Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (1) (2024), 53 – 61
DOI : 10.15672/hujms.1180981

Research Article

Notes on some classes of spirallike functions associated with the q-integral operator

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Abstract

The object of the present paper is to find the essential properties for certain subfamilies of analytic and spirallike functions which are generated by q-integral operator. Further, we derive membership relations for functions belong to these subfamilies, and also we determine coefficient estimates.

Mathematics Subject Classification (2020). 30C45, 05A30

Keywords. spirallike function, convolution, q-integral operator

1. Introduction

Stand by $\mathbb{A}$ the family of functions $f(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k$ analytic in the open unit disk $\mathcal{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with $f(0) = 0 = f'(0) - 1$. A function $f \in \mathbb{A}$ is named univalent in $\mathcal{D}$ provided that it does not take the same value twice. Stand by $\mathbb{S}$ the subfamily of $\mathbb{A}$ involving univalent functions. Further, for the function g with the Taylor series $g(\zeta) = \zeta + b_2\zeta^2 + \cdots = \zeta + \sum_{k=2}^{\infty} b_k \zeta^k$, the convolution $f * g$ is expressed by

$$(f * g)(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k b_k \zeta^k.$$

On the other hand, for analytic functions f and g in $\mathcal{D}$, we ensure that f is subordinate to g, expressed by $f \prec g$, for a Schwarz function $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}(0) = 0, |\mathbf{\Lambda}(\zeta)| < 1$ and $f(\zeta) = g(\mathbf{\Lambda}(\zeta))$ $(\zeta \in \mathcal{D})$.

Now, we shall deal with a subfamily of $\mathbb{S}$ which is of special interest in its own right, namely the spirallike functions.

For $-\infty < t < \infty$ and $\vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the logarithmic ϑ-spiral curve is expressed by $w = w_0 \exp(-e^{-i\vartheta} t)$, where w_0 is a nonzero complex number. We must mention here that 0-spirals are radial half-lines. For an analytic function, we can call it ϑ-spirallike provided that its range is ϑ-spirallike. Stand by $\mathcal{S}_\vartheta$ the family of ϑ-spirallike functions. Analytically, $f \in \mathbb{A}$ belongs to the family $\mathcal{S}_\vartheta$ iff $\Re\left(e^{i\vartheta}\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > 0$ [18].

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Received: 27.09.2022; Accepted: 10.02.2023

The study of q-calculus in Geometric Function Theory was partially provided by Srivastava [19]. This application is still among the most popular subject of many mathematicians today [1–4, 6, 8, 12, 14, 15, 20]. In the course of the paper, suppose $0 < q < 1$ and the definitions deal with the complex-valued function f.

The q-derivative of f is expressed by [9]:

$$D_q f(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(q\zeta)}{(1-q)\zeta}, & \zeta \neq 0 \\ \\ f'(0), & \zeta = 0 \end{cases}.$$

When f is differentiable at ζ, we arrive $\lim_{q\to 1^-} D_q f(\zeta) = f'(\zeta)$.

The q-integral of f is expressed by [10]:

$$\int_0^\zeta f(u)d_q u = \zeta(1-q)\sum_{k=0}^{\infty} q^k f(\zeta q^k),$$

provided the series converges.

Next, the q-gamma function is expressed by

$$\Gamma_q(u) = (1-q)^{1-u}\prod_{k=0}^{\infty}\frac{1-q^{k+1}}{1-q^{k+u}} \ (u > 0)$$

with

$$\Gamma_q(u+1) = [u]_q\Gamma_q(u), \quad \Gamma_q(u+1) = [u]_q!, \tag{1.1}$$

where $u \in \mathbb{N}$ and

$$[u]_q! = \begin{cases} [u]_q[u-1]_q \dots [2]_q[1]_q, & u \geq 1 \\ 1, & u = 0. \end{cases}$$

If we set $q \to 1^-$, we find $\Gamma_q(u) \to \Gamma(u)$ [9].

