuations in movement of stock prices with arbitrage opportunities being excluded.

The rest of the paper is organized as follows: In Section 2, we recall the definition and some properties of the nth-order fBm; while Section 3 is devoted to our main results. The following notations are systematically used: $x_+ = \max(x,0)$, $x_- = \max(-x,0)$ for all $x \in \mathbb{R}$ and the symbol $\triangleq$ denotes the equality in terms of finite dimensional distributions; while $g(x) = \mathcal{O}(f(x))$ and $g(x) \sim f(x)$ (as $x \to x^*$) are respectively used to say that $x \mapsto g(x)/f(x)$ is bounded on neighbourhood of x^* and $\lim\limits_{x\to x^*} g(x)/f(x) = 1$.

2. nth-order fractional Brownian motion

In [20], the nth-order fBm (hereafter B_H^n, $H \in (n-1,n)$, $n \geq 1$ is integer) is defined as a zero mean Gaussian process starting at zero with the integral representation

$$\begin{aligned} B_H^n(t) &= \frac{1}{\mathbf{\Gamma}(H+1/2)}\int_{-\infty}^{0}\left[(t-s)^{H-1/2} - \sum_{j=0}^{n-1}\binom{H-1/2}{j}(-s)^{H-1/2-j}t^j\right]dB(s) \\ &\quad + \frac{1}{\mathbf{\Gamma}(H+1/2)}\int_0^t (t-s)^{H-1/2}\,dB(s), \end{aligned} \tag{2.1}$$

where $B(t)$ is two-sided standard Brownian motion (Bm), $\mathbf{\Gamma}(x)$ stands for the usual Gamma function and

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-(j-1))}{j!}, \quad \binom{\alpha}{0} = 1 \text{ (by convention)}.$$

In the case $n = 1$, the standard fBm is retrieved, as formula (2.1) reduces to the Mandelbrot-Van Ness representation [17] of the fBm. The process B_H^n satisfies the following properties (for more details and proofs, see [8, 9, 20]).

- **(i)** B_H^n is self-similar with exponent H, i.e., $B_H^n(ct) \triangleq c^H B_H^n(t)$, for every $c > 0$.
- **(ii)** B_H^n has derivatives up to order $n-1$ vanishing at zero and the $(n-1)$th derivative coincides with the standard fBm, that is, $\dfrac{d^{n-1}}{dt^{n-1}}(B_H^n(t)) = B^1_{H-(n-1)}(t)$.

(iii) B_H^n exhibits long-range dependence and stationarity of increments is achieved at order n, that is, the increments $\Delta_s^k B_H^n$, $s>0$, $k=0,\cdots,n-1$ are nonstationary and $\Delta_s^n B_H^n$ is a stationary process. Here $\Delta_l^k g(x)$ stands for increments of a function $g(x)$ at order k with explicit form

$$\Delta_l^k g(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g(x+jl) \quad \text{and} \quad \Delta_l^0 g(x) = g(x).$$

(iv) The covariance function of the process B_H^n is given by

$$G_{H,n}(t,s) = (-1)^n \frac{C_H^n}{2} \left\{ |t-s|^{2H} - \sum_{j=0}^{n-1} (-1)^j \binom{2H}{j} \left[\left(\frac{t}{s}\right)^j |s|^{2H} + \left(\frac{s}{t}\right)^j |t|^{2H} \right] \right\}, \tag{2.2}$$

where C_H^n is a nonnegative constant defined recursively by

$$C_H^1 = 1/\left(\Gamma(2H+1)\sin(\pi H)\right)$$

and for $n \geq 2$

$$C_H^n = \frac{C_{H-(n-1)}^1}{(2H)(2H-1)\cdots(2H-(2n-3))}. \tag{2.3}$$

In particular,

$$\mathbb{V}ar\left(B_H^n(t)\right) = C_H^n \binom{2H-1}{n-1} |t|^{2H}.$$

(v) For any $n \geq 2$ the process B_H^n is a special semimartingale with finite variation.
(vi) B_H^n is a Markov process if and only if $n=1$ and $H=1/2$.
(vii) B_H^n can be extended to an α-order fBm U_H^α (see [8]) defined as

$$U_H^\alpha(t) = \frac{1}{\mathbf{\Gamma}(\alpha+1)} \int_0^t (t-s)^\alpha dB_H(s), \quad H \in (0,1), \; \alpha \in (-1,\infty), \tag{2.4}$$

whenever this integral exists. Here B_H denotes a one-sided fBm. In the case $\alpha=0$, we retrieve the standard fBm B_H. If $\alpha=n-1$, then U_H^α coincides with the nth-order fBm with Hurst parameter $H' = H+(n-1)$.

3. Main results

We define the nth-order sub-fBm S_H^n as $S_H^n(t) = \left(B_H^n(t)+B_H^n(-t)\right)/\sqrt{2}$, for all $t \geq 0$ and $H \in (n-1,n)$, where B_H^n is a two-sided nth-order fBm defined as centered Gaussian with covariance function (2.2). Clearly, the case $n=1$ corresponds to the usual sub-fBm. Before we establish some properties of S_H^n that are of great importance, we introduce a definition of long-range dependence for non stationary processes.

Definition 3.1. Let $s>0$ be fixed and $t>s$. Then a process X is said to have long-range dependence property if

$$Corr\left(X(s),X(t)\right) \sim c(s)t^{-d}, \quad \text{as } t \to \infty,$$

where $c(s)$ is a constant depending on s and $d \in (0,1)$. Here $Corr\left(X(s),X(t)\right)$ stands for the correlation function of the process X.

