On the $n$th-Order subfractional Brownian motion

Mohamed El Omari*, Mohamed Mabdaoui

Chouaib Doukkali University, Polydisciplinary Faculty of Sidi Bennour, B.P. 299, Jabrane Khalid Jabrane Street, 24000 El Jadida, Morocco

Abstract

In the present work, we introduce the $n$th-Order subfractional Brownian motion $S_H^n = \{S_H^n(t), t \geq 0\}$ with Hurst index $H \in (n - 1, n)$ and order $n \geq 1$; then we examine some of its basic properties: self-similarity, long-range dependence, non Markovian nature and semimartingale property. A local law of iterated logarithm for $S_H^n$ is also established.

Mathematics Subject Classification (2020). 60G05, 60G15, 60G17, 60G18, 60G22

Keywords. Gaussian self-similar process, non Markovian process, subfractional Brownian motion, semimartingale property, local law of the iterated logarithm

1. Introduction

The self-similarity and long range-dependence have become two important aspects of stochastic models. The first one means that as scale is changing, the process looks like identical. For this reason, it has been applied in image processing for modeling texture having multiscale patterns such as natural scenes [13, 19], bone texture radiographs [16], or rough surfaces [28]. The long range-dependence is strongly related to long memory phenomena arising in a variety of different scientific fields, including hydrology [18], biology [5], medicine [14], economics [11] or traffic network [25]. The fractional Brownian motion (fBm) is the best known and most widely used self-similar process that exhibits the long-range dependence. Thus, it is not surprising that a large number of publications are devoted to the study of fBm and its generalizations (see, e.g., [7–10] and references therein).

The two-sided fBm with Hurst index $H \in (0, 1)$ is formally defined as a centered Gaussian process $B_H = \{B_H(t), t \in \mathbb{R}\}$ having the covariance function

$$
\mathbb{E}(B_H(t)B_H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t - s|^{2H}\right), \quad \text{for all } t, s \in \mathbb{R}.
$$

The fBm is of stationary increments and reduces to the standard Brownian motion (Bm) in the case $H = 1/2$. Compared to the extensive studies on fBm, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments. As an extension of Brownian motion, the authors in [2] introduced and studied a rather special class of self-similar Gaussian processes which they call sub-fractional Brownian motions (sub-fBm). As mentioned by the authors, these

*Corresponding Author.

Email addresses: elomari.m@ucd.ac.ma (M. El Omari), mabdaoui.m@ucd.ac.ma (M. Mabdaoui)

Received: 27.09.2022; Accepted: 30.05.2023
processes have properties analogous to those of fBm, and they are intermediate between Bm and fBm in the sense that their increments on nonoverlapping intervals are more weakly correlated and their covariance decays faster than for fBm. The sub-fBm is formally defined as a zero mean Gaussian process \( S_H^n, H \in (0, 2) \) with covariance

\[
R_H^n(t, s) = (2 - 2H) \left( \frac{t^{2H}}{(t+s)^{2H}} \right), \quad \text{for all } t, s \geq 0.
\]

The existence of \( S_H^n \) for all \( H \in (0, 1) \) follows from the equality in distribution \( (S_H^n(t))_{t \geq 0} \triangleq c_H (B_H(t) + B_H(-t))_{t \geq 0} \), where \( B_H \) is a two-sided fBm and \( c_H \) is some nonnegative constant. It is important to note that \( S_H^n \) is semimartingale if \( H = 1/2 \) or \( H \in (1, 2) \). This kind of processes arises from occupation time fluctuations of branching particle systems with Poisson initial condition. More works on sub-fBm can be found in [2–4, 22–24]. In [20], the authors introduced \( n \)th-order fBms as extensions of the standard fBm. Such extensions are very smooth as the order \( n \) increases and they exhibit long-range dependence; while the stationarity of increments is achieved at the order \( n \). One of the main features of \( n \)th-order fBm’s is their ability to describe a wide class of \( 1/f^\alpha \)-nonstationary signals with the range \( \alpha \in (1, \infty) \). It is shown in [9] that \( n \)th-order fBm’s are semimartingales whenever \( n \geq 2 \). Some extensions of them can be found in [8, 9]. Motivated by this kind of processes we introduce the \( n \)th-order sub-fBm \( S_H^n \) and establish some of its basic properties. In comparison with the fBm, \( S_H^n \) extends the usual sub-fBm and share many properties with the \( n \)th-order fBm. Especially, the semimartingale property required for modeling fluctuations in movement of stock prices with arbitrage opportunities being excluded.

The rest of the paper is organized as follows: In Section 2, we recall the definition and some properties of the \( n \)th-order fBm; while Section 3 is devoted to our main results. The following notations are systematically used: \( x_+ = \max(x, 0), x_- = \max(-x, 0) \) for all \( x \in \mathbb{R} \) and the symbol \( \triangleq \) denotes the equality in terms of finite dimensional distributions; while \( g(x) = O(f(x)) \) and \( g(x) \sim f(x) \) (as \( x \to x^* \)) are respectively used to say that \( x \mapsto g(x) / f(x) \) is bounded on neighbourhood of \( x^* \) and \( \lim_{x \to x^*} g(x) / f(x) = 1 \).

