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On p,q-Harmonic Numbers

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Abstract

In this study, we examined a new generalization of well-known number sequence which is called harmonic numbers. We defined p,q-harmonic numbers which is also a generalization of q-harmonic numbers and deduced some properties and identities related to this number sequence by using some combinatorial operations.

Keywords: *q-calculus;p,q-analogue;harmonic numbers;q-harmonic numbers*
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1. Introduction

Quantum calculus or q-calculus plays an important role in combintorics, number theory and physics. Its analysis and some applications can be found in [1,2]. There are q-analogs of the factorial, binomial coefficient, derivative, integral, Fibonacci numbers, and so on. In [13], q-analog of an integer n is given by

$$[n]_q := \frac{1-q^n}{1-q} = 1+q+q^n+q^2+...q^{n-1}, \tag{1.1}$$

with $0<q<1$. It is also denoted by $[n]$. From this definition, we can write q-analogue of n by finite sunmmation as follows ;

$$[n]_q = \sum_{k=0}^{n-1} q^k. \tag{1.2}$$

q-Analogs are based on the fact that

$$\lim_{q\to 1^-} \frac{1-q^n}{1-q} = n.$$

As usual binomial coefficient, q-Binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where

$$[n]_q! = [n]_q[n-1]_q[n-2]_q...[1]_q$$

the q-factorial. Its triangular recurrence relation is given as follows:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q.$$

From this relation, we obtain the the recursive formula of usual binomial coefficient as $q\to 1^-$. Also we have the following horizontal and vertical recurrence relations for q-binomial coefficcients respectively,

$$\binom{n+1}{k+1}_q = \sum_{j=k}^{n} q^{j-k}\binom{j}{k}_q$$

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and

$$\binom{n}{k}_q = \sum_{j=0}^{n-k} (-1)^j q^{jk+\binom{j+1}{2}} \binom{n+1}{k+j+1}_q.$$

Furthermore, as $q \to 1$, the first relation is reduced to Chu Shih Chieh's identity which is given by

$$\binom{n}{k} = \binom{n+1}{k+1} - \binom{n+1}{k+2} + ... + (-1)^{n-k} \binom{n+1}{n+1}.$$

q-Pochhammer symbol or q-shifted factorial which is known as the q-analogue of falling factorial is defined by

$$(a;q)_n = ([a]_q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

with $([a]_q)_0 = 1$.
Post-Quantum calculus or p,q-calculus is constructed by expanding q-analog into two components p and q as a generalization of q-calculus and its applications and interesting properties can be found in [4,8,9].
p,q-analogue of a nonnegative integer n is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q} = \sum_{k=0}^{n-1} p^{n-k-1} q^k, \tag{1.3}$$

where $0 < q < p \leq 1$. By this definition the symmetry property, that is,

$$[n]_{p,q} = [n]_{q,p}$$

can be seen. One can reduce p,q-number to q-number by taking $p = 1$ in (1.3). As for q-binomial coefficient, the p,q-factorial and p,q-binomial coefficient are determined and defined by

$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}[n-2]_{p,q}...[1]_{p,q}$$

and

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$

respectively. p,q-Binomial coefficients and p,q-derivatives are studied by R. Corcino in [11]. Author gave the triangular recurrence relations

$$\binom{n+1}{k}_{p,q} = p^k \binom{n}{k}_{p,q} + q^{n-k+1} \binom{n}{k-1}_{p,q}$$

and

$$\binom{n+1}{k}_{p,q} = q^k \binom{n}{k}_{p,q} + p^{n-k+1} \binom{n}{k-1}_{p,q},$$

with $\binom{0}{0}_{p,q} = \binom{n}{n}_{p,q} = 1 = 1$ and $\binom{n}{k}_{p,q} = 0$ if $k > n$. Taking $p = 1$, it can be seen that the first relation is reduced to triangular recurrence relation of q-binomial coefficient which is mentioned above. The first values of $\binom{n}{k}_{p,q}$ are given in the following table.