Theorem 3.2. *Let S_H^n be the nth-order sub-fBm with $H \in (n-1,n)$. The following statements hold.*

(i) *S_H^n is a centered Gaussian process with the covariance function*

$$\mathfrak{R}_H^n(t,s) = \frac{(-1)^n C_H^n}{2} \left(|t-s|^{2H} + |t+s|^{2H} - 2\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{2H}{2j} \left[t^{2j} s^{2H-2j} + s^{2j} t^{2H-2j} \right] \right),$$

for all $t,s \geq 0$, where C_H^n is a nonnegative given in (2.3).

(ii) *The process S_H^n is self-similar with index H. i.e.,*

$$\{S_H^n(at), \quad t \geq 0\} \triangleq \left\{a^H S_H^n(t), \quad t \geq 0\right\}, \quad \textit{for every } a > 0.$$

(iii) *The process S_H^n is differentiable for any $n \geq 2$ and can be rewritten as*

$$S_H^n(t) = \int_0^t \int_0^s S_{H-2}^{n-2}(u)duds, \quad \textit{for all } t \geq 0 \textit{ and } n \geq 3.$$

(iv) *The process S_H^n is semimartingale for any $n \geq 2$.*

(v) *The process S_H^n admits the following representation $S_H^n(t) = \int_{\mathbb{R}} K_H(t,s)dB(s)$, where*

$$K_H(t,s) = \frac{1}{\sqrt{2}\Gamma(H-1/2)}\left[(t-s)_+^{2H} + (t+s)_-^{2H} - 2\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{H-1/2}{2j}(-s)_+^{H-1/2-2j}t^{2j}\right].$$

(vi) *The process S_H^n has long-range dependence property in terms of Definition 3.1 for all $n \geq 1$ and $H \in (n-1, n)$.*

Proof.

- The first statement **(i)** follows by definition of the process S_H^n and the use of equation (2.2); while the statement **(ii)** can be readily verified by using the form of its covariance function given in **(i)**. In fact, one has $\mathcal{R}_H^n(ct, cs) = c^{2H}\mathcal{R}_H^n(t,s)$, for all $t, s \geq 0$ and any $c > 0$.
- **(iii)-(iv)**: First, note that S_H^n is differentiable for every $n \geq 2$ (this is inherited from B_H^n), and simple computations leads to **(iii)**. This suggests that S_H^n is semimartingale. Indeed, one can prove this property in the same way as in [9, Theorem 2.1]. Finally, to get **(v)** it sufficies to combine the the Mandelbrot-Van Ness repesentations of both $B_H^n(t)$ and $B_H^n(-t)$.
- **(vi)**: Let $t > s$ and $s > 0$ fixed. By expanding both $\left(1-\frac{s}{t}\right)^{2H}$ and $\left(1+\frac{s}{t}\right)^{2H}$ in the correlation form of $S_H^n(t)$, it follows that as $t \to \infty$

$$\begin{aligned} Corr\left(S_H^n(s), S_H^n(t)\right) &= \frac{\mathcal{R}_H^n(t,s)}{\sqrt{\mathcal{R}_H^n(t,t)\mathcal{R}_H^n(s,s)}} \\ &\sim \begin{cases} -2d_{n,H}\binom{2H}{n-1}\left(\frac{s}{t}\right)^{H-(n-1)}, & \text{when } n \text{ is odd,} \\ 2d_{n,H}\binom{2H}{n}\left(\frac{s}{t}\right)^{n-H}, & \text{when } n \text{ is even} \end{cases} \\ \text{where } \; d_{n,H} &= \left(2^{2H} - 4\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{2H}{2j}\right)^{-1} \quad \text{and } n-1 < H < n. \end{aligned}$$

□

Theorem 3.3. *The nth order sub-fBm S_H^n is Markovian if and only if $(n,H) = (1, 1/2)$.*

Since our processes of interest are centered Gaussian, we will use the following lemma, for which the proof can be found separately in [21, (1.13)-Chapter III] and [12].

Lemma 3.4. *Let $X = \{X(t), \ t \geq 0\}$ be a centered Gaussian process with covariance function $\mathcal{R}(t,s)$. The following statements hold.*

(i) *The process X is Markovian if and only if $\mathcal{R}(t,s)\mathcal{R}(u,u) = \mathcal{R}(t,u)\mathcal{R}(u,s)$, for every $t > u > s$.*

(ii) *If X is a Markov process then we have* $\mathcal{R}(t,s)=\dfrac{\mathcal{R}(t,u)\mathcal{R}(s,s)}{\mathcal{R}(u,u)}$, *for every* $t>u>s$.

Proof of Theorem 3.3. In the case $n=1$ we retrieve the usual sub-fBm S_H^1, $H\in(0,1)$, which is known to be Markovian if and only if $H=1/2$ (e.g., [2]). We shall only prove that S_H^n with $H\in(n-1,n)$ is not a Markov process for every $n\geq 2$. To do so we follow [9] and establish the following statements:

- **(i)** If the process S_H^n, $H\in(n-1,n)$ is Markovian, then the processes S_{H-2k}^{n-2k} with $n-2k\geq 1$ and k is an integer, are Markovian as well.
- **(ii)** The processes S_H^2 and $S_{5/2}^3$ are non Markovian.