2. \( n \)th-order fractional Brownian motion

In [20], the \( n \)th-order fBm (hereafter \( B_H^n, H \in (n-1, n) \), \( n \geq 1 \) integer) is defined as a zero mean Gaussian process starting at zero with the integral representation

\[
B_H^n(t) = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{0} \left( (t-s) \Gamma^{-1/2} - \sum_{j=0}^{n-1} \binom{H-1/2}{j} (-s)^{H-1/2-j} t^j \right) dB(s)
+ \frac{1}{\Gamma(H+1/2)} \int_{0}^{t} (t-s)^{H-1/2} dB(s),
\]

where \( B(t) \) is two-sided standard Brownian motion (Bm), \( \Gamma(x) \) stands for the usual Gamma function and

\[
\binom{\alpha}{j} = \frac{\alpha(\alpha-1) \cdots (\alpha-j+1)}{j!}, \quad \binom{\alpha}{0} = 1 \text{ (by convention)}.
\]

In the case \( n = 1 \), the standard fBm is retrieved, as formula (2.1) reduces to the Mandelbrot-Van Ness representation [17] of the fBm. The process \( B_H^n \) satisfies the following properties (for more details and proofs, see [8, 9, 20]).

(i) \( B_H^n \) is self-similar with exponent \( H \), i.e., \( B_H^n(ct) \triangleq c^H B_H^n(t) \), for every \( c > 0 \).
(ii) \( B_H^n \) has derivatives up to order \( n-1 \) vanishing at zero and the \( (n-1) \)th derivative coincides with the standard fBm, that is, \( \frac{d^{n-1}}{dt^{n-1}} (B_H^n(t)) = B_H^n(t) \).
(iii) $B^n_H$ exhibits long-range dependence and stationarity of increments is achieved at order $n$, that is, the increments $\Delta^k_s B^n_H$, $s > 0$, $k = 0, \cdots, n - 1$ are nonstationary and $\Delta^k_s B^n_H$ is a stationary process. Here $\Delta^k_s g(x)$ stands for increments of a function $g(x)$ at order $k$ with explicit form

$$\Delta^k_s g(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g(x+jl)$$

and $\Delta^0_s g(x) = g(x)$.

(iv) The covariance function of the process $B^n_H$ is given by

$$G^n_{H,n}(t,s) = (-1)^n \frac{C^n_H}{2} \left\{ |t-s|^{2H} - \sum_{j=0}^{n-1} (-1)^j \left( \binom{n}{j} \right)^2 \left[ \frac{t}{s} \right]^j |s|^{2H} + \left( \frac{s}{t} \right)^j |t|^{2H} \right\},$$

where $C^n_H$ is a nonnegative constant defined recursively by

$$C^n_1 = 1/ (\Gamma(2H + 1) \sin(\pi H))$$

and for $n \geq 2$

$$C^n_H = \frac{C^{n-1}_H}{(2H)(2H-1) \cdots (2H-(2n-3))}.$$  

In particular,

$$\text{Var} (B^n_H(t)) = C^n_H \left( \frac{2H-1}{n-1} \right) |t|^{2H}.$$  

(v) For any $n \geq 2$ the process $B^n_H$ is a special semimartingale with finite variation.

(vi) $B^n_H$ is a Markov process if and only if $n = 1$ and $H = 1/2$.

(vii) $B^n_H$ can be extended to an $\alpha$-order fBm $U^n_H$ (see [8]) defined as

$$U^n_H(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^\alpha dB^n_H(s), \quad H \in (0,1), \quad \alpha \in (-1, \infty),$$

whenever this integral exists. Here $B_H$ denotes a one-sided fBm. In the case $\alpha = 0$, we retrieve the standard fBm $B_H$. If $\alpha = n-1$, then $U^n_H$ coincides with the $n$th-order fBm with Hurst parameter $H' = H + (n-1)$.

3. Main results

We define the $n$th-order sub-fBm $S^n_H$ as $S^n_H(t) = (B^n_H(t) + B^n_H(-t)) / \sqrt{2}$, for all $t \geq 0$ and $H \in (n-1,n)$, where $B^n_H$ is a two-sided $n$th-order fBm defined as centered Gaussian with covariance function (2.2). Clearly, the case $n = 1$ corresponds to the usual sub-fBm. Before we establish some properties of $S^n_H$ that are of great importance, we introduce a definition of long-range dependence for non stationary processes.

**Definition 3.1.** Let $s > 0$ be fixed and $t > s$. Then a process $X$ is said to have long-range dependence property if

$$\text{Corr} (X(s), X(t)) \sim c(s)t^{-d}, \quad \text{as} \ t \to \infty,$$

where $c(s)$ is a constant depending on $s$ and $d \in (0,1)$. Here $\text{Corr} (X(s), X(t))$ stands for the correlation function of the process $X$.

**Theorem 3.2.** Let $S^n_H$ be the $n$th-order sub-fBm with $H \in (n-1,n)$. The following statements hold.

(i) $S^n_H$ is a centered Gaussian process with the covariance function

$$S^n_H(t,s) = \frac{(-1)^n C^n_H}{2} \left[ |t-s|^{2H} + |t+s|^{2H} - 2 \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2H-2j)^{2H-2j} \right],$$

for all $t,s \geq 0$, where $C^n_H$ is a nonnegative given in (2.3).
The process $S^n_H$ is self-similar with index $H$. i.e.,

$$\{S^n_H(at), \ t \geq 0\} \triangleq \{a^HS^n_H(t), \ t \geq 0\}, \text{ for every } a > 0.$$

(iii) The process $S^n_H$ is differentiable for every $n \geq 2$ and can be rewritten as

$$S^n_H(t) = \int_0^t \int_0^s S^n_{H-2}(u)du, \text{ for all } t \geq 0 \text{ and } n \geq 3.$$

(iv) The process $S^n_H$ is semimartingale for every $n \geq 2$.

