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A NEW PERSPECTIVE ON BICOMPLEX NUMBERS WITH LEONARDO NUMBER COMPONENTS

Murat TURAN[1], Sıddıka ÖZKALDI KARAKUŞ[2] and Semra KAYA NURKAN[3]

[1,2]Department of Mathematics, Faculty of Science, Bilecik Şeyh Edebali University, Bilecik, TÜRKİYE
[3]Department of Mathematics, Faculty of Arts and Sciences, Uşak University, Uşak, TÜRKİYE

Abstract. In the present paper, the bicomplex Leonardo numbers will be introduced with the use of Leonardo numbers and some important algebraic properties including recurrence relation, generating function, Catalan's and Cassini's identities, Binet's formula, sum formulas will also be obtained.

1. Introduction

It is an old and interesting problem to obtain a natural extension of complex numbers and many mathematicians have studied this by defining multicomplex numbers, and corresponding function theory. One of these extensions is quaternions which have been described by S.W. Hamilton [7], and the other one is bicomplex numbers which have been described by C. Segre [17] in order to formulate physical problems in a 4-dimensional space. There are some differences between these extensions in the perspective of commutativity and forming a division algebra. Namely, quaternions are non commutative and form a division algebra, while bicomplex numbers are commutative and do not form a division algebra.

In [15], Price has developed the bicomplex algebra and function theory. Indeed, bicomplex algebra is a 2-dimensional Clifford algebra, satisfy the commutative multiplication on $\mathbb{C}$ and also has important applications in image processing, geometry and theoretical physics (see [15,16]).

It is well known that a complex number $x \in \mathbb{C}$ is represented as $x = x_1 + x_2 i$, such that $x_1, x_2 \in \mathbb{R}$, $i^2 = -1$ and a bicomplex numbers $x \in \mathbb{C}_2$ is written as

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[1] murat199869@gmail.com; 0000-0001-9684-7924
[2] siddika.karakus@bilecik.edu.tr; 0000-0002-2699-4109
[3] semra.kaya@usak.edu.tr-Corresponding author; 0000-0001-6473-4458.

$$x = x_1 + x_2 i + x_3 j + x_4 ij \tag{1}$$

by the basis $1, i, j, ij$ where $x_s \in \mathbb{R}$, $1 \leq s \leq 4$ and $i^2 = j^2 = -1$, $ij = ji$ with $(ij)^2 = 1$.

Notice that $ij \in \mathbb{C}_2$, but $ij \notin \mathbb{C}$. From equation (1), the space of the bicomplex numbers $\mathbb{C}_2$ can be seen of dimension 4 over $\mathbb{R}$, since the space of complex numbers $\mathbb{C}$ is of dimension 2 over $\mathbb{R}$.

For any two bicomplex numbers $z = z_1 + z_2 i + z_3 j + z_4 ij$ and $w = w_1 + w_2 i + w_3 j + w_4 ij$, addition, multiplication and scalar multiplication of an element in $\mathbb{C}_2$ by a real number c are given, respectively

$$z + w = (z_1 + w_1) + (z_2 + w_2)\, i + (z_3 + w_3)\, j + (z_4 + w_4)\, ij$$

$$\begin{aligned} z \cdot w &= (z_1 w_1 - z_2 w_2 - z_3 w_3 + z_4 w_4) + (z_1 w_2 + z_2 w_1 - z_3 w_4 - z_4 w_3)\, i \\ &+ (z_1 w_3 + z_3 w_1 - z_2 w_4 - z_4 w_2)\, j + (z_1 w_4 + z_4 w_1 + z_2 w_3 + z_3 w_2)\, ij \end{aligned} \tag{2}$$

$$cz = cz_1 + cz_2 i + cz_3 j + cz_4 ij.$$

Note that there is a big difference between $\mathbb{C}$ and $\mathbb{C}_2$: $\mathbb{C}$ form a field while $\mathbb{C}_2$ do not since it contains divisors of zero. Now we are ready to turn our main topic since we have given brief overview of bicomplex numbers. For more details, we refer the reader to [13, 15] which are dealing with bicomplex analysis.