n/k	1	2	3	...
1	1			
2	p^2+pq+q^2	p^2+pq+q^2		
3	$p^3+pq^2+qp^2+q^3$	$(p^2+pq+q^2)(p^2+q^2)$	$p^3+pq^2+qp^2+q^3$	
$\vdots$	...	...	...	$\ddots$

For $m = 1,2,...,$ p,q-shifted factorial is given by

$$([a]_{p,q})_n = [a]_{p,q}[a+1]_{p,q}...[a+n-1]_{p,q},$$

with $([a]_{p,q})_0 = 1$. Also p,q-analogue of the exponential operator exists in the form

$$exp_{p,q}(z) = \sum_{m=0}^{\infty} \frac{z^m}{[m]_{p,q}!},$$

for all z (see [6]). In literature, p,q-derivative and p,q-hypergeometric funcitons are studied and some interesting properties are deduced in [12,14]. For example, Sahai et al. examined the generalized p,q-hypergeometric series which is given by

$$r\psi_s(a_1,...,a_r;b_1,...,b_s;p,q;z) = \sum_{m=0}^{\infty} \frac{([a_1]_{p,q})_m...([a_r]_{p,q})_m}{([b_1]_{p,q})_m...([b_s]_{p,q})_m} \left[\frac{(-1)^m q^{-\frac{1}{2}\binom{m}{2}}}{(p^{\frac{1}{2}} - q^{-\frac{1}{2}})^m}\right]^{1+s-r} \frac{z^m}{[m]_{p,q}!},$$

where $([a]_{p,q})_m$ is the p,q-shifted factorial, $p>0, q>0$ and $pq<1$.
Harmonic numbers have been studied for many years and are also called harmonic series which is related to the Riemann Zeta function. n-th harmonic number and alternatig harmonic number are defined by the finite summation as

$$H_n=\sum_{k=1}^{n}\frac{1}{k},\qquad I_n=\sum_{k=1}^{n}\frac{(-1)^k}{k},$$

respectively with $H_0=I_0=0$. The generating function of harmonic numbers is given by

$$\sum_{k\geq 0}H_k z^k=-\frac{\log(1-z)}{1-z}$$

and a more general form of the generating function is also given by

$$\sum_{k\geq m}(H_k-H_m)\binom{k}{m}z^{k-m}=-\frac{\log(1-z)}{(1-z)^{m+1}},$$

for a natural number m. Important identities involving harmonic numbers can be seen in [7,15]. For $r\geq 1$, hyperharmonic number of order r is defined by

$$H_n^{(r)}=\sum_{k=1}^{n}\frac{1}{k^s}$$

and satisfy the reccurence relation

$$H_n^{(r)}=\sum_{k=1}^{n}H_k^{(r-1)},$$

where $H_0^{(r)}=\frac{1}{n}$ and $H_n^{(0)}=0$ if $n\leq 0$ and $r<0$. It can be observed that $H_n^{(1)}=H_n$. Some special identities and properties for harmonic numbers are given by Anthony Sofo in [3].
q-Harmonic numbers and alternating q-harmonic numbers are given by

$$H_n(q)=\sum_{k=1}^{n}\frac{1}{[k]_q},\quad \tilde{H}_n(q)=\sum_{k=1}^{n}\frac{q^k}{[k]_q} \tag{1.4}$$

and

$$I_n(q)=\sum_{k=1}^{n}\frac{(-1)^k}{[k]_q},$$

respectively. The first values of q-harmonic numbers are

$$H_0(q)=0,\qquad H_1(q)=1,\qquad H_2(q)=\frac{q+2}{q+1},\qquad H_3(q)=\frac{4q+3q^2+q^3+3}{2q+2q^2+q^3+1},\dots$$

$$\tilde{H}_0(q)=0,\qquad \tilde{H}_1(q)=q,\qquad \tilde{H}_2(q)=q\frac{2q+1}{q+1},\qquad \tilde{H}_3(q)=\frac{3q+4q^2+3q^3+1}{2q+2q^2+q^3+1},\dots$$

and

$$I_0(q)=0,\qquad I_1(q)=-1,\qquad I_2(q)=-\frac{q}{q+1},\qquad I_3(q)=-\frac{2q+q^2+q^3+1}{2q+2q^2+q^3+1},\dots$$