(v) The process $S^n_H$ admits the following representation $S^n_H(t) = \int_{\mathbb{R}} K_H(t,s)dB(s)$, where

$$K_H(t,s) = \frac{1}{\sqrt{2\Gamma(H-1/2)}} \left( (t-s)^{2H} + (t+s)^{2H} - 2 \sum_{j=0}^{n-1} \left( \frac{H-1/2}{2j} \right)(-s)^{H-1/2-2j} t^{2j} \right).$$

(vi) The process $S^n_H$ has long-range dependence property in terms of Definition 3.1 for all $n \geq 1$ and $H \in (n-1, n)$.

Proof.

- The first statement (i) follows by definition of the process $S^n_H$ and the use of equation (2.2); while the statement (ii) can be readily verified by using the form of its covariance function given in (i). In fact, one has $\mathcal{R}^n_H(ct,cs) = c^{2H}\mathcal{R}^n_H(t,s)$, for all $t, s \geq 0$ and any $c > 0$.

- (iii)-(iv): First, note that $S^n_H$ is differentiable for every $n \geq 2$ (this is inherited from $B^n_H$), and simple computations leads to (iii). This suggests that $S^n_H$ is semimartingale. Indeed, one can prove this property in the same way as in [9, Theorem 2.1]. Finally, to get (v) it suffices to combine the the Mandelbrot-Van Ness representations of both $B^n_H(t)$ and $B^n_H(-t)$.

- (vi): Let $t > s$ and $s > 0$ fixed. By expanding both $\left( 1 - \frac{s}{t} \right)^{2H}$ and $\left( 1 + \frac{s}{t} \right)^{2H}$ in the correlation form of $S^n_H(t)$, it follows that as $t \to \infty$

$$\text{Corr} \left( S^n_H(s), S^n_H(t) \right) = \frac{\mathcal{R}^n_H(t,s)}{\sqrt{\mathcal{R}^n_H(t,t)\mathcal{R}^n_H(s,s)}} \sim \begin{cases} -2d_{n,H} \left( \frac{2H}{n-1} \right) \left( \frac{s}{t} \right)^{n-(n-1)} H^{-1}, & \text{when } n \text{ is odd,} \\ 2d_{n,H} \left( \frac{2H}{n} \right) \left( \frac{s}{t} \right)^n H, & \text{when } n \text{ is even} \end{cases} \text{ and } n-1 < H < n.$$ 

Theorem 3.3. The nth order sub-fBm $S^n_H$ is Markovian if and only if $(n,H) = (1,1/2)$.

Since our processes of interest are centered Gaussian, we will use the following lemma, for which the proof can be found separately in [10, 1.13]-Chapter III] and [12].

Lemma 3.4. Let $X = \{X(t), \ t \geq 0\}$ be a centered Gaussian process with covariance function $\mathcal{R}(t,s)$. The following statements hold.

(i) The process $X$ is Markovian if and only if $\mathcal{R}(t,s)\mathcal{R}(u,u) = \mathcal{R}(t,u)\mathcal{R}(u,s)$, for every $t > u > s$. 

\[ \square \]
(ii) If $X$ is a Markov process then we have $\mathcal{R}(t, s) = \frac{\mathcal{R}(t, u)\mathcal{R}(s, u)}{\mathcal{R}(u, u)}$, for every $t > u > s$.

**Proof of Theorem 3.3.** In the case $n = 1$ we retrieve the usual sub-fBm $S^1_H$, $H \in (0, 1)$, which is known to be Markovian if and only if $H = 1/2$ (e.g., [2]). We shall only prove that $S^n_H$ with $H \in (n - 1, n)$ is not a Markov process for every $n \geq 2$. To do so we follow [9] and establish the following statements:

(i) If the process $S^n_H$, $H \in (n - 1, n)$ is Markovian, then the processes $S^{n-2k}_H$ with $n - 2k \geq 1$ and $k$ is an integer, are Markovian as well.

(ii) The processes $S^n_H$ and $S^n_{1/2}$ are non Markovian.