The use of the covariance function $\mathcal{R}_H^n(t,s)$ as given in **(i)**-Theorem 3.2 will complicate our computations. Instead, we shall use its integral form which follows from **(iii)**-Theorem 3.2. We have

$$\mathcal{R}_H^n(t,s)=\int_0^t\int_0^s\int_0^x\int_0^y \mathcal{R}_{H-2}^{n-2}(\xi,\zeta)d\xi d\zeta dxdy.$$

If S_H^n is Markovian, then by **(i)**-Lemma 3.4 we have

$$\mathcal{R}_H^n(t,s)\mathcal{R}_H^n(u,u) \;=\; \mathcal{R}_H^n(t,u)\mathcal{R}_H^n(u,s),\ \textit{for all } t>u>s;$$

or

$$\begin{aligned}\int_0^t\int_0^s\int_0^x\int_0^y \mathcal{R}_{H-2}^{n-2}(\xi,\zeta)\,d\xi d\zeta dxdy &= \frac{1}{\mathcal{R}_{H-2}^{n-2}(u,u)}\left[\int_0^t\int_0^u\int_0^x\int_0^y \mathcal{R}_{H-2}^{n-2}(\xi,\zeta)\,d\xi d\zeta dxdy\right]\\ &\quad\times\left[\int_0^s\int_0^u\int_0^x\int_0^y \mathcal{R}_{H-2}^{n-2}(\xi,\zeta)\,d\xi d\zeta dxdy\right].\end{aligned}\tag{3.1}$$

Differentiating equality (3.1) twice with respect to t and twice with respect to s, we obtain

$$\begin{aligned}\mathcal{R}_{H-2}^{n-2}(t,s) &= \frac{1}{\mathcal{R}_{H-2}^{n-2}(u,u)}\left[\int_0^u\int_0^x \mathcal{R}_{H-2}^{n-2}(t,\xi)\,d\xi dx\right]\\ &\quad\times\left[\int_0^u\int_0^y \mathcal{R}_{H-2}^{n-2}(s,\zeta)\,d\zeta dy\right],\quad \text{for all } t>u>s.\end{aligned}\tag{3.2}$$

Let a, b be nonnegative numbers such that $s<a<u<b<t$, then from (3.2) we get

$$\begin{aligned}\mathcal{R}_{H-2}^{n-2}(a,b) &= \frac{1}{\mathcal{R}_{H-2}^{n-2}(u,u)}\left[\int_0^u\int_0^x \mathcal{R}_{H-2}^{n-2}(b,\xi)\,d\xi dx\right]\\ &\quad\times\left[\int_0^u\int_0^y \mathcal{R}_{H-2}^{n-2}(a,\zeta)\,d\zeta dy\right].\end{aligned}\tag{3.3}$$

Multiplying equations (3.2) and (3.3), side by side, we obtain

$$\begin{aligned}\mathcal{R}_{H-2}^{n-2}(t,s)\,.\mathcal{R}_{H-2}^{n-2}(a,b) &= \left[\frac{1}{\mathcal{R}_{H-2}^{n-2}(u,u)}\int_0^u\int_0^y \mathcal{R}_{H-2}^{n-2}(s,\zeta)\,d\zeta dy\times\int_0^u\int_0^x \mathcal{R}_{H-2}^{n-2}(b,\xi)\,d\xi dx\right]\\ &\quad\times\left[\frac{1}{\mathcal{R}_{H-2}^{n-2}(u,u)}\int_0^u\int_0^x \mathcal{R}_{H-2}^{n-2}(t,\xi)\,d\xi dx\times\int_0^u\int_0^y \mathcal{R}_{H-2}^{n-2}(a,\zeta)\,d\zeta dy\right],\end{aligned}$$

and this implies

$$\mathcal{R}_{H-2}^{n-2}(t,s)\,\mathcal{R}_{H-2}^{n-2}(a,b)=\mathcal{R}_{H-2}^{n-2}(t,a)\,\mathcal{R}_{H-2}^{n-2}(b,s)\,,\tag{3.4}$$

using the fact that the covariance function is continuous and taking the limit $(a\to u;\ b\to u)$ in (3.4) we obtain

$$\mathcal{R}_{H-2}^{n-2}(t,s)\,\mathcal{R}_{H-2}^{n-2}(u,u)=\mathcal{R}_{H-2}^{n-2}(t,u)\,\mathcal{R}_{H-2}^{n-2}(u,s)\,,$$

thereby S_{H-2}^{n-2} is Markovian and **(i)** is then established.
Let $n=2$ and $H\in(1,2)$. After some computations it follows from **Theorem 3.2-(i)**

$$\mathcal{R}_H^2(t,s)=\frac{C_H^2}{2}\left[|t-s|^{2H}+(t+s)^{2H}-2\left(t^{2H}+s^{2H}\right)\right].$$

For $t>0$ fixed, we set $\Psi_t:\ s\longmapsto \mathcal{R}_H^2(t,s)/\mathcal{R}_H^2(s,s)$. If S_H^2 is Markovian then by virtue of **(ii)**-Lemma 3.4, we get

$$\frac{\mathcal{R}_H^2(t,s)}{\mathcal{R}_H^2(s,s)}=\frac{\mathcal{R}_H^2(t,u)}{\mathcal{R}_H^2(u,u)},\ \text{for all } t>u>s,$$

which means that Ψ_t must be a constant function on the interval $(0,t)$. Observe that

$$\Psi_t(s)=\frac{1}{(2^{2H}-4)}\left[\left(\frac{t}{s}+1\right)^{2H}+\left(\frac{t}{s}-1\right)^{2H}-2\left(\frac{t}{s}\right)^{2H}-2\right].$$

By standard calculus we check that the function Ψ_t is not canstant, and this yields a contradiction. Hence S_H^2 is non Markovian. Finally, the covariance of $S_{\frac{5}{2}}^3$ has an explicit form as

$$\mathcal{R}_{\frac{5}{2}}^3(t,s)=C_{\frac{5}{2}}^3\left[(t\wedge s)^5-5(t\wedge s)^4(t\vee s)+10(t\wedge s)^3(t\vee s)^2\right],\ \text{for all } t,s\geq 0.$$