Some important identities and properties are given by Kızılateş and Tuglu in [5]. For example, for $n\geq 1$

$$\sum_{k=1}^{n-1}q^k[k]_q\tilde{H}_k(q)=\frac{[n]_q^2}{[2]_q}\left(q\tilde{H}_k(q)-\frac{q^3}{[2]_q}\right).$$

In [10], Ömür et al. gave the following equations;

$$\sum_{k=1}^{n}q^{-k}\tilde{H}_k(q)=\frac{q}{q-1}\left(H_n(q)-q^{-n-1}\tilde{H}_n(q)\right), \tag{1.5}$$

$$\sum_{k=1}^{n}q^{-2k}\tilde{H}_k(q)=\frac{q^2}{q+1}\left(q^{-2n-2}[2n+2]_q\tilde{H}_n(q)-q^{-n}[n]_q-n\right). \tag{1.6}$$

Authors also investigated the congruences of q-harmonic numbers. For example, for a prime number p they give the following congruence:

$$\sum_{k=d+1}^{p-1}(-q)^k\tilde{H}_{p-k-1}(q)\equiv\frac{1}{[2]_q}\left((-q)^{d+1}H_d(q)-2Q_p(2,q)-I_d(q)-\frac{1-q}{2}(p-(-1)^d+(p-1)(-q)^{d+1})\right)\left(mod[p]_q\right). \tag{1.7}$$

2. Some Identities Involving p,q-Harmonic Numbers

In this section, firstly we define the p,q-harmonic numbers and investigate their some properties.

Definition 2.1. *For $p \neq q$, p,q-harmonic numbers and alternating p,q-harmonic numbers are defined by*

$$H_n(p,q)=\sum_{k=1}^{n}\frac{1}{[k]_{p,q}},\quad \tilde{H}_n(p,q)=\sum_{k=1}^{n}\frac{p^k}{[k]_{p,q}},\quad \tilde{H}_n(q,p)=\sum_{k=1}^{n}\frac{q^k}{[k]_{p,q}} \tag{2.1}$$

and

$$I_n(p,q)=\sum_{k=1}^{n}\frac{(-1)^k}{[k]_{p,q}}, \tag{2.2}$$

respectively.

Setting $p=1$ in (2.1) and (2.2), q-Harmonic numbers can be found.
By the fact that $[n]_{p,q}=[n]_{q,p}$, we clearly observe that $H_n(p,q)=H_n(q,p)$ and $I_n(p,q)=I_n(q,p)$. But this is not true for $\tilde{H}_n(p,q)$. By some elementary operations, we have

$$\tilde{H}_n(q,p)=\sum_{k=1}^{n}\frac{q^k}{[k]_{p,q}}=\sum_{k=1}^{n}\left(\frac{p^k}{[k]_{p,q}}-(p-q)\right)=\sum_{k=1}^{n}\frac{p^k}{[k]_{p,q}}-k(p-q).$$

That is

$$\tilde{H}_n(q,p)=\tilde{H}_n(p,q)-n(p-q). \tag{2.3}$$

Therefore, the order of p and q is important for the number $\tilde{H}_n(p,q)$.

Proposition 2.1. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n}[k]_{p,q}=\frac{1}{1-p}\left([n]_q-p[n]_{p,q}\right). \tag{2.4}$$

Proof. By the definition of $[k]_{p,q}$, we can write

$$\sum_{k=1}^{n}[k]_{p,q}=\sum_{k=1}^{n}\sum_{i=1}^{k}p^{k-i}q^{i-1}=q^{-1}\sum_{k=1}^{n}p^k\sum_{i=1}^{k}p^{-i}q^i$$

. By changing the sums and some elementary operations, we have

$$\sum_{k=1}^{n}[k]_{p,q}=q^{-1}\left(\sum_{i=1}^{n}p^{-i}q^i\sum_{k=i}^{n}p^k\right)=q^{-1}\left(\sum_{i=1}^{n}p^{-i}q^i\left(\sum_{k=1}^{n}p^k-\sum_{k=1}^{i-1}p^k\right)\right).$$