The use of the covariance function $\mathcal{R}^n_H(t, s)$ as given in (i)-Theorem 3.2 will complicate our computations. Instead, we shall use its integral form which follows from (iii)-Theorem 3.2. We have

$$\mathcal{R}^n_H(t, s) = \int_0^t \int_0^s \int_0^y \mathcal{R}^{n-2}(\xi, \zeta) d\xi d\zeta dy.$$ 

If $S^n_H$ is Markovian, then by (i)-Lemma 3.4 we have

$$\mathcal{R}^n_H(t, s)\mathcal{R}^n_H(u, u) = \mathcal{R}^n_H(t, u)\mathcal{R}^n_H(u, s), \text{ for all } t > u > s;$$

or

$$\int_0^t \int_0^s \int_0^y \mathcal{R}^{n-2}_H(\xi, \zeta) d\xi d\zeta dy = \frac{1}{\mathcal{R}^{n-2}_H(u, u)} \left[ \int_0^t \int_0^u \int_0^y \mathcal{R}^{n-2}_H(\xi, \zeta) d\xi d\zeta dy \right]$$

$$\times \left[ \int_0^u \int_0^y \mathcal{R}^{n-2}_H(\xi, \zeta) d\xi d\zeta dy \right]. \quad (3.1)$$

Differentiating equality (3.1) twice with respect to $t$ and twice with respect to $s$, we obtain

$$\mathcal{R}^{n-2}_H(t, s) = \frac{1}{\mathcal{R}^{n-2}_H(u, u)} \left[ \int_0^u \int_0^y \mathcal{R}^{n-2}_H(t, \xi) d\xi dx \right]$$

$$\times \left[ \int_0^u \int_0^y \mathcal{R}^{n-2}_H(s, \zeta) d\zeta dy \right], \text{ for all } t > u > s. \quad (3.2)$$

Let $a, b$ be nonnegative numbers such that $s < a < u < b < t$, then from (3.2) we get

$$\mathcal{R}^{n-2}_H(a, b) = \frac{1}{\mathcal{R}^{n-2}_H(u, u)} \left[ \int_0^u \int_0^y \mathcal{R}^{n-2}_H(b, \xi) d\xi dx \right]$$

$$\times \left[ \int_0^u \int_0^y \mathcal{R}^{n-2}_H(a, \zeta) d\zeta dy \right]. \quad (3.3)$$

Multiplying equations (3.2) and (3.3), side by side, we obtain

$$\mathcal{R}^{n-2}_H(t, s)\mathcal{R}^{n-2}_H(a, b) = \left[ \frac{1}{\mathcal{R}^{n-2}_H(u, u)} \int_0^u \int_0^y \mathcal{R}^{n-2}_H(s, \zeta) d\zeta dy \times \int_0^u \int_0^y \mathcal{R}^{n-2}_H(b, \xi) d\xi dx \right]$$

$$\times \left[ \frac{1}{\mathcal{R}^{n-2}_H(u, u)} \int_0^u \int_0^y \mathcal{R}^{n-2}_H(t, \xi) d\xi dx \times \int_0^u \int_0^y \mathcal{R}^{n-2}_H(a, \zeta) d\zeta dy \right],$$

and this implies

$$\mathcal{R}^{n-2}_H(t, s)\mathcal{R}^{n-2}_H(a, b) = \mathcal{R}^{n-2}_H(t, a)\mathcal{R}^{n-2}_H(b, s), \quad (3.4)$$

using the fact that the covariance function is continuous and taking the limit $(a \to u; \ b \to u)$ in (3.4) we obtain

$$\mathcal{R}^{n-2}_H(t, s)\mathcal{R}^{n-2}_H(u, u) = \mathcal{R}^{n-2}_H(t, u)\mathcal{R}^{n-2}_H(u, s),$$
thereby $S_{H-2}^n$ is Markovian and (i) is then established.
Let $n = 2$ and $H \in (1, 2)$. After some computations it follows from Theorem 3.2-(i)
\[
\mathcal{R}_H^2(t, s) = \frac{C_H^2}{2} \left[|t - s|^{2H} + (t + s)^{2H} - 2 \left(2^{2H} + s^{2H}\right)\right].
\]
For $t > 0$ fixed, we set $\Psi_t : s \mapsto \mathcal{R}_H^2(t, s)/\mathcal{R}_H^2(s, s)$. If $S_{H}^2$ is Markovian then by virtue of (ii)-Lemma 3.4, we get
\[
\frac{\mathcal{R}_H^2(t, s)}{\mathcal{R}_H^2(s, s)} = \frac{\mathcal{R}_H^2(t, u)}{\mathcal{R}_H^2(u, u)} \quad \text{for all } t > u > s,
\]
which means that $\Psi_t$ must be a constant function on the interval $(0, t)$. Observe that
\[
\Psi_t(s) = \frac{1}{(2^{2H} - 4)} \left[\left(\frac{t}{s} + 1\right)^{2H} + \left(\frac{t}{s} - 1\right)^{2H} - 2 \left(\frac{t}{s}\right)^{2H} - 2\right].
\]
By standard calculus we check that the function $\Psi_t$ is not constant, and this yields a contradiction. Hence $S_{H}^2$ is non Markovian. Finally, the covariance of $S_{H}^2$ has an explicit form as
\[
\mathcal{R}_H^2(t, s) = C_H^2 \left[(t \lor s)^5 - 5(t \lor s)^4(t \lor s) + 10(t \lor s)^3(t \lor s)^2\right], \quad \text{for all } t, s \geq 0.
\]
For $t = 2$, $u = 1$ and $s = \frac{1}{2}$; with simple calculations we find
\[
\mathcal{R}_H^2\left(2, \frac{1}{2}\right)^3 \mathcal{R}_H^2\left(1, 1\right) \neq \mathcal{R}_H^2\left(2, 1\right)^3 \mathcal{R}_H^2\left(1, \frac{1}{2}\right).
\]
\[\square\]