For $t=2$, $u=1$ and $s=\dfrac{1}{2}$; with simple calculations we find

$$\mathcal{R}_{\frac{5}{2}}^3\left(2,\frac{1}{2}\right)\mathcal{R}_{\frac{5}{2}}^3(1,1)\neq\mathcal{R}_{\frac{5}{2}}^3(2,1)\,\mathcal{R}_{\frac{5}{2}}^3\left(1,\frac{1}{2}\right).$$

□

Theorem 3.5. *Consider the nth order sub-fBm S_H^n. Then, with probability one, the following limit*

$$c_H^n:=\limsup_{u\to 0^+}\frac{|S_H^n(ut)|}{\Phi_H(u)},\quad \textit{exists for all } t,\in(0,T], \tag{3.5}$$

where Φ_H is $(n-1)$-times continuously differentiable function such that

$$\Phi_H^{(n-1)}(u)=u^{H-n+1}\left(2\log\log(u^{-1})\right)^{1/2},\quad \textit{for all } u>0.$$

Proof. We split the proof of this theorem into three steps. First, we show that (3.5) holds in the case $n=1$, which corresponds to the usual sub-fBm with $\Phi_H(u)=u^H\left(2\log\log(u^{-1})\right)^{1/2}$, $H\in(0,1)$. Note that another form of the law of iterated logarithm for the sub-fBm can be found in [26]. Second, we establish the statement (3.5) for every $n\geq 3$ odd. Finally, we show (3.5) for every $n\geq 2$ in a similar fashion as done in the previous steps.

Step 1. Let $n=1$ and $H\in(0,1)$. In this case, we adopt Arcones's notations [1] and verify the conditions **(i)-(ix)** of [1, Theorem 4.1]. Let $u\in[0,1]$, $t\in\mathbf{T}=[0,T]$ and consider the pseudometric $\rho(u,v)=\sqrt{\mathbb{E}\left(S_H^1(u)-S_H^1(v)\right)^2}$. Set $\tau(u)=u$ and $w(u)=u^H$. Clearly, $\rho(0,T)=\sqrt{\mathbb{E}\left(S_H^1(T)\right)^2}=\sqrt{2-2^{2H-1}}T^H<\infty$, thus **(v)** follows immediately. It is not hard to see that the conidtions **(i)**, **(vii)-(ix)** are satisfied. For the condition **(ii)** let $t,s\in\mathbf{T}$ and $u\in(0,1]$. By self-similarity of the process S_H^1 we have

$$\begin{aligned}\lim_{u\to 0^+}\mathbb{E}\left[\frac{S_H^1(\tau(u)s)S_H^1(\tau(u)t)}{w^2(u)}\right] &= \lim_{u\to 0^+}\frac{\mathbb{E}\left(S_H^1(us)S_H^1(ut)\right)}{u^{2H}},\\ &= \mathbb{E}\left(S_H^1(s)S_H^1(t)\right)=\mathcal{R}_H^1(t,s),\end{aligned}$$

(iii) Let $m \geq 1$ (integer), $r > \varepsilon > 0$, $t_1, \cdots, t_m \in \mathbf{T} \setminus \{0\}$ and $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$. Set $\mathcal{J}^+ := \{$the set of pairs (j,k), $1 \leq j, l \leq m$ for which $\lambda_j \lambda_k \geq 0\}$. For

$$v \in \left[ue^{-\left(\log(u^{-1})\right)^r},\ ue^{-\left(\log(u^{-1})\right)^\varepsilon}\right],$$

we have

$$\begin{aligned}
\mathcal{A}^{r,\varepsilon}_{j,k} &:= \mathbb{E}\left(\frac{S^1_H(ut_j)S^1_H(vt_k)}{w(u)w(v)}\right) = \frac{\mathcal{R}^1_H(ut_j, vt_k)}{u^H v^H}, \\
&= (t_j t_k)^H \left[x_{jk}^{2H} + x_{jk}^{-2H} - \frac{1}{2}\left((x_{jk} + x_{jk}^{-1})^{2H} + \left|x_{jk} - x_{jk}^{-1}\right|^{2H}\right)\right], \text{ with } x_{jk} = \sqrt{\frac{ut_j}{vt_k}}, \\
&\leq (t_j t_k)^H \left(x_{jk}^{2H} + x_{jk}^{-2H} - \left|x_{jk} - x_{jk}^{-1}\right|^{2H}\right) \leq 3(t_j t_k)^H \left(x_{jk} \wedge x_{jk}^{-1}\right)^{2H \wedge (2-2H)}, \\
&\leq 3(t_j t_k)^H \left(\frac{ut_j}{vt_k} \wedge \frac{vt_k}{ut_j}\right)^{H \wedge (1-H)}. \qquad (3.6)
\end{aligned}$$

The two inequalities in (3.6) are justified by the following facts, respectively: $a \wedge b \leq (a+b)/2 \leq a \vee b$, for all $a, b \in \mathbb{R}$ and $x^{2H} + x^{-2H} - \left|x - x^{-1}\right|^{2H} \leq 3(x \wedge x^{-1})^{2H \wedge (2-2H)}$, for all $x > 0$ and $H \in (0,1)$. Since $v/u \leq e^{-(\log(u^{-1}))^\varepsilon}$ we get

$$\begin{aligned}
\mathcal{A}^{r,\varepsilon}_{j,k} &\leq 3(t_j t_k)^H \left(\frac{t_j \vee t_k}{t_j \wedge t_k}\right)^{H \wedge (1-H)} \left(\frac{u}{v} \wedge \frac{v}{u}\right)^{H \wedge (1-H)}, \\
&\leq 3(t_j \vee t_k)^{2H} \left(\frac{v}{u}\right)^{H \wedge (1-H)} \leq 3(t_j \vee t_k)^{2H} e^{-(H \wedge (1-H))(\log(u^{-1}))^\varepsilon}, \\
&\longrightarrow 0, \quad \text{as } u \to 0^+. \qquad (3.7)
\end{aligned}$$