The q-beta function

$$B_q(u,s) = \int_0^1 \zeta^{u-1}(1-q\zeta)_q^{s-1} d_q\zeta, \quad (u, s > 0) \tag{1.2}$$

is the q-analogue of Euler's formula [10] with

$$B_q(u,s) = \frac{\Gamma_q(u)\Gamma_q(s)}{\Gamma_q(u+s)}. \tag{1.3}$$

Next, the q-binomial coefficients are expressed by [7]

$$\binom{k}{n}_q = \frac{[k]_q!}{[n]_q![k-n]_q!}. \tag{1.4}$$

In a recent study [11], the generalized q-integral operator $\chi^{\alpha}_{\beta,q} : \mathbb{A} \to \mathbb{A}$ is expressed by

$$\chi^{\alpha}_{\beta,q} f(\zeta) = \binom{\alpha+\beta}{\beta}_q \frac{[\alpha]_q}{\zeta^\beta}\int_0^\zeta \left(1 - \frac{qu}{\zeta}\right)_q^{\alpha-1} u^{\beta-1} f(u) d_q u \quad (\alpha > 0, \beta > -1).$$

From (1.1), (1.2), (1.3) and (1.4), they arrive

$$\chi^{\alpha}_{\beta,q} f(\zeta) = \zeta + \sum_{k=2}^{\infty}\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)} a_k\zeta^k.$$

For some special values, we find the following integral operators previously known.

(i) If $\alpha = 1$, the q-Bernardi integral operator $J_{\beta,q}f$ is obtained [13]

$$J_{\beta,q}f(\zeta) = \frac{[1+\beta]_q}{\zeta^\beta}\int_0^\zeta u^{\beta-1}f(u)d_q u = \sum_{k=1}^{\infty}\frac{[1+\beta]_q}{[k+\beta]_q} a_k\zeta^k.$$

(ii) If $\alpha = 1,\ q \to 1^-$, the Bernardi integral operator is obtained [5]

$$J_\beta f(\zeta) = \frac{1+\beta}{\zeta^\beta}\int_0^\zeta u^{\beta-1} f(u)du = \sum_{k=1}^{\infty} \frac{1+\beta}{k+\beta} a_k \zeta^k.$$

(iii) If $\alpha = 1,\ \beta = 0,\ q \to 1^-$, the Alexander integral operator is obtained [16]

$$J_0 f(\zeta) = \int_0^\zeta \frac{f(u)}{u} du = \zeta + \sum_{k=2}^{\infty} \frac{1}{k} a_k \zeta^k.$$

Now, we introduce the new subfamilies $S^\alpha_{\beta,q}[A,B]$ and $K^\alpha_{\beta,q}[A,B]$ of analytic functions:

$$S^\alpha_{\beta,q}[A,B] = \left\{ f \in \mathbb{S} : e^{i\vartheta} \frac{\zeta\left(\chi^\alpha_{\beta,q} f(\zeta)\right)'}{\chi^\alpha_{\beta,q} f(\zeta)} \prec \cos\vartheta \left(\frac{1+A\zeta}{1+B\zeta}\right) + i \sin\vartheta, \zeta \in \mathcal{D} \right\}$$

and

$$K^\alpha_{\beta,q}[A,B] = \left\{ f \in \mathbb{S} : e^{i\vartheta} \left(1 + \frac{\zeta\left(\chi^\alpha_{\beta,q} f(\zeta)\right)''}{\left(\chi^\alpha_{\beta,q} f(\zeta)\right)'}\right) \prec \cos\vartheta \left(\frac{1+A\zeta}{1+B\zeta}\right) + i \sin\vartheta, \zeta \in \mathcal{D} \right\},$$

where $|\vartheta| < \frac{\pi}{2},\ -1 \le B < A \le 1$.

We know that there is a relation between the families $S^\alpha_{\beta,q}[A,B]$ and $K^\alpha_{\beta,q}[A,B]$ such as

$$f \in K^\alpha_{\beta,q}[A,B] \iff zf' \in S^\alpha_{\beta,q}[A,B]. \tag{1.5}$$

Note that

1) Letting $q \to 1^-$ and $\alpha = 1$, we arrive the families $S_\beta[A,B]$ and $K_\beta[A,B]$ involving Bernardi integral operator given in (ii).