**Theorem 3.5.** Consider the $n$th order sub-fBM $S_{H}^n$. Then, with probability one, the following limit
\[
c_H^n := \lim_{u \to 0^+} \left|\frac{S_H^n(ut)}{\Phi_H(u)}\right|, \quad \text{exists for all } t, \in (0, T],
\]
where $\Phi_H$ is $(n - 1)$-times continuously differentiable function such that
\[
\Phi_H^{(n-1)}(u) = u^{H-n+1} \left(2 \log \log(u^{-1})\right)^{1/2}, \quad \text{for all } u > 0.
\]

**Proof.** We split the proof of this theorem into three steps. First, we show that (3.5) holds in the case $n = 1$, which corresponds to the usual sub-fBM with $\Phi_H(u) = u^H \left(2 \log \log(u^{-1})\right)^{1/2}$, $H \in (0, 1)$. Note that another form of the law of iterated logarithm for the sub-fBM can be found in [26]. Second, we establish the statement (3.5) for every $n \geq 3$ odd. Finally, we show (3.5) for every $n \geq 2$ in a similar fashion as done in the previous steps.

**Step 1.** Let $n = 1$ and $H \in (0, 1)$. In this case, we adopt Arcones’ notations [1] and verify the conditions (i)-(ix) of [1, Theorem 4.1]. Let $u \in [0, 1]$, $t \in T = [0, T]$ and consider the pseudometric $\rho(u, v) = \sqrt{\mathbb{E} \left(S_H^1(u) - S_H^1(v)^2\right)}$. Set $\tau(u) = u$ and $w(u) = u^H$. Clearly, $\rho(0, T) = \sqrt{\mathbb{E} \left(S_H^1(T)^2\right)} = \sqrt{2 - 2^{2H-1}T^H} < \infty$, thus (v) follows immediately. It is not hard to see that the conditions (i), (vii)-(ix) are satisfied. For the condition (ii) let $t, s \in T$ and $u \in [0, 1]$. By self-similarity of the process $S_H^1$, we have
\[
\lim_{u \to 0^+} \mathbb{E} \left[\frac{S_H^1(\tau(u)s)S_H^1(\tau(ut))}{w^2(u)}\right] = \lim_{u \to 0^+} \mathbb{E} \left[\frac{S_H^1(us)S_H^1(ut)}{u^{2H}}\right],
\]
\[
= \mathbb{E} \left(S_H^1(s)S_H^1(t)\right) = \mathcal{R}_H(t, s),
\]
(iii) Let $m \geq 1$ (integer), $r > \varepsilon > 0$, $t_1, \ldots, t_m \in \mathbf{T} \setminus \{0\}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Set $\mathcal{J} := \{\text{the set of pairs } (j, k), \ 1 \leq j, l \leq m \text{ for which } \lambda_j \lambda_k \geq 0\}$. For
\[ v \in \left[u e^{-\left(\frac{\log(u-1)}{r}\right)}, u e^{-\left(\frac{\log(u-1)}{r}\right)}\right], \]
we have
\[
\mathcal{A}_{\overline{r}, \varepsilon}^{\tau, \kappa} := \mathbb{E}\left(\frac{S_H^1(u t_j) S_H^1(v t_k)}{w(u) w(v)}\right) = \frac{\mathcal{R}_H^1(u t_j, v t_k)}{u^H v^H},
\]
\[
= (t_j t_k)^H \left[ x_{jk}^{2H} + x_{jk}^{-2H} - \frac{1}{2} (x_{jk} + x_{jk}^{-1})^{2H} + \frac{1}{2} (x_{jk} - x_{jk}^{-1})^{2H}\right], \]
with $x_{jk} = \sqrt{\frac{u t_j}{v t_k}}$,
\[
\leq (t_j t_k)^H \left[ x_{jk}^{2H} + x_{jk}^{-2H} - \frac{1}{2} (x_{jk} + x_{jk}^{-1})^{2H} + \frac{1}{2} (x_{jk} - x_{jk}^{-1})^{2H}\right] \leq 3(t_j t_k)^H \left( x_{jk} \wedge x_{jk}^{-1}\right)^{(H \wedge (2H)}(2H - 2H)
\]
\[
\leq 3(t_j t_k)^H \left( \frac{u t_j}{v t_k} \wedge \frac{v t_k}{u t_j}\right)^{(H \wedge (1-H))}, \tag{3.6}
\]
The two inequalities in (3.6) are justified by the following facts, respectively: $a \wedge b \leq (a + b)/2 \leq a \vee b$, for all $a, b \in \mathbb{R}$ and $x^{2H} + x^{-2H} - |x - x^{-1}|^{2H} \leq 3(x \wedge x^{-1})^{2H} \wedge (2H - 2H)$, for all $x > 0$ and $H \in (0, 1)$. Since $v/u \leq e^{-\left(\frac{\log(u-1)}{r}\right)e}$ we get
\[
\mathcal{A}_{\overline{r}, \varepsilon}^{\tau, \kappa} \leq 3(t_j t_k)^H \left( \frac{t_j \vee t_k}{t_j \wedge t_k} \wedge \frac{v}{u}\right)^{(H \wedge (1-H))},
\]
\[
\leq 3(t_j \vee t_k)^H \left( \frac{v}{u}\right)^{(H \wedge (1-H))} \leq 3(t_j \vee t_k)^H e^{-(H \wedge (1-H))(\log(u-1))},
\]
\[ \rightarrow 0, \text{ as } u \rightarrow 0^+. \tag{3.7}
\]
Observing that $\mathcal{A}_{\overline{r}, \varepsilon}^{\tau, \kappa} \geq 0$ and using (3.7) we obtain
\[
\sup_{u \in (\log(u-1))^{0^+}} \sum_{j,k=1}^{m} \lambda_j \lambda_k \mathcal{A}_{\overline{r}, \varepsilon}^{\tau, \kappa} \leq \sup_{u \in (\log(u-1))^{0^+}} \sum_{(j,k) \in \mathcal{J}^+} \lambda_j \lambda_k \mathcal{A}_{\overline{r}, \varepsilon}^{\tau, \kappa},
\]
\[
\leq 3 \sum_{(j,k) \in \mathcal{J}^+} \lambda_j \lambda_k (t_j \vee t_k)^H e^{-(H \wedge (1-H))(\log(u-1))},
\]
\[ \rightarrow 0, \text{ as } u \rightarrow 0^+ \text{ and } r \rightarrow 1^-.
\]
(iv) Let $\varepsilon > 0$ and recall that $\tau(u) = u$, $w(u) = u^H$. We know that $S_H^1$ is self-similar with index $H$ and $\sigma_T^2 := \sup \mathbb{E}[S_H^1(\tau(u) t)]^2, t \in \mathbf{T} = (u T)^H \mathbb{E} S_H^1(1)^2$, therefore by [6, Lemma 12.18] we assert that there is a nonnegative constant $C$ (depending only on $T$) such that
\[
\mathbb{P}\left( \sup_{t \in \mathbf{T}} \frac{|S_H^1(\tau(u) t)|}{w(u) (2 \log(u-1))^{1/2}} > \varepsilon \right) \leq C e^{-\frac{\log(u-1)\varepsilon^2}{2T^2 H \mathbb{E} S_H^1(1)^2}},
\]
\[ \rightarrow 0, \text{ as } u \rightarrow 0^+. \]