Observing that $\mathcal{A}^{r,\varepsilon}_{j,k} \geq 0$ and using (3.7) we obtain

$$\begin{aligned}
\sup_{ue^{-(\log(u^{-1}))^r} \leq v \leq ue^{-(\log(u^{-1}))^\varepsilon}} \sum_{j,k=1}^m \lambda_j \lambda_k \mathcal{A}^{r,\varepsilon}_{j,k} &\leq \sup_{ue^{-(\log(u^{-1}))^r} \leq v \leq ue^{-(\log(u^{-1}))^\varepsilon}} \sum_{(j,k) \in \mathcal{J}^+} \lambda_j \lambda_k \mathcal{A}^{r,\varepsilon}_{j,k}, \\
&\leq 3 \sum_{(j,k) \in \mathcal{J}^+} \lambda_j \lambda_k (t_j \vee t_k)^{2H} e^{-(H \wedge (1-H))(\log(u^{-1}))^\varepsilon}, \\
&\longrightarrow 0, \quad \text{as } u \to 0^+ \text{ and } r \to 1^-.
\end{aligned}$$

(iv) Let $\epsilon > 0$ and recall that $\tau(u) = u$, $w(u) = u^H$. We know that S^1_H is self-similar with index H and $\sigma^2_{\mathbf{T}} := \sup\left\{\mathbb{E}[S^1_H(\tau(u)t)]^2,\ t \in \mathbf{T}\right\} = (uT)^{2H}\mathbb{E}\left|S^1_H(1)\right|^2$, therefore by [6, Lemma 12.18] we assert that there is a nonnegative constant C (depending only on T) such that

$$\begin{aligned}
\mathbb{P}\left(\sup_{t \in \mathbf{T}} \frac{|S^1_H(\tau(ut))|}{w(u)\left(2\log\log(u^{-1})\right)^{1/2}} > \epsilon\right) &\leq C \exp\left[-\frac{\log\log(u^{-1})\epsilon^2}{2T^{2H}\mathbb{E}\left|S^1_H(1)\right|^2}\right], \\
&\longrightarrow 0, \quad \text{as } u \to 0^+.
\end{aligned}$$

(vi) Let $\eta > 0$ and $\delta > 0$ (to be chosen later). Straightforward computations lead to $\left\|S^1_H(\theta^n t) - S^1_H(\theta^n s)\right\|^2 = (\theta^n)^{2H}\rho(t,s)^2$. Thereby $\sup_{\substack{\rho(t,s) \leq \delta \\ t,s \in \mathbf{T}}} \left\|S^1_H(\theta^n t) - S^1_H(\theta^n s)\right\|^2 = (\theta^n)^{2H}\delta^2$.

Hence,

$$\sum_{n=1}^\infty \exp\left[\frac{-\eta(w^2(\theta^n)\log(n)}{\sup_{\substack{\rho(t,s) \leq \delta \\ t,s \in \mathbf{T}}} \left\|S^1_H(\theta^n t) - S^1_H(\theta^n s)\right\|^2}\right] = \sum_{n=1}^\infty \frac{1}{n^{\eta/\delta^2}} < \infty.$$

We choose $\delta < \sqrt{\eta}$ so that the last inequality holds true. At this stage, we assert that all conditions of [1, Theorem 4.1] are fulfilled and the statement (3.5) holds for $n = 1$.

Step 2. For the case $n \geq 2$ (odd) we apply the generalized L'hôpital's rule (e.g., [15] or [27, Theorem 6]) recursively to get

$$\begin{aligned}
0 \leq \limsup_{u\to 0^+} \frac{|S_H^n(ut)|}{\Phi_H(u)} &\leq \limsup_{u\to 0^+} \frac{\int_0^{ut}\int_0^x \left|S_{H-2}^{n-2}(y)\right| dydx}{\Phi_H(u)}, \\
&\leq t^2 \limsup_{u\to 0^+} \frac{\left|S_{H-2}^{n-2}(tu)\right|}{\Phi_H^{(2)}(u)} \leq \cdots \leq t^{n-1} \limsup_{u\to 0^+} \frac{|S_{H'}^1(tu)|}{\Phi_H^{(n-1)}(u)}, \\
&\quad H' = H - (n-1) \in (0,1), \\
&= t^{n-1} \limsup_{u\to 0^+} \frac{|S_{H'}^1(tu)|}{u^{H'}\left(2\log\log(u^{-1})\right)^{1/2}} < \infty.
\end{aligned}$$

Hence, the statement (3.5) holds true for every $n \geq 1$ odd.