One of the well known as below and most examined sequences is Fibonacci and also they are many notable sequences of integers. In the existing literature, one can find many papers on Fibonacci and Lucas numbers, (see [8, 9, 11]). Moreover, they have been examined on different number systems, for example, quaternions and hybrid numbers [2, 6, 10, 14, 22]. It is benefical to recall the definitions of Fibonacci and Lucas sequences: for $n \geq 0$,

$$F_{n+2} = F_{n+1} + F_n$$

$$L_{n+2} = L_{n+1} + L_n$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$, respectively. The Binet's formulas for F_n and L_n are

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi} \tag{3}$$

$$L_n = \phi^n + \psi^n, \tag{4}$$

where ϕ and ψ are the roots of the characteristic equation $x^2 - x - 1 = 0$.

In the present paper, we deal with Leonardo sequence which has similar properties with Fibonacci sequence and denote the *nth* Leonardo numbers by L_{e_n}. Some properties of Leonardo numbers have been given by Catarino and Borges in [4] and it is noteworthy to recall that Leonardo sequence is given via this recurrence relation: for $n \geq 2$,

$$L_{e_n} = L_{e_{n-1}} + L_{e_{n-2}} + 1, \tag{5}$$

with the initial conditions $L_{e_0} = L_{e_1} = 1$. One can find many sequences of integers indexed in *The Online Encyclopedia of Integer Sequences*, being in this case $\{L_{e_n}\}$: $A001595$ in [19].

Also, the following relation holds for Leonardo numbers for $n \geq 2$,

$$L_{e_{n+1}} = 2L_{e_n} - L_{e_{n-2}}. \tag{6}$$

The Binet formula of the Leonardo numbers is

$$L_{e_n} = \frac{2\phi^{n+1} - 2\psi^{n+1} - \phi + \psi}{\phi - \psi} \tag{7}$$

where ϕ and ψ are roots of characteristic equation $x^3 - 2x^2 + 1 = 0$.

By Binet formula, the relationship between Leonardo and Fibonacci numbers is

$$L_{e_n} = 2F_{n+1} - 1 \tag{8}$$

where F_n is nth Fibonacci number.

In [4], Cassini's, Catalan's and d'Ocagne's identities have been obtained for Leonardo numbers by Catarino et /it al. Moreover they have presented the 2-dimensional recurrences relations and matrix representation of Leonardo numbers. In [18], Shannon have defined generalized Leonardo numbers which are considered Asveld's extension and Horadam's generalized sequence.

Now, we are ready to recall some identities involving Fibonacci, Lucas and Leonardo numbers as follows, for more details related to them, please refer [1, 4, 11, 22]:

$$F_n + L_n = 2F_{n+1} \tag{9}$$

$$F_{n+r}F_{n+s} - F_nF_{n+r+s} = (-1)^n F_rF_s \tag{10}$$

$$L_{e_{n+m}} + (-1)^m L_{e_{n-m}} = L_m(L_{e_n} + 1) - 1 - (-1)^m \tag{11}$$

$$L_{e_{n+m}} - (-1)^m L_{e_{n-m}} = L_{n+1}(L_{e_{m-1}} + 1) - 1 + (-1)^m \tag{12}$$

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r} F_r^2 \tag{13}$$

$$L_{r+s} - (-1)^s L_{r-s} = 5F_rF_s \tag{14}$$

$$\sum_{k=1}^{n} (-1)^{k-1} F_{k+1} = (-1)^{n-1} F_n. \tag{15}$$

In the existing literature, there are also different generalizations of Fibonacci and Lucas numbers. One of these generalizations is the bicomplex Fibonacci and Lucas numbers and they have been defined by Nurkan et /it al. and some properties have been presented in [14]. They have defined the bicomplex version of Fibonacci and bicomplex Lucas numbers as follows:

$$BF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}ij \tag{16}$$

$$BL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}ij \tag{17}$$

where F_n and L_n are nth Fibonacci and Lucas sequences, respectively. Furthermore, Torunbalcı have defined the bicomplex Fibonacci quaternions in [21] by

$$\begin{aligned} BF_n &= F_n + iF_{n+1} + jF_{n+2} + ijF_{n+3}, \\ &= F_n + iF_{n+1} + (F_{n+2} + iF_{n+3})\, j \end{aligned}$$

where i, j and ij satisfy the conditions $i^2 = j^2 = -1$, $ij = ji$ with $(ij)^2 = 1$.