(vi) Let $\eta > 0$ and $\delta > 0$ (to be chosen later). Straightforward computations lead to
\[
\left| S_H^1(\theta^n t) - S_H^1(\theta^n s) \right|^2 = (\theta^n)^{2H} \rho(t, s)^2.
\]
Thereby
\[
\sup_{\rho(t,s) \leq \delta} \left\| S_H^1(\theta^n t) - S_H^1(\theta^n s) \right\|^2 = (\theta^n)^{2H} \delta^2.
\]
Hence,
\[
\sum_{n=1}^{\infty} \exp\left[ -\eta (w^2(\theta^n) \log(n)) \sup_{\rho(t,s) \leq \delta} \left\| S_H^1(\theta^n t) - S_H^1(\theta^n s) \right\|^2 \right] = \sum_{n=1}^{\infty} \frac{1}{\eta^2 / \delta^2} < \infty.
We choose $\delta < \sqrt{\eta}$ so that the last inequality holds true. At this stage, we assert that all conditions of [1, Theorem 4.1] are fulfilled and the statement (3.5) holds for $n = 1$.

**Step 2.** For the case $n \geq 2$ (odd) we apply the generalized L’hôpital’s rule (e.g., [15] or [27, Theorem 6]) recursively to get

$$0 \leq \limsup_{u \to 0^+} \frac{|S^n_H(ut)|}{\Phi_H(u)} \leq \limsup_{u \to 0^+} \frac{\int_0^ut \int_0^t |S_{H-2}^{n-2}(y)| dydx}{\Phi_H(u)},$$

$$\leq t^2 \limsup_{u \to 0^+} \frac{|S_{H-2}^{n-2}(tu)|}{\Phi_H(2)(u)} \leq \cdots \leq t^{n-1} \limsup_{u \to 0^+} \frac{|S_{H}^{1}(tu)|}{\Phi_H^{(n-1)}(u)},$$

$$H' = H - (n - 1) \in (0, 1),$$

$$= t^{n-1} \limsup_{u \to 0^+} \frac{|S_{H'}^{1}(tu)|}{u^{H'}(2 \log \log(u^{-1}))^{1/2}} < \infty.$$ 

Hence, the statement (3.5) holds true for every $n \geq 1$ odd.

**Step 3.** When $n$ is even, we shall consider $\mathcal{R}_H^2(t, s)$ at first stage, then use L’hôpital’s rule and the recurrent form of the covariance $\mathcal{R}_H^2(t, s)$ to get the general result for an even integer $n \geq 2$. For $n = 2$, the conditions (i)-(ii) and (iv)-(ix) of [1, Theorem 4.1] can be verified in similar fashion as done in Step 1. For the condition (iii), using the same notations, the terms $\mathcal{A}_{k,j}^{r,e}$ are of the form

$$\mathcal{A}_{k,j}^{r,e} := \mathbb{E} \left( \frac{S_H^2(ut_j)S_H^2(vt_k)}{w(u)w(v)} \right) = \frac{\mathcal{R}_H^2(ut_j, vt_k)}{u^H v^H},$$

$$= \frac{C_H^2}{2} (\log t_k)^H \left[ |y_{jk} - y_{jk}^{-1}|^{2H} + (y_{jk} + y_{jk}^{-1})^{2H} - 2 \left( y_{jk}^{2H} + y_{jk}^{-2H} \right) \right],$$

with $y_{jk} = \sqrt{\frac{vt_k}{ut_j}}$, $H \in (1, 2)$

$$= \frac{C_H^2}{2} (\log t_k)^H \left[ y_{jk}^{-2H} \sum_{l=1}^{\infty} \left( \frac{2H}{2l} \right) y_{jk}^{2l} - y_{jk}^{2H} \right],$$

$$\leq \frac{C_H^2}{2} (\log t_k)^H \left( H(2H - 1)y_{jk}^{-2H} + O(y_{jk}^{2H}) \right) \to 0, \quad \text{as } u \to 0^+. $$