Step 3. When n is even, we shall consider $\mathcal{R}_H^2(t,s)$ at first stage, then use L'hôpital's rule and the recurrent form of the covariance $\mathcal{R}_H^n(t,s)$ to get the general result for an even integer $n \geq 2$. For $n = 2$, the conditions **(i)**-**(ii)** and **(iv)**-**(ix)** of [1, Theorem 4.1] can be verified in similar fashion as done in Step 1. For the condition **(iii)**, using the same notations, the terms $\mathcal{A}_{j,k}^{r,\varepsilon}$ are of the form

$$\begin{aligned}
\mathcal{A}_{j,k}^{r,\varepsilon} &:= \mathbb{E}\left(\frac{S_H^2(ut_j)S_H^2(vt_k)}{w(u)w(v)}\right) = \frac{\mathcal{R}_H^2(ut_j, vt_k)}{u^H v^H}, \\
&= \frac{C_H^2}{2}(t_j t_k)^H \left[\left|y_{jk} - y_{jk}^{-1}\right|^{2H} + (y_{jk} + y_{jk}^{-1})^{2H} - 2\left(y_{jk}^{2H} + y_{jk}^{-2H}\right)\right], \\
&\qquad \text{with } y_{jk} = \sqrt{\frac{vt_k}{ut_j}}, \quad H \in (1,2) \\
&= C_H^2 (t_j t_k)^H \left[y_{jk}^{-2H} \sum_{l=1}^{\infty} \binom{2H}{2l} y_{jk}^{4l} - y_{jk}^{2H}\right], \\
&\leq C_H^2 (t_j t_k)^H \left(H(2H-1) y_{jk}^{4-2H} + \mathcal{O}(y_{jk}^{8-2H})\right) \longrightarrow 0, \quad \text{as } u \to 0^+.
\end{aligned}$$

The last two inequalities follow by expanding $|1 - y_{jk}|^{2H}$, $(1 + y_{jk})^{2H}$ and the fact that $v/u \to 0$ implies $y_{jk} \to 0$. By observing that $\mathcal{A}_{j,k}^{r,\varepsilon} \geq 0$ in this case, we conclude that **(iii)** holds true as well. The proof of Theorem 3.5 is then complete. □

Proposition 3.6. *The limit given in (3.5) is strictly positive.*

Proof. Unlike the fBm (e.g., [6, Proposition 12.19]) for which we know that

$$\limsup_{u\to 0^+} \frac{|B_H(u)|}{u^H \left(2\log\log u^{-1}\right)^{1/2}} = 1,$$

it is not clear how to specify the value of c_H^n (3.5). This is due to the complexity of covariance structure of S_H^n. Note that Theorem 3.5 states that $c_H^n \in [0, \infty)$. To establish Proposition 3.6 we split the proof into three steps:

Step 1. If $n=1$, then the process of interest reduces to the usual sub-fBm S_H^1 with $\Phi_H(u)=u^H\left(2\log\log(u^{-1})\right)^{1/2}$, $H\in(0,1)$. Observe that

$$\begin{aligned} c_H^1 &= \limsup_{u\to 0^+}\frac{|S_H^1(ut)|}{\Phi_H(u)}, \\ &\geq \limsup_{k\to\infty}\frac{\left|S_H^1(tr^k)\right|}{r^{kH}\sqrt{2\log\left(-k\log(r)\right)}}, \quad \text{for som fixed } r\in(0,1), \\ &\geq \limsup_{k\to\infty}\frac{\xi_k}{\sqrt{2\log(k)}}, \quad \text{where } \xi_k := S_H^1(tr^k)/r^{kH}. \end{aligned}$$

In the last inequality we used the fact that $\log\left(-k\log(r)\right)\sim\log(k)$, as $k\to\infty$. To conclude we shall show that

$$\limsup_{k\to\infty}\frac{\xi_k}{\sqrt{2\log(k)}}\geq\sqrt{\mathcal{R}_H^1(t,t)}>0. \tag{3.8}$$

Consider the sequence $\{\zeta_k=\mathcal{R}_H^1(t,t)^{-1/2}\xi_k \ : \ k\geq 1\}$. It is not hard to see that $\{\zeta_k\}_k$ is jointly normal with $\mathbb{E}(\zeta_k)=0$ and $\mathbb{V}ar(\zeta_k)=1$ (this is inherited from the Gaussianity of S_H^1). For $k,m\in(p,2p]$ with $k<m$ we have

$$\begin{aligned} \mathbb{E}\left(\zeta_k\zeta_m\right) &= Corr\left(S_H^1(r^k),S_H^1(r^m)\right), \\ &= r^{(m-k)H}+r^{(k-m)H}-\frac{1}{2}\left[\left(r^{(m-k)/2}+r^{(k-m)/2}\right)^{2H}+\left|r^{(m-k)/2}-r^{(k-m)/2}\right|^{2H}\right], \\ &\leq r^{(m-k)H}+r^{(k-m)H}-\left|r^{(m-k)/2}-r^{(k-m)/2}\right|^{2H}, \\ &\leq 3r^{(m-k)(H\wedge(1-H))}\leq 3r^{H\wedge(1-H)}\longrightarrow 0, \text{ as } r\to 0. \end{aligned}$$

The last three inequalities are justified by the following facts, respectively: $a\wedge b\leq(a+b)/2\leq a\vee b$, for all $a,b\in\mathbb{R}$ and $x^{2H}+x^{-2H}-\left|x-x^{-1}\right|^{2H}\leq 3(x\wedge x^{-1})^{2H\wedge(2-2H)}$, for all $x>0$, $H\in(0,1)$, and $(m-k)\in\{1,\cdots,p-1\}$. As result we can choose r small enough so that

$$\limsup_{p\to\infty}\max\{\mathbb{E}\left(\zeta_k\zeta_m\right) \ : \ k,m\in(p,2p], \ k\neq m\}<\frac{\delta}{2},$$

with $\delta\in(0,1)$. According to [6, Lemma 12.20], it follows that with probability one $\limsup_{k\to\infty}\frac{\zeta_k}{\sqrt{2\log(k)}}\geq 1-\delta$ or equivalently $\limsup_{k\to\infty}\frac{\xi_k}{\sqrt{2\log(k)}}\geq\sqrt{\mathcal{R}_H^1(t,t)}(1-\delta)>0$. Hence, (3.8) follows by the arbitrariness of δ.