2) Letting $q \to 1^-$, $\alpha = 1$ and $\beta = 0$, we arrive the families $S[A,B]$ and $K[A,B]$ involving Alexander integral operator given in (iii).

This paper deals with the new subfamilies $S^\alpha_{\beta,q}[A,B]$ and $K^\alpha_{\beta,q}[A,B]$ of analytic functions involving a generalized q-integral operator and its several properties.

2. Convolution properties

To present convolution properties, we express Lemma 2.1 due to Silverman and Silvia [17].

Lemma 2.1. *The function f is in $S^*[A,B]$ iff for all $\zeta \in \mathcal{D}$ and all η, $|\eta| = 1$,*

$$\frac{1}{\zeta}\left[f * \frac{\zeta + \frac{\eta - A}{A-B}\zeta^2}{(1-\zeta)^2}\right] \neq 0. \tag{2.1}$$

Lemma 2.2. *The function f is in $S^\alpha_{\beta,q}[A,B]$ iff for all $\zeta \in \mathcal{D}$ and all η, $|\eta| = 1$,*

$$\frac{1}{\zeta}\left[f * \left(\zeta + \sum_{k=2}^{\infty} \frac{(k-\varepsilon)\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{(1-\varepsilon)\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}\zeta^k\right)\right] \neq 0,$$

where

$$\varepsilon = \frac{e^{i\vartheta} + (A\cos\vartheta + iB\sin\vartheta)\eta}{e^{i\vartheta}(1+B\eta)}. \tag{2.2}$$

Proof. An application of Lemma 2.1 exhibits that $f \in S^{\alpha}_{\beta,q}[A,B]$ iff

$$e^{i\vartheta}\frac{\zeta\left(\chi^{\alpha}_{\beta,q}f(\zeta)\right)'}{\chi^{\alpha}_{\beta,q}f(\zeta)} \neq \cos\vartheta\left(\frac{1+A\eta}{1+B\eta}\right)+i\sin\vartheta$$

$$\Leftrightarrow \zeta\left(\chi^{\alpha}_{\beta,q}f(\zeta)\right)' - \chi^{\alpha}_{\beta,q}f(\zeta)\left(\frac{e^{i\vartheta}+(A\cos\vartheta+iB\sin\vartheta)\eta}{e^{i\vartheta}(1+B\eta)}\right) \neq 0 \quad (\zeta\in\mathcal{D},\ |\eta|=1). \tag{2.3}$$

Since

$$\zeta f' = f * \frac{\zeta}{(1-\zeta)^2}, \qquad f = f * \frac{\zeta}{1-\zeta},$$

we arrive

$$\chi^{\alpha}_{\beta,q}f(\zeta) = f(\zeta) * h(\zeta) * \frac{\zeta}{1-\zeta}$$

$$\zeta\left(\chi^{\alpha}_{\beta,q}f(\zeta)\right)' = f(\zeta) * h(\zeta) * \frac{\zeta}{(1-\zeta)^2},$$

where $h(\zeta) = \zeta + \sum_{k=2}^{\infty}\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}\zeta^k$.

By substituting ε given by (2.2), we find that the relation (2.3) is equivalent to

$$f(\zeta) * h(\zeta) * \left(\frac{\zeta}{(1-\zeta)^2} - \frac{\varepsilon\zeta}{1-\zeta}\right) \neq 0. \tag{2.4}$$

On the other hand, by extensions of $\frac{\zeta}{(1-\zeta)^2}$ and $\frac{\zeta}{(1-\zeta)}$, we find

$$\left(\frac{\zeta}{(1-\zeta)^2} - \frac{\varepsilon\zeta}{1-\zeta}\right) = \zeta + \sum_{k=2}^{\infty}\frac{k-\varepsilon}{1-\varepsilon}\zeta^k. \tag{2.5}$$

By substituting (2.5) in (2.4), the proof is complete. □

Theorem 2.3. *A necessary and sufficient condition for the function f to be in $S^{\alpha}_{\beta,q}[A,B]$ is that*

$$1 - \sum_{k=2}^{\infty}\frac{(k-1)(e^{i\vartheta}+iB\eta\sin\vartheta)-(A-kB)\eta\cos\vartheta}{(A-B)\eta\cos\vartheta} \times \frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}a_k\zeta^{k-1} \neq 0.$$