The last two inequalities follow by expanding $|1 - y_{jk}|^{2H}$, $(1 + y_{jk})^{2H}$ and the fact that $v/u \to 0$ implies $y_{jk} \to 0$. By observing that $\mathcal{A}_{k,j}^{r,e} \geq 0$ in this case, we conclude that (iii) holds true as well. The proof of Theorem 3.5 is then complete. \hfill $\square$

**Proposition 3.6.** The limit given in (3.5) is strictly positive.

**Proof.** Unlike the fBm (e.g., [6, Proposition 12.19]) for which we know that

$$\limsup_{u \to 0^+} \frac{|B_H(u)|}{u^H (2 \log \log u^{-1})^{1/2}} = 1,$$

it is not clear how to specify the value of $c_H^0$ (3.5). This is due to the complexity of covariance structure of $S_H^n$. Note that Theorem 3.5 states that $c_H^0 \in [0, \infty)$. To establish Proposition 3.6 we split the proof into three steps:
Step 1. If $n = 1$, then the process of interest reduces to the usual sub-fBm $S'_H$ with $\Phi_H(u) = u^H (2 \log \log(u^{-1}))^{1/2}$, $H \in (0, 1)$. Observe that

$$c_H^1 = \limsup_{u \to 0^+} \frac{|S_H^1(ut)|}{\Phi_H(u)} ,$$

$$\geq \limsup_{k \to \infty} \frac{|S_H^1(tr^k)|}{r^{kH} \sqrt{2 \log(-k \log(r))}} ,$$

where $\xi_k := S_H^1(tr^k)/r^{kH}$.

In the last inequality we used the fact that $\log(-k \log(r)) \sim \log(k)$, as $k \to \infty$. To conclude we shall show that

$$\limsup_{k \to \infty} \frac{\xi_k}{\sqrt{2 \log(k)}} \geq \sqrt{R_H^1(t, t)} > 0 . \quad \text{(3.8)}$$

Consider the sequence $\{\xi_k = R_H^1(t, t)^{-1/2} \xi_k : k \geq 1\}$. It is not hard to see that $\{\xi_k\}$ is jointly normal with $\mathbb{E}(\xi_k) = 0$ and $\text{Var}(\xi_k) = 1$ (this is inherited from the Gaussianity of $S_H^1$). For $m, k \in (p, 2p]$ with $k < m$ we have

$$\mathbb{E}(\xi_k \xi_m) = \text{Corr}(S'_H(r^k), S'_H(r^m)) ,$$

$$= r^{(m-k)H} + r^{(k-m)H} - \frac{1}{2} \left[ \left( r^{(m-k)/2} + r^{(k-m)/2} \right)^{2H} + \left( r^{(m-k)/2} - r^{(k-m)/2} \right)^{2H} \right] ,$$

$$\leq r^{(m-k)H} + r^{(k-m)H} - r^{(m-k)/2} - r^{(k-m)/2} \right)^{2H} ,$$

$$\leq 3r^{(m-k)(H \wedge (1-H))} \leq 3r^{H \wedge (1-H)} \to 0, \quad \text{as} \quad r \to 0 .$$

The last three inequalities are justified by the following facts, respectively: $a \wedge b \leq (a + b)/2 \leq a \vee b$, for all $a, b \in \mathbb{R}$ and $x^{2H} + x^{-2H} - \left| x - x^{-1} \right|^{2H} \leq 3(x \wedge x^{-1})^{2H \wedge (2-2H)}$, for all $x > 0$, $H \in (0, 1)$, and $(m-k) \in \{1, \cdots, p - 1\}$. As result we can choose $r$ small enough so that

$$\limsup_{p \to \infty} \max \{ \mathbb{E}(\xi_k \xi_m) : k, m \in (p, 2p], k \neq m \} < \frac{\delta}{2} ,$$

with $\delta \in (0, 1)$. According to [6, Lemma 12.20], it follows that with probability one

$$\limsup_{k \to \infty} \frac{\xi_k}{\sqrt{2 \log(k)}} \geq 1 - \delta$$

or equivalently

$$\limsup_{k \to \infty} \frac{\xi_k}{\sqrt{2 \log(k)}} \geq \sqrt{R_H^1(t, t)(1 - \delta)} > 0 .$$

Hence, (3.8) follows by the arbitrariness of $\delta$.