Step 2. In the case $n=2$, the normalizing function is defined as

$$\Phi_H(z)=\int_0^z x^{H-1}\left(2\log\log(x^{-1})\right)^{1/2}dx,$$

and for every $r\in(0,e^{-1})$, we have

$$\begin{aligned} \Phi_H(r^k) &= \int_0^{r^k}x^{H-1}\left(2\log\log(x^{-1})\right)^{1/2}dx, \quad \text{with } H\in(1,2), \\ &= r^{kH}\int_0^1 y^{H-1}\left(2\log\log((r^ky)^{-1})\right)^{1/2}dy, \quad \text{(By change of variables } y=x/r^k), \\ &\leq r^{kH}\left(\int_0^{r^*}+\int_{r^*}^1\right)y^{H-1}\sqrt{2\log\left(-k\log(r)-\log(y)\right)}dy, \end{aligned} \tag{3.9}$$

where $r^*=e^{-a(r)}$ and $a(r)=-\log(r)/(-\log(r)-1)$.

Let $k > 1/(-\log(r))$. On the set $\{y \leq r^*\}$ we have

$$\begin{cases} \log(u+v) \leq \log(u) + \log(v) \\ \text{with } u = -k\log(r) \text{ and } v = -\log(y). \end{cases} \tag{3.10}$$

Just observe that $u > 1$ and

$$\begin{aligned} v = -\log(y) &\geq -\log(r^*) = a(r), \\ &= \frac{-\log(r)}{-\log(r)-1} \geq \frac{-\log(r)}{-\log(r)-1/k}, \\ &\geq \frac{u}{u-1}. \end{aligned}$$

On the set $\{y > r^*\}$ we have $-k\log(r) - \log(y) < -k\log(r) + a(r)$. Combining this fact with (3.9) and (3.10) yields

$$\begin{aligned} \Phi_H(r^k) &\leq r^{kH}\left\{\int_0^{r^*} y^{H-1}\sqrt{2\log\left(-k\log(r)\right)}dy + \int_0^{r^*} y^{H-1}\sqrt{2\log\log(1/y)}dy \right. \\ &\qquad \left. + \int_{r^*}^1 y^{H-1}\sqrt{2\log\left(-k\log(r)+a(r)\right)}dy\right\}, \\ &\leq r^{kH}\left\{2H^{-1}\sqrt{2\log\left(-k\log(r)+a(r)\right)} + \Phi_H(1)\right\}, \end{aligned}$$

Now using the fact: $\log\left(-k\log(r)+a(r)\right) \sim \log(k)$ as $k \uparrow \infty$, we can find $k_0 \geq 1$ and $C > 0$ such that $\Phi_H(r^k) \leq Cr^{kH}\sqrt{2\log(k)}$ for all $k \geq k_0$. Using this inequality we obatin

$$\begin{aligned} c_H^2 &= \limsup_{u\to 0^+} \frac{|S_H^2(ut)|}{\Phi_H(u)}, \\ &\geq C^{-1}\limsup_{k\to\infty} \frac{\left|S_H^2(tr^k)\right|}{r^{kH}\sqrt{2\log(k)}}, \quad \text{for som fixed } r \in (0,1). \end{aligned}$$

Once again, we consider the sequence $\{\eta_k = \mathcal{R}_H^2(t,t)^{-1/2}S_H^2(tr^k)/r^{kH} \ : \ k \geq 1\}$ and show that $\limsup_{k\to\infty} \frac{\eta_k}{\sqrt{2\log(k)}} \geq 1-\delta$. This follows immediately by [6, Lemma 12.20]. In fact, for $r \in (0, 1/2)$, $k, m \in (p, 2p]$ with $k < m$ and by straightforward computations we get

$$\begin{aligned} \mathbb{E}\left(\eta_k\eta_m\right) &= Corr\left(S_H^2(r^k), S_H^2(r^m)\right), \\ &= \frac{\left|r^{(m-k)/2} + r^{(k-m)/2}\right|^{2H} + \left|r^{(m-k)/2} - r^{(k-m)/2}\right|^{2H} - 2\left(r^{(m-k)H} + r^{(k-m)H}\right)}{2^{2H}-4} \\ &= \frac{r^{(k-m)H}\left[\left(1+r^{(m-k)}\right)^{2H} + \left(1-r^{(m-k)}\right)^{2H} - 2\left(1+r^{2H(m-k)}\right)\right]}{2^{2H}-4} \\ &\leq Lr^{(2-H)(m-k)} \leq Lr^{2-H} \longrightarrow 0, \text{ as } r \to 0, \end{aligned}$$

where $L = 2(2^{2H}-4)^{-1}\left(H(2H-1) + \sum_{j=2}^{\infty}\binom{2H}{2j}\left(\frac{1}{2}\right)^{2j-2}\right)$, which implies the condition (12.31) in [6, Lemma 12.20]. Now, we can conclude that $c_H^2 \geq C^{-1}\sqrt{\mathcal{R}_H^2 1(t,t)} > 0$.