In [12] the authors have investigated Leonardo Pisano polynomials and hybrinomials with the use of the Leonardo Pisano numbers and hybrid numbers. They have also obtained the basic algebraic properties and some identities of these polynomials and hybrinomials.

With the motivation of these mentioned papers, here, we introduce the bicomplex numbers with Leonardo number components. We also aim to obtain generating function, Binet's formula, recurrence relation, summation formula, Catalan's, Cassini's and other identities.

For more details about Leonardo numbers, see [3–5, 12, 20].

2. Bicomplex Leonardo Numbers

In this section, by introducing the bicomplex Leonardo numbers, we study Binet's formula, summation formulas, Catalan's and Cassini's identities and generating function.

Definition 1. *For $n \geqslant 1$, the nth bicomplex Leonardo numbers are defined by*

$$\mathbb{BL}_{e_n} = L_{e_n} + L_{e_{n+1}} i + L_{e_{n+2}} j + L_{e_{n+3}} ij. \tag{18}$$

Throughout the paper, nth bicomplex Leonardo numbers is denoted by $\mathbb{BL}_{e_n}$.

From the recurrence relation (5) and the definition of bicomplex Leonardo numbers (18), for $n \geq 2$ we get

$$\begin{aligned} \mathbb{BL}_{e_n} &= (L_{e_{n-1}} + L_{e_{n-2}} + 1) + (L_{e_n} + L_{e_{n-1}} + 1)i \\ &\quad +(L_{e_{n+1}} + L_{e_n} + 1)j + (L_{e_{n+2}} + L_{e_{n+1}} + 1)ij, \\ &= \mathbb{BL}_{e_{n-1}} + \mathbb{BL}_{e_{n-2}} + C. \end{aligned}$$

For the sake of the shortness, we express $1 + i + j + ij$ by C along the paper. Also initial conditions are $\mathbb{BL}_{e_0} = 1 + i + 3j + 5ij$ and $\mathbb{BL}_{e_1} = 1 + 3i + 5j + 9ij$.

Another recurrence relation of bicomplex Leonardo numbers can also be given by

$$\mathbb{BL}_{e_{n+1}} = 2\mathbb{BL}_{e_n} - \mathbb{BL}_{e_{n-2}} \tag{19}$$

for $n \geqslant 2$. Using the definition of bicomplex Leonardo numbers (18) and the recurrence relation of Leonardo numbers (6), for $n \geqslant 2$ we get

$$\begin{aligned} \mathbb{BL}_{e_{n+1}} &= 2L_{e_n} - L_{e_{n-2}} + (2L_{e_{n+1}} - L_{e_{n-1}})i \\ &\quad +(2L_{e_{n+2}} - L_{e_n})j + (2L_{e_{n+3}} - L_{e_{n+1}})ij \\ &= 2\mathbb{BL}_{e_n} - \mathbb{BL}_{e_{n-2}} \end{aligned}$$

with the initial values $\mathbb{BL}_{e_0} = 1 + i + 3j + 5ij$ and $\mathbb{BL}_{e_1} = 1 + 3i + 5j + 9ij$.