Step 2. In the case $n = 2$, the normalizing function is defined as

$$\Phi_H(z) = \int_0^z x^{H-1} (2 \log \log(x^{-1}))^{1/2} \, dx ,$$

and for every $r \in (0, e^{-1})$, we have

$$\Phi_H(r^k) = \int_0^{r^k} x^{H-1} (2 \log \log(x^{-1}))^{1/2} \, dx , \quad \text{with} \quad H \in (1, 2) ,$$

$$= r^{kH} \int_0^{1} y^{H-1} (2 \log \log((r^k y)^{-1}))^{1/2} \, dy , \quad \text{(By change of variables} \quad y = x/r^k , \quad \text{)}$$

$$\leq r^{kH} \left( \int_0^{r^*} + \int_{r^*}^{1} \right) y^{H-1} \sqrt{2 \log(-k \log(r) - \log(y))} \, dy , \quad \text{(3.9)}$$

where $r^* = e^{-a(r)}$ and $a(r) = -\log(r)/(-\log(r) - 1)$.
Let $k > 1/(- \log(r))$. On the set \{ $y \leq r^*$ \} we have
\[
\begin{align*}
\log(u + v) & \leq \log(u) + \log(v) \\
\text{with } u = -k \log(r) \text{ and } v = -\log(y).
\end{align*}
\] (3.10)
Just observe that $u > 1$ and
\[
v = -\log(y) \geq -\log(r^*) = a(r),
\]
\[
\frac{-\log(r)}{-\log(r) - 1} \geq -\frac{\log(r) - 1/k}{u - 1}.
\]
On the set \{ $y > r^*$ \} we have $-k \log(r) - \log(y) < -k \log(r) + a(r)$. Combining this fact with (3.9) and (3.10) yields
\[
\Phi_H(r^k) \leq r^{kh} \left\{ \int_0^{r^*} \int_0^y \sqrt{2 \log(-k \log(r))} \, dy + \int_0^{r^*} \int_0^y \sqrt{2 \log(-k \log(r) + a(r))} \, dy \right\},
\]
\[
\leq r^{kh} \left\{ 2H^{-1} \sqrt{2 \log(-k \log(r) + a(r))} + \Phi_H(1) \right\}.
\]
Now using the fact: $\log(-k \log(r) + a(r)) \sim \log(k)$ as $k \uparrow \infty$, we can find $k_0 \geq 1$ and $C > 0$ such that $\Phi_H(r^k) \leq Cr^{kh} \sqrt{2 \log(k)}$ for all $k \geq k_0$. Using this inequality we obtain
\[
c_H^2 = \limsup_{u \to 0^+} \frac{|S_H^2(ut)|}{\Phi_H(u)} \geq C^{-1} \limsup_{k \to \infty} \frac{|S_H^2(tr^k)|}{r^{kh} \sqrt{2 \log(k)}} \quad \text{for som fixed } r \in (0, 1).
\]
Once again, we consider the sequence $\{ \eta_k = R_H^2(t, t) - 1/2S_H^2(tr^k)/r^{kh} : k \geq 1 \}$ and show that $\limsup_{k \to \infty} \frac{\eta_k}{\sqrt{2 \log(k)}} \geq 1 - \delta$. This follows immediately by [6, Lemma 12.20]. In fact, for $r \in (0, 1/2)$, $k, m \in (p, 2p)$ with $k < m$ and by straightforward computations we get
\[
E(\eta_k \eta_m) = corr(S_H^2(r^k), S_H^2(r^m)),
\]
\[
= \frac{|r^{(m-k)/2} + r^{(m-k)/2}|^2 + |r^{(m-k)/2} - r^{(m-k)/2}|^2}{22H - 4} - 2 \left( r^{(m-k)H} + r^{(k-m)H} \right)
\]
\[
= \frac{r^{(m-k)H} \left( (1 + r^{(m-k)})^{2H} + (1 - r^{(m-k)})^{2H} - 2 (1 + r^{2H(m-k)}) \right)}{22H - 4}
\]
\[
\leq Lr^{(2H)(m-k)} \leq Lr^{2H} \to 0, \text{ as } r \to 0,
\]
where $L = 2(2^{2H} - 4)^{-1} \left( H(2H - 1) + \sum_{j=2}^{\infty} \left( \frac{2H}{2j} \right) \left( \frac{1}{2} \right)^{2j-2} \right)$, which implies the condition (12.31) in [6, Lemma 12.20]. Now, we can conclude that $c_H^2 \geq C^{-1} \sqrt{R_H^2(1, t)} > 0$.

**Step 3. (General Case)** Fix $n \geq 3$ (odd or even). We shall suppose that $c_H^1$ given in (3.5) equals zero and obtain a contradiction. Clearly, the aforementioned hypothesis implies $\lim_{u \to 0^+} \frac{S_H^1(ut)}{\Phi_H(u)} = 0$, then by using the usual L'hôpital’s rule recursively we obtain $c_H^1 = 0$ (if $n$ is odd with $c_H^1$ being the quantity associated with $S_H^1$, $H' \in (0, 1)$) or $c_H^2 = 0$ (if $n$ is even with $c_H^2$ being the quantity associated with $S_H^2$, $H' \in (1, 2)$). This
clearly contradicts results of the two previous cases. Note that the process of differentiation should occur according to (iii)-Theorem 3.2.

Acknowledgment. We would like to thank the editorial staff and the reviewers for the comments that helped in improving this work.

References

[8] M. El Omari, An α-order fractional Brownian motion with Hurst index $H \in (0,1)$ and $\alpha \in \mathbb{R}_+$, Sankhya A 85 (1), 572-599, 2023.