Step 3. (General Case) Fix $n \geq 3$ (odd or even). We shall suppose that c_H^n given in (3.5) equals zero and obtain a contradiction. Clearly, the aforementioned hypothesis implies $\lim_{u\to 0^+} \frac{S_H^n(ut)}{\Phi_H(u)} = 0$, then by using the usual L'hôpital's rule recursively we obtain $c_{H'}^1 = 0$ (if n is odd with $c_{H'}^1$ being the quantity associated with $S_{H'}^1$, $H' \in (0,1)$) or $c_{H'}^2 = 0$ (if n is even with $c_{H'}^2$ being the quantity associated with $S_{H'}^2$, $H' \in (1,2)$). This clearly contradicts results of the two previous cases. Note that the process of differentiation should occur according to **(iii)**-Theorem 3.2. □

Acknowledgment. We would like to thank the editorial staff and the reviewers for the comments that helped in improving this work.

References

[1] M.A. Arcones, *On the law of the iterated logarithm for Gaussian processes*, J. Theoret. Probab. **8** (4), 877-903, 1995.

[2] T. Bojdecki, L. Gorostiza and A. Talarczyk, *Sub-fractional Brownian motion and its relation to occupation times*, Statist. Probab. Lett. **69** (4), 405-419, 2004.

[3] T. Bojdecki, L. Gorostiza and A. Talarczyk, *Limit theorems for occupation time fluctuations of branching systems i: long-range dependence*, Stochastic Process. Appl. **116** (1), 1-18, 2006.

[4] T. Bojdecki, L. Gorostiza and A. Talarczyk, *Some extensions of fractional Brownian motion and sub-fractional Brownian motion related to particle systems*, Electron. Commun. Probab. **12**, 161-172, 2007.

[5] J.J. Collins and C.J. De Luca, *Upright, correlated random walks: A statistical-biomechanics approach to the human postural control system*, Chaos **5** (1), 57-63, 1995.

[6] R.M. Dudley, R. Norvaia and R. Norvaia, *Concrete Functional Calculus*, Springer, 2011.

[7] M. El Omari, *Notes on spherical bifractional Brownian motion*, Mod. Stoch. Theory Appl. **9** (3), 339-355, 2022.

[8] M. El Omari, *An α-order fractional Brownian motion with Hurst index $H \in (0,1)$ and $\alpha \in \mathbb{R}_+$*, Sankhya A **85** (1), 572-599, 2023.

[9] M. El Omari, *Mixtures of higher-order fractional Brownian motions*, Comm. Statist. Theory Methods **52** (12), 4200-4215, 2023.

[10] M. El Omari, *Parameter estimation for nth-order mixed fractional Brownian motion with polynomial drift*, J. Korean Stat. Soc. **52**, 450-461, 2023.

[11] C.W.J. Granger, *The typical spectral shape of an economic variable*, Econometrica **34** (1), 150-161, 1966.

[12] D.P. Huy, *A remark on non-Markov property of a fractional Brownian motion*, Vietnam J. Math. **31** (3), 237-240, 2003.

[13] R. Jennane and R. Harba, *Fractional Brownian motion: A model for image texture*, EUSIPCO Signal Process. **3**, 1389-1392, 1994.

[14] W.S. Kuklinski, K. Chandra, U.E. Ruttirmann and R.L. Webber, *Application of fractal texture analysis to segmentation of dental radiographs*, Proc. SPIE 1092, Medical Imaging III: Image Processing **1092**, 111-117, 1989.

[15] C.M. Lee, *Generalizations of l'Hôpitals rule*, Proc. Amer. Math. Soc. **66** (2), 315-320, 1977.

[16] T. Lundahl, W.J. Ohley, S.M. Kay and R. Siffert, *Fractional Brownian motion: A maximum likelihood estimator and its application to image texture*, IEEE Trans. Med. Imag. **5** (3), 152-161, 1986.

[17] B.B. Mandelbrot and J.W. Van Ness, *Fractional Brownian motions, fractional noises and applications*, SIAM Rev. **10** (4), 422-437, 1968.

[18] A.I. McLeod and K.W. Hipel, *Preservation of the rescaled adjusted range: 1. A reassessment of the Hurst phenomenon*, Water Resour. Res. **14** (3), 491-508, 1978.

[19] A.P. Pentland, *Fractal-based description of natural scenes*, IEEE Trans. Pattern Anal. Mach. Intell. **6** (6), 661-674, 1984.

[20] E. Perrin, R. Harba, C. Berzin-Joseph, I. Iribarren, and A. Bonami, *nth-order fractional Brownian motion and fractional gaussian noises*, IEEE Trans. Signal Process. **49** (5), 1049-1059, 2001.

[21] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer Science & Business Media, 1999.

[22] G. Shen, C. Chen and L. Yan, *Remarks on sub-fractional Bessel processes*, Acta Math. Sci. Ser. **31** (5), 1860-1876, 2011.

[23] C. Tudor, *Some properties of the sub-fractional Brownian motion*, Stochastics **79** (5), 431-448, 2007.

[24] C. Tudor, *On the Wiener integral with respect to a sub-fractional Brownian motion on an interval*, J. Math. Anal. Appl. **351** (1), 456-468, 2009.

[25] W. Willinger, M.S. Taqqu, W.E. Leland and D.V. Wilson, *Self-similarity in high-speed packet traffic: analysis and modeling of ethernet traffic measurements*, Statist. Sci. **10** (1), 67-85, 1995.

[26] H. Qi and L. Yan, *A law of iterated logarithm for the subfractional Brownian motion and an application*, J. Inequal. Appl. **2018** (96), 1-18, 2018.

[27] W.H. Young, *On indeterminate forms*, Proc. Lond. Math. Soc. **2** (1), 40-76, 1910.

[28] M. Zabat, M. Vayer-Besançon, R. Harba, S. Bonnamy and H. Van Damme, *Surface topography and mechanical properties of smectite films*, Trends in Colloid and Interface Science XI, 96-102, Springer, Berlin, 1997.