Theorem 1. *The generation function for the bicomplex Leonardo numbers denoted by $g\mathbb{BL}_{e_n}(t)$ is*

$$g\mathbb{BL}_{e_n}(t) = \frac{\mathbb{BL}_{e_0} + t(-1 + i - j - ij) + t^2(1 - i - j - 3ij)}{1 - 2t + t^3}.$$

Proof. Let the formal power series expression of the generating function for $\{\mathbb{BL}_{e_n}\}_{n=0}^{\infty}$ be as

$$g\mathbb{BL}_{e_n}(t) = \sum_{n=0}^{\infty} \mathbb{BL}_{e_n} t^n. \tag{20}$$

That is

$$g\mathbb{BL}_{e_n}(t) = \mathbb{BL}_{e_0} + \mathbb{BL}_{e_1} t + \mathbb{BL}_{e_2} t^2 + ... + \mathbb{BL}_{e_k} t^k + ...$$

Then, we have

$$\begin{aligned}
\left(1 - 2t + t^3\right) g\mathbb{BL}_{e_n}(t) &= \left(1 - 2t + t^3\right) \begin{pmatrix} \mathbb{BL}_{e_0} + \mathbb{BL}_{e_1} t \\ +\mathbb{BL}_{e_2} t^2 + ... + \mathbb{BL}_{e_k} t^k + ... \end{pmatrix} \\
\left(1 - 2t + t^3\right) g\mathbb{BL}_{e_n}(t) &= \mathbb{BL}_{e_0} + \mathbb{BL}_{e_1} t + \mathbb{BL}_{e_2} t^2 + ... + \\
&\quad -2\mathbb{BL}_{e_0} t - 2\mathbb{BL}_{e_1} t^2 - 2\mathbb{BL}_{e_2} t^3 - ... \\
&\quad +\mathbb{BL}_{e_0} t^3 + \mathbb{BL}_{e_1} t^4 + \mathbb{BL}_{e_2} t^5 + ... \\
&= \mathbb{BL}_{e_0} + t\left(\mathbb{BL}_{e_1} - 2\mathbb{BL}_{e_0}\right) + t^2\left(\mathbb{BL}_{e_2} - 2\mathbb{BL}_{e_1}\right) \\
&\quad +t^3\left(\mathbb{BL}_{e_3} - 2\mathbb{BL}_{e_2} + \mathbb{BL}_{e_0}\right) + ... \\
&\quad +t^{k+1}\left(\mathbb{BL}_{e_{k+1}} - 2\mathbb{BL}_{e_k} + \mathbb{BL}_{e_{k-2}}\right) + ...
\end{aligned}$$

Since the recurrence relation of bicomplex numbers (19) and also by using initial conditions, we get

$$\begin{aligned}
g\mathbb{BL}_{e_n}(t)(1 - 2t + t^3) &= (1 + i + 3j + 5ij) \\
&\quad +t(-1 + i - j - ij) + t^2(1 - i - j - 3ij).
\end{aligned}$$

Therefore, we get the generating function for $\{\mathbb{BL}_{e_n}\}_{n=0}^{\infty}$ as

$$\sum_{n=0}^{\infty} \mathbb{BL}_{e_n} t^n = \frac{\mathbb{BL}_{e_0} + t(-1 + i - j - ij) + t^2(1 - i - j - 3ij)}{1 - 2t + t^3}.$$

□

Theorem 2. *For any integer $n \geqslant 0$, we have*

$$\mathbb{BL}_{e_n} = 2BF_{n+1} - C. \tag{21}$$

Here BF_n is nth bicomplex Fibonacci number.

Proof. Using the definition of bicomplex Leonardo numbers (18) and the recurrence relation between Leonardo and Fibonacci numbers (8) we get

$$\begin{aligned}
\mathbb{BL}_{e_n} &= L_{e_n} + L_{e_{n+1}}i + L_{e_{n+2}}j + L_{e_{n+3}}ij, \\
&= (2F_{n+1} - 1) + (2F_{n+2} - 1)i \\
&\quad +(2F_{n+3} - 1)j + (2F_{n+4} - 1)ij \\
&= 2(F_{n+1} + F_{n+2}i + F_{n+3}j + F_{n+4}ij) - C \\
&= 2BF_{n+1} - C.
\end{aligned}$$

□

Theorem 3. *For any integer $n \geqslant 0$, the Binet's formula for $\mathbb{BL}_{e_n}$ is as follows:*

$$\mathbb{BL}_{e_n} = 2\left(\frac{\underline{\Phi}\phi^{n+1} - \underline{\Psi}\psi^{n+1}}{\phi - \psi}\right) - C \tag{22}$$

where $\underline{\Phi} = 1 + \phi i + \phi^2 j + \phi^3 ij$ and $\underline{\Psi} = 1 + \psi i + \psi^2 j + \psi^3 ij$.