Proof. Notice that

$$\frac{k-\varepsilon}{1-\varepsilon} = -\frac{(k-1)(e^{i\vartheta}+iB\eta\sin\vartheta)-(A-kB)\eta\cos\vartheta}{(A-B)\eta\cos\vartheta}. \tag{2.6}$$

By using (2.6), we can write the relation (2.1) as

$$\frac{1}{\zeta}\left[\zeta - \sum_{k=2}^{\infty}\frac{(k-1)(e^{i\vartheta}+iB\eta\sin\vartheta)-(A-kB)\eta\cos\vartheta}{(A-B)\eta\cos\vartheta} \times \frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}a_k\zeta^k\right] \neq 0. \tag{2.7}$$

Simplifying relation (2.7), we obtain the desired condition. □

Lemma 2.4. *The function f is in $K^{\alpha}_{\beta,q}[A,B]$ iff for all $\zeta\in\mathcal{D}$ and all η, $|\eta|=1$,*

$$\frac{1}{\zeta}\left[f * \left(\zeta + \sum_{k=2}^{\infty}\frac{(k-\varepsilon)\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{(1-\varepsilon)\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}k\zeta^k\right)\right] \neq 0,$$

where

$$\varepsilon = \frac{e^{i\vartheta}+(A\cos\vartheta+iB\sin\vartheta)\eta}{e^{i\vartheta}(1+B\eta)}.$$

Proof. Set

$$f \prec h \Leftrightarrow f(0) = h(0), \quad f(\mathcal{D}) \subset h(\mathcal{D}), \quad \zeta \in \mathcal{D}.$$

Note that

$$\zeta h'(\zeta) = \zeta + \sum_{k=2}^{\infty} \frac{(k-\varepsilon)\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{(1-\varepsilon)\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)} k\zeta^k. \tag{2.8}$$

From the identity $\zeta f * h = f * \zeta h'$ and the fact that

$$f \in K^{\alpha}_{\beta,q}[A,B] \quad \Leftrightarrow \quad \zeta f' \in S^{\alpha}_{\beta,q}[A,B],$$

from Lemma 2.1, we arrive

$$\frac{1}{\zeta}[\zeta f'(\zeta) * h(\zeta)] \neq 0 \quad \Leftrightarrow \quad \frac{1}{\zeta}[f(\zeta) * \zeta h'(\zeta)] \neq 0. \tag{2.9}$$

By substituting relation (2.8) in (2.9), we have the desired result. □

Theorem 2.5. *A necessary and sufficient condition for the function f to be in $K^{\alpha}_{\beta,q}[A,B]$ is that*

$$1 - \sum_{k=2}^{\infty} \frac{(k-1)(e^{i\vartheta} + iB\eta\sin\vartheta) - (A-kB)\eta\cos\vartheta}{(A-B)\eta\cos\vartheta} \times \frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)} k a_k \zeta^{k-1} \neq 0.$$

Proof. By using Lemma 2.2 and in a similar way of Theorem 2.3, we solve the theorem. □

3. Coefficient estimates

In the following, as an application of previous theorems, we derive coefficient estimates and inclusion properties for a function to be in the families $S^{\alpha}_{\beta,q}[A,B]$ and $K^{\alpha}_{\beta,q}[A,B]$.

Lemma 3.1 ([21])**.** *Let the parameters A, B and α, as well as the integer k, limited by $-1 \leq B < A \leq 1$, $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$ and $k \in \mathbb{N}\backslash\{1\}$ be fixed. Assume*

$$[A-(k-1)B]^2\cos^2\vartheta + (k-2)^2\left(B^2\sin^2\vartheta - 1\right) \geq 0.$$

Then

$$\frac{1}{(k-1)^2}\left[(A-B)^2\cos^2\vartheta + \sum_{n=2}^{k-1}\left|(A-nB)^2\cos^2\vartheta + (n-1)^2\left(B^2\sin^2\vartheta - 1\right)\right| \times \prod_{j=0}^{n-2}\frac{\left|(A-B)\,e^{-i\vartheta}\cos\vartheta - jB\right|^2}{(j+1)^2}\right] = \prod_{j=0}^{k-2}\frac{\left|(A-B)\,e^{-i\vartheta}\cos\vartheta - jB\right|^2}{(j+1)^2}$$

We can express following coefficient estimates by using Lemma 3.1.