Since $0<q<p\leq 1$, we can use the geometric sum formula and we get

$$\sum_{k=1}^{n}[k]_{p,q}=q^{-1}\left(\sum_{i=1}^{n}p^{-i}q^i\left(\frac{1-p^{n+1}}{1-p}-\frac{1-p^i}{1-p}\right)\right)=\frac{q^{-1}}{1-p}\left(\sum_{i=1}^{n}q^i-\sum_{i=1}^{n}p^{n-i+1}q^i\right).$$

Finally using (1.1) and (1.3), we get the result. □

Lemma 2.2. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n}q^kH_k(p,q)=\sum_{k=1}^{n}q^kH_k(q,p)=\frac{1}{1-q}\left(\tilde{H}_n(q,p)-q^{n+1}H_n(p,q)\right). \tag{2.5}$$

Proof. By (2.1), we obtain

$$\sum_{k=1}^{n}q^kH_k(p,q)=\sum_{k=1}^{n}q^k\sum_{i=1}^{k}\frac{1}{[i]_{p,q}}.$$

By changing the sums, we get

$$\sum_{k=1}^{n}q^kH_k(p,q)=\sum_{i=1}^{n}\frac{1}{[i]_{p,q}}\left(\sum_{k=1}^{n}q^k-\sum_{k=1}^{i-1}q^k\right).$$

Using the definition (1.1), we write

$$\sum_{k=1}^{n}q^kH_k(p,q)=H_n(p,q)[n+1]_q-\sum_{i=1}^{n}\frac{[i]_q}{[i]_{p,q}}$$

which equals to

$$H_n(p,q)[n+1]_q-\frac{1}{1-q}\sum_{i=1}^{n}\frac{1-q^i}{[i]_{p,q}}=H_n(p,q)[n+1]_q-\frac{1}{1-q}\left(\sum_{i=1}^{n}\frac{1}{[i]_{p,q}}-\sum_{i=1}^{n}\frac{q^i}{[i]_{p,q}}\right).$$

Finally, we complete the proof by using (2.1). □

As a result of this Lemma, we can obtain the following equation by replacing q and p with eachother in (2.5).

Corollary 2.1. *For $n \geq 1$,*

$$\sum_{k=1}^{n} p^k H_k(p,q) = \sum_{k=1}^{n} p^k H_k(q,p) = \frac{1}{1-p}\left(\tilde{H}_n(p,q) - p^{n+1}H_n(p,q)\right). \tag{2.6}$$

Lemma 2.3. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n} q^{-k}\tilde{H}_k(q,p) = \frac{q}{q-1}\left(H_n(p,q) - q^{-n-1}\tilde{H}_n(q,p)\right). \tag{2.7}$$

Proof. By using (1.1), (1.3) and (2.1), the proof is similar to the proof of the Lemma 2.2. □

One can clearly observe that by taking $p = 1$ in (2.7), we obtain the equation (1.5). Moreover, we can obtain the following equation from (2.7) by interchanging q and p with eachother.

Corollary 2.2. *For $n \geq 1$,*

$$\sum_{k=1}^{n} p^{-k}\tilde{H}_k(p,q) = \frac{p}{p-1}\left(H_n(p,q) - p^{-n-1}\tilde{H}_n(p,q)\right). \tag{2.8}$$

Lemma 2.4. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n} p^{-k}\tilde{H}_k(q,p) = \frac{p}{p-1}\left(H_n(p,q) - p^{-n-1}\tilde{H}_n(p,q) - \frac{p-q}{p-1}\left(1 - p^{-n-1}(p-n+np)\right)\right). \tag{2.9}$$

Proof. Using the identity (2.3) and some elementary operations, we obtain

$$\sum_{k=1}^{n} p^{-k}\tilde{H}_k(q,p) = \sum_{k=1}^{n} p^{-k}\left(\tilde{H}_k(p,q) - k(p-q)\right) \tag{2.10}$$

which equals to

$$\sum_{k=1}^{n} p^{-k}\tilde{H}_k(p,q) - (p-q)\sum_{k=1}^{n} p^{-k}k.$$

The first sum is obtained in the Corollary 2.2. For the second sum we can write

$$\sum_{k=1}^{n} p^{-k}k = \sum_{k=1}^{n} p^{-k}\sum_{i=1}^{k} 1.$$

Then by changing sums we get

$$\sum_{k=1}^{n} p^{-k}k = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} p^{-k} - \sum_{k=1}^{i-1} p^{-k}\right).$$