Proof. By using the definition of bicomplex Leonardo numbers (18) and the Binet's formula of Leonardo numbers (7), we get

$$\mathbb{BL}_{e_n} = 2\begin{pmatrix} \frac{\phi^{n+1}-\psi^{n+1}}{\phi-\psi} + \frac{\phi^{n+2}-\psi^{n+2}}{\phi-\psi}i \\ +\frac{\phi^{n+3}-\psi^{n+3}}{\phi-\psi}j + \frac{\phi^{n+4}-\psi^{n+4}}{\phi-\psi}ij \end{pmatrix} - (1 + i + j + ij).$$

If the expressions $\underline{\Phi} = 1 + \phi i + \phi^2 j + \phi^3 ij$, $\underline{\Psi} = 1 + \psi i + \psi^2 j + \psi^3 ij$ are used in the last equation, we can easily obtained the result. □

Theorem 4. *The summation formulas for $\mathbb{BL}_{e_n}$ are as follows for $n \geq 0$,*

1) $\sum\limits_{k=0}^{n} \mathbb{BL}_{e_k} = \mathbb{BL}_{e_{n+2}} - (n+2)C - (2i + 4j + 8ij)$,

2) $\sum\limits_{k=0}^{n} \mathbb{BL}_{e_{2k}} = \mathbb{BL}_{e_{2n+1}} - nC - (2i + 2j + 4ij)$,

3) $\sum\limits_{k=0}^{n} \mathbb{BL}_{e_{2k+1}} = \mathbb{BL}_{e_{2n+2}} - (n+2)C - (2j + 4ij)$.

Also for $n \geq 1$

4) $\sum\limits_{r=0}^{n} (-1)^{r-1}\,\mathbb{BL}_{e_r} = \begin{cases} -\left(\mathbb{BL}_{e_{n-1}} + 2 + 2j + 2ij\right), & n \text{ is even} \\ \mathbb{BL}_{e_{n-1}} - 1 + i - j - ij, & n \text{ is odd.} \end{cases}$

Proof. With the use of the sums and products of terms of the Leonardo sequence proposition (3.1) in [4] and also the definition of bicomplex Leonardo numbers, the proof of (1), (2) and (3) follows easily.

In order to prove (4); we obtain

$$\sum_{r=0}^{n} (-1)^{r-1}\,\mathbb{BL}_{e_r} = \sum_{r=0}^{n} (-1)^{r-1}\,L_{e_r} + i\sum_{r=0}^{n} (-1)^{r-1}\,L_{e_{r+1}}$$

$$+j\sum_{r=0}^{n}(-1)^{r-1}L_{e_{r+2}}+ij\sum_{r=0}^{n}(-1)^{r-1}L_{e_{r+3}}$$

from the definition of $\mathbb{BL}_{e_n}$. Then by using (5), (8) and (15) we get

$$\sum_{r=0}^{n}(-1)^{r-1}\mathbb{BL}_{e_r}=\begin{cases}(-2BF_n-1+i-j-ij), & n \text{ is even}\\ (2BF_n-2-2j-2ij), & n \text{ is odd.}\end{cases}$$

where BF_n is nth bicomplex Fibonacci number. Taking into account (21) we complete the proof. □

Now the following interesting identities in accordance with the Binet's formula (22) for $\{L_{e_n}\}$ can be presented as follows:

Theorem 5. ***(Catalan's Identity)*** *For positive integers n and r with $n \geq r$, we have*

$$\begin{aligned}\mathbb{BL}_{e_n}^2-\mathbb{BL}_{e_{n-r}}\mathbb{BL}_{e_{n+r}} &= (\mathbb{BL}_{e_{n-r}}+\mathbb{BL}_{e_{n+r}}-2\mathbb{BL}_{e_n})C \\ &\quad +12(-1)^{n-r+1}(2j+ij)F_r^2.\end{aligned} \tag{23}$$

Proof. First, by using (22) to left hand side (LHS) then, we get

$$\begin{aligned}LHS &= \left(2\left(\frac{\underline{\Phi}\phi^{n+1}-\underline{\Psi}\psi^{n+1}}{\phi-\psi}\right)-C\right)\left(2\left(\frac{\underline{\Phi}\phi^{n+1}-\underline{\Psi}\psi^{n+1}}{\phi-\psi}\right)-C\right) \\ &- \left(2\left(\frac{\underline{\Phi}\phi^{n-r+1}-\underline{\Psi}\psi^{n-r+1}}{\phi-\psi}\right)-C\right)\left(2\left(\frac{\underline{\Phi}\phi^{n+r+1}-\underline{\Psi}\psi^{n+r+1}}{\phi-\psi}\right)-C\right).\end{aligned} \tag{24}$$

By considering ϕ, ψ, $\underline{\Phi}=1+\phi i+\phi^2 j+\phi^3 ij$ and $\underline{\Psi}=1+\psi i+\psi^2 j+\psi^3 ij$ then, we also have

$$\underline{\Phi}.\underline{\Psi}=6j+3ij. \tag{25}$$

By taking into account (13) and (25) in (LHS), one can get

$$\begin{aligned}LHS &= (\mathbb{BL}_{e_{n-r}}+\mathbb{BL}_{e_{n+r}}-2\mathbb{BL}_{e_n})C \\ &\quad +12(-1)^{n-r+1}(2j+ij)F_r^2\end{aligned}$$

which completes the proof. □

Remark that, if one takes in the case $r=1$ in (23) and using the relation (1.5), then Catalan's identity reduces to Cassini's identity for $\mathbb{BL}_{e_n}$.

Corollary 1. ***(Cassini's Identity)*** *For $n \geq 1$, we have*

$$\begin{aligned}\mathbb{BL}_{e_n}^2-\mathbb{BL}_{e_{n-1}}\mathbb{BL}_{e_{n+1}} &= (\mathbb{BL}_{e_{n-1}}-\mathbb{BL}_{e_{n-2}})C \\ &\quad +12(-1)^n(2j+ij).\end{aligned}$$

Theorem 6. *The following holds between the Fibonacci numbers and bicomplex Leonardo numbers*

$$\begin{aligned}\mathbb{BL}_{e_{k+m}}\mathbb{BL}_{e_{k+s}}-\mathbb{BL}_{e_k}\mathbb{BL}_{e_{k+m+s}} &= (\mathbb{BL}_{e_k}-\mathbb{BL}_{e_{k+m}}+\mathbb{BL}_{e_{k+m+s}}-\mathbb{BL}_{e_{k+s}})C\\ &\quad +12(-1)^{k+1}F_mF_s\left(2j+ij\right).\end{aligned}$$

Here k, m, and s be positive integers.