Theorem 3.2. *Let $f \in S^{\alpha}_{\beta,q}[A,B]$. Assume*

$$(A-(k-1)\,B)^2\cos^2\vartheta \geq (k-2)^2\left(1 - B^2\sin^2\vartheta\right).$$

Then

$$|a_k| \leq \frac{\Gamma_q(\alpha+\beta+k)\,\Gamma_q(\beta+1)}{\Gamma_q(\beta+k)\,\Gamma_q(\alpha+\beta+1)} \prod_{j=0}^{k-2} \frac{\left|(A-B)\cos\vartheta e^{-i\vartheta} - Bj\right|}{j+1}. \tag{3.1}$$

This result is sharp.

Proof. Let us put

$$I(\zeta)=\chi^{\alpha}_{\beta,q}f(\zeta)=\zeta+\sum_{k=2}^{\infty}B_k\zeta^k,$$

where $B_k=\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}$.

Since $f\in S^{\alpha}_{\beta,q}[A,B]$, there exists a function $\mathbf{\Lambda}(\zeta)$ such that

$$e^{i\vartheta}\sec\vartheta\frac{\zeta I'(\zeta)}{I(\zeta)}-i\tan\vartheta=\frac{1+A\mathbf{\Lambda}(\zeta)}{1+B\mathbf{\Lambda}(\zeta)}.$$

Then, we get

$$(1+i\tan\vartheta)\sum_{k=2}^{\infty}(k-1)B_k\zeta^k=\left((A-B)\zeta+\sum_{k=2}^{\infty}\left[A-kB-i(k-1)B\tan\vartheta\right]B_k\zeta^k\right)\mathbf{\Lambda}(\zeta).$$

The above equality may be rewritten (for $m\in\mathbb{N}$) as follows:

$$(1+i\tan\vartheta)\sum_{k=2}^{m}(k-1)B_k\zeta^k+\sum_{k=m+1}^{\infty}d_k\zeta^k=\left(\sum_{k=1}^{m-1}\left[A-kB-i(k-1)B\tan\vartheta\right]B_k\zeta^k\right)\mathbf{\Lambda}(\zeta).$$

Hence,

$$\sum_{k=m+1}^{\infty}d_k\zeta^k=\left(\sum_{k=1}^{m-1}\left[A-kB-i(k-1)B\tan\vartheta\right]B_k\zeta^k\right)\mathbf{\Lambda}(\zeta)-(k-1)(1+i\tan\vartheta)\sum_{k=2}^{m}B_k\zeta^k$$

and both two sums on the RHS are convergent and the sum on the LHS is convergent in $\mathcal{D}$ for $k=2,3,\cdots$. By appealing to Parseval's Theorem, we arrive

$$(k-1)^2|B_k|^2\le\sum_{m=1}^{k-1}\left\{[A-mB]^2\cos^2\vartheta+\left(B^2\sin^2\vartheta-1\right)(m-1)^2\right\}|B_m|^2, \tag{3.2}$$

where $B_1=1$.

We need to show the inequality given in (3.1) is true by using the principle of mathematical induction. Actually, we see that it holds for $k=2$, as

$$|B_2|\le(A-B)\cos\vartheta.$$

Suppose that inequality (3.1) holds true for $k=m-1$ for some fixed m, it means that the inequality

$$|B_{m-1}|\le\prod_{j=0}^{m-2}\frac{\left|(A-B)\cos\vartheta e^{-i\vartheta}-Bj\right|}{(j+1)} \tag{3.3}$$

holds true. Then, we have to show that (3.1) is true for $k=m$. It is easy to see this relation considering (3.2), (3.3) and Lemma 3.1. After necessarily calculations, we have for $k=m$ that