By geometric sum formula we have

$$\sum_{k=1}^{n} p^{-k}k = \frac{p(p-q)}{(p-1)^2}\left(1 - p^{-n-1}(p-n+np)\right). \tag{2.11}$$

Substituting (2.9) and (2.11) in (2.10), we complete the proof. □

We can obtain the following equation by writing q and p interchangeably in (2.9).

Corollary 2.3. *For $n \geq 1$,*

$$\sum_{k=1}^{n} q^{-k}\tilde{H}_k(p,q) = \frac{q}{q-1}\left(H_n(p,q) - q^{-n-1}\tilde{H}_n(q,p) - \frac{q-p}{q-1}\left(1 - q^{-n-1}(q-n+nq)\right)\right). \tag{2.12}$$

Now we give the main results in the following theorems.

Theorem 2.5. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n} [k]_{p,q}H_k(p,q) = \frac{1}{p-q}\left(\frac{\tilde{H}_n(p,q)}{1-p} - \frac{\tilde{H}_n(q,p)}{1-q} + \left(\frac{q^{n+1}}{1-q} - \frac{p^{n+1}}{1-p}\right)H_n(q,p)\right). \tag{2.13}$$

Proof. By the definition (1.3), we can write

$$\sum_{k=1}^{n} [k]_{p,q}H_k(p,q)$$

as

$$\frac{1}{p-q}\sum_{k=1}^{n}\left(p^k - q^k\right)H_k(p,q) = \frac{1}{p-q}\left(\sum_{k=1}^{n} p^k H_k(p,q) - \sum_{k=1}^{n} q^k H_k(p,q)\right).$$

Finally, using (2.5) and (2.6) completes the proof. □

Theorem 2.6. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n}[k]_{p^{-1},q^{-1}}\tilde{H}_k(q,p)=\frac{pq}{p-q}\left(\frac{(p-q)H_n(p,q)}{(p-1)(q-1)}-\left(\frac{q^{-n}}{q-1}-\frac{p^{-n}}{p-1}\right)\tilde{H}_n(p,q)+nq^{-n}\frac{p-q}{q-1}+\frac{p(p-q)}{(p-1)^2}\left(1-p^{-n-1}(p-n+np)\right)\right). \tag{2.14}$$

Proof. By (1.3), we write the sum

$$\sum_{k=1}^{n}[k]_{p^{-1},q^{-1}}\tilde{H}_k(q,p)$$

as

$$\frac{pq}{p-q}\sum_{k=1}^{n}\left(p^{-k}-q^{-k}\right)\tilde{H}_k(q,p)=\frac{pq}{p-q}\left(\sum_{k=1}^{n}p^{-k}\tilde{H}_k(q,p)-\sum_{k=1}^{n}q^{-k}\tilde{H}_k(q,p)\right).$$

Then by using the equations (2.7) and (2.9) we complete the proof. □

Theorem 2.7. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n}[k]_{p^{-1},q^{-1}}\tilde{H}_k(p,q)=\frac{pq}{q-p}\left(\frac{(q-p)H_n(p,q)}{(1-q)(1-p)}-\left(\frac{p^{-n}}{p-1}-\frac{q^{-n}}{q-1}\right)\tilde{H}_n(q,p)+np^{-n}\frac{q-p}{p-1}+\frac{q(q-p)}{(q-1)^2}\left(1-q^{-n-1}(q-n+nq)\right)\right). \tag{2.15}$$

Proof. The proof can be done similarly by using (2.8) and (2.10). Also by writing q and p interchangeably in (2.14), the desired result is obtained. □

3. Conclusion

In this paper, we examined a new generalization of harmonic numbers and some summations of this numbers. We have achieved results that will lead to our next works. As for usual harmonic numbers and q-harmonic numbers, new identities for p,q-harmonic numbers can be deduced and various congruences of these numbers can be investigate.

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