Proof. By using the Binet's formula (22) to left hand side (LHS), we get

$$\begin{aligned}LHS &= \left(\frac{2\underline{\Phi}\phi^{k+m+1}-2\underline{\Psi}\psi^{k+m+1}}{\phi-\psi}-C\right)\left(\frac{2\underline{\Phi}\phi^{k+s+1}-2\underline{\Psi}\psi^{k+s+1}}{\phi-\psi}-C\right)\\ &\quad -\left(\frac{2\underline{\Phi}\phi^{k+1}-2\underline{\Psi}\psi^{k+1}}{\phi-\psi}-C\right)\left(\frac{2\underline{\Phi}\phi^{k+m+s+1}-2\underline{\Psi}\psi^{k+m+s+1}}{\phi-\psi}-C\right)\\ &= (\mathbb{BL}_{e_k}-\mathbb{BL}_{e_{k+m}}+\mathbb{BL}_{e_{k+m+s}}-\mathbb{BL}_{e_{k+s}})C\\ &\quad +\frac{4\underline{\Phi}.\underline{\Psi}}{(\phi-\psi)^2}\left(\phi^{k+1}\psi^{k+1}\left(-\phi^m\psi^s-\phi^s\psi^m+\psi^{m+s}+\phi^{m+s}\right)\right).\end{aligned}$$

Then with the use of Vajda's identity for Fibonacci numbers (10) and (25), we have

$$\begin{aligned}LHS &= (\mathbb{BL}_{e_k}-\mathbb{BL}_{e_{k+m}}+\mathbb{BL}_{e_{k+m+s}}-\mathbb{BL}_{e_{k+s}})C\\ &\quad +12(-1)^{k+1}F_mF_s\left(2j+ij\right).\end{aligned}$$

□

Theorem 7. *The following identities between the Lucas, Leonardo, bicomplex Lucas and bicomplex Leonardo numbers are as follows:*

$$\mathbb{BL}_{e_{n+m}}+(-1)^m\mathbb{BL}_{e_{n-m}} = L_m\mathbb{BL}_{e_n}+(L_m-(-1)^m-1)C \tag{26}$$

$$\mathbb{BL}_{e_{n+m}}-(-1)^m\mathbb{BL}_{e_{n-m}} = (L_{e_{m-1}}+1)BL_{n+1}+((-1)^m-1)C. \tag{27}$$

Here n and m are positive integers, with $n \geq m$.

Proof. For the proof of (26), by using the definition of bicomplex Leonardo numbers to left hand side (LHS), we get

$$\begin{aligned}LHS &= (L_{e_{n+m}}+(-1)^mL_{e_{n-m}})+(L_{e_{n+m+1}}+(-1)^mL_{e_{n-m+1}})i\\ &\quad +(L_{e_{n+m+2}}+(-1)^mL_{e_{n-m+2}})j+(L_{e_{n+m+3}}+(-1)^mL_{e_{n-m+3}})ij.\end{aligned}$$

Taking into account (11), we obtain

$$LHS = L_m\mathbb{BL}_{e_n}+C(L_m-(-1)^m-1).$$

By taking into account (1.11) the proof of (27) can obtained in a similar manner. □

Theorem 8. *The following identity between the Fibonacci and bicomplex Leonardo numbers as follows:*

$$\mathbb{BL}_{e_{m+r}}\mathbb{BL}_{e_{m-r}}-\mathbb{BL}_{e_{m+s}}\mathbb{BL}_{e_{m-s}} = \left(\mathbb{BL}_{e_{m+s}}-\mathbb{BL}_{e_{m+r}}+\mathbb{BL}_{e_{m-s}}-\mathbb{BL}_{e_{m-r}}\right)C$$

$$+12\left(2j+ij\right)\left(\left(-1\right)^{m-s+1}F_s^2+\left(-1\right)^{m-r}F_r^2\right)$$

Here m, r and s are positive integers with $m\geq r$ and $m\geq s$.