$$|B_m|^2\le\frac{1}{(m-1)^2}\sum_{k=1}^{m-1}\left\{[A-kB]^2\cos^2\vartheta\ +\left(B^2\sin^2\vartheta-1\right)(k-1)^2\right\}|B_k|^2$$

$$\le\frac{1}{(m-1)^2}\left\{(A-B)^2\cos^2\vartheta+\sum_{k=2}^{m-1}\left|[A-kB]^2\cos^2\vartheta+\left(B^2\sin^2\vartheta-1\right)(n-1)^2\right||B_k|^2\right\}$$

$$\leq \frac{1}{(m-1)^2}\left\{(A-B)^2\cos^2\vartheta+\sum_{k=2}^{m-1}\left|[A-kB]^2\cos^2\vartheta+(k-1)^2\left(B^2\sin^2\vartheta-1\right)\right|\right.$$
$$\left.\times\prod_{j=0}^{k-2}\frac{\left|(A-B)\cos\vartheta e^{-i\vartheta}-Bj\right|}{(j+1)}\right\}$$

$$=\prod_{j=0}^{m-2}\frac{\left|(A-B)\cos\vartheta e^{-i\vartheta}-Bj\right|}{(j+1)}.$$

We see that (3.1) is true for every k. To examine the sharpness in (3.1), it is enough to consider the function

$$I(\zeta)=\chi_{\beta,q}^{\alpha}f(\zeta)=\frac{\zeta}{(1+B\zeta)^{\frac{(B-A)e^{-i\vartheta}\cos\vartheta}{B}}}.$$

□

We can also prove the following theorem by using relation given in (1.5)

Theorem 3.3. *Let $f\in K_{\beta,q}^{\alpha}[A,B]$. Assume*

$$(A-(k-1)B)^2\cos^2\vartheta\geq(k-2)^2\left(1-B^2\sin^2\vartheta\right).$$

Then

$$|a_k|\leq\frac{\Gamma_q(\alpha+\beta+k)\,\Gamma_q(\beta+1)}{\Gamma_q(\beta+k)\,\Gamma_q(\alpha+\beta+1)}\prod_{j=0}^{k-2}\frac{\left|(A-B)\cos\vartheta e^{-i\vartheta}-Bj\right|}{j+2}.$$

This is also sharp estimate.

Theorem 3.4. *If $f\in S_{\beta,q}^{\alpha}[A,B]$, then*

$$\sum_{k=2}^{\infty}\left(|(k(B+1)-1|+|A\cos\vartheta+iB\sin\vartheta|\right)\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}|a_k|\leq(A-B)\cos\vartheta,$$

Proof. Since

$$\left|1-\textstyle\sum_{k=2}^{\infty}\frac{(k-1)(e^{i\vartheta}+iB\eta\sin\vartheta)-(A-kB)\eta\cos\vartheta}{(A-B)\eta\cos\vartheta}\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}a_k\right|$$
$$\geq 1-\textstyle\sum_{k=2}^{\infty}\left|\frac{(k-1)(e^{i\vartheta}+iB\eta\sin\vartheta)-(A-kB)\eta\cos\vartheta}{(A-B)\eta\cos\vartheta}\right|\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}|a_k|$$

and

$$\left|\frac{(k-1)(e^{i\vartheta}+iB\eta\sin\vartheta)-(A-kB)\eta\cos\vartheta}{(A-B)\eta\cos\vartheta}\right|=\frac{\left|(k-1)(e^{i\vartheta}+iB\sin\vartheta)-(A-kB)\cos\vartheta\right|}{(A-B)\cos\vartheta}$$
$$\leq\frac{|(k(B+1)-1|+|A\cos\vartheta+iB\sin\vartheta|}{(A-B)\cos\vartheta},$$

the outcome follows from Theorem 2.3. □

Theorem 3.5. *If $f\in K_{\beta,q}^{\alpha}[A,B]$, then*

$$\sum_{k=2}^{\infty}\left(|(k(B+1)-1|+|A\cos\vartheta+iB\sin\vartheta|\right)\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}k\,|a_k|\leq(A-B)\cos\vartheta.$$

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