Proof. By using (22) to left hand side (LHS), we have

$$\begin{aligned}
LHS &= \left(2\frac{\underline{\Phi}\phi^{m+r+1}-\underline{\Psi}\psi^{m+r+1}}{\phi-\psi}-C\right)\left(2\frac{\underline{\Phi}\phi^{m-r+1}-\underline{\Psi}\psi^{m-r+1}}{\phi-\psi}-C\right)\\
&\quad-\left(2\frac{\underline{\Phi}\phi^{m+s+1}-\underline{\Psi}\psi^{m+s+1}}{\phi-\psi}-C\right)\left(2\frac{\underline{\Phi}\phi^{m-s+1}-\underline{\Psi}\psi^{m-s+1}}{\phi-\psi}-C\right)\\
&= (\mathbb{BL}_{e_{m+s}}-\mathbb{BL}_{e_{m+r}}+\mathbb{BL}_{e_{m-s}}-\mathbb{BL}_{e_{m-r}})C\\
&\quad-\frac{4\underline{\Phi\Psi}}{(\phi-\psi)^2}\left(\phi^m\psi^m\left(-\phi^r\psi^{-r}-\phi^{-r}\psi^r+\phi^s\psi^{-s}+\phi^{-s}\psi^s\right)\right).\\
&= (\mathbb{BL}_{e_{m+s}}-\mathbb{BL}_{e_{m+r}}+\mathbb{BL}_{e_{m-s}}-\mathbb{BL}_{e_{m-r}})C\\
&\quad-4\underline{\Phi}.\underline{\Psi}\left[\frac{\phi^m\psi^m}{(\phi-\psi)^2}\left(\phi^s\psi^{-s}+\phi^{-s}\psi^s-2\right)-\frac{\phi^m\psi^m}{(\phi-\psi)^2}\left(\phi^r\psi^{-r}+\phi^{-r}\psi^r-2\right)\right]
\end{aligned}$$

On the other hand, we can show that

$$\begin{aligned}
F_m^2-F_{m+r}F_{m-r} &= \left(-1\right)^{m-r}F_r^2\\
&= \frac{\phi^m\psi^m}{(\phi-\psi)^2}\left(\phi^r\psi^{-r}+\phi^{-r}\psi^r-2\right).
\end{aligned}$$

Also by using above equation and (25) in (LHS), we get

$$\begin{aligned}
LHS &= \left(\mathbb{BL}_{e_{m+s}}-\mathbb{BL}_{e_{m+r}}+\mathbb{BL}_{e_{m-s}}-\mathbb{BL}_{e_{m-r}}\right)C\\
&\quad+12\left(2j+ij\right)\left(\left(-1\right)^{m-s+1}F_s^2+\left(-1\right)^{m-r}F_r^2\right).
\end{aligned}$$

□

Theorem 9. *The following identity between the Lucas and bicomplex Leonardo numbers is provided:*

$$\begin{aligned}
\mathbb{BL}_{e_n}\mathbb{BL}_{e_m}-\mathbb{BL}_{e_s}\mathbb{BL}_{e_r} &= \left(\mathbb{BL}_{e_s}-\mathbb{BL}_{e_n}+\mathbb{BL}_{e_r}-\mathbb{BL}_{e_m}\right)C\\
&\quad+\frac{12}{5}\left(2j+ij\right)\left(\left(-1\right)^mL_{n-m}-\left(-1\right)^rL_{s-r}\right).
\end{aligned}$$

Here n, m, s and r are positive integers with $n\geq m$, $s\geq r$ and $n+m=s+r$.

Proof. By using (22) to left hand side (LHS), we get

$$\begin{aligned}
LHS &= \left(2\frac{\underline{\Phi}\phi^{n+1}-\underline{\Psi}\psi^{n+1}}{\phi-\psi}-C\right)\left(2\frac{\underline{\Phi}\phi^{m+1}-\underline{\Psi}\psi^{m+1}}{\phi-\psi}-C\right)\\
&\quad-\left(2\frac{\underline{\Phi}\phi^{s+1}-\underline{\Psi}\psi^{s+1}}{\phi-\psi}-C\right)\left(2\frac{\underline{\Phi}\phi^{r+1}-\underline{\Psi}\psi^{r+1}}{\phi-\psi}-C\right)\\
&= \left(\mathbb{BL}_{e_s}-\mathbb{BL}_{e_n}+\mathbb{BL}_{e_r}-\mathbb{BL}_{e_m}\right)C
\end{aligned}$$

$$+\frac{4\underline{\Phi}.\underline{\Psi}}{(\phi-\psi)^2}\left(\phi^n\psi^m+\phi^m\psi^n-\phi^s\psi^r-\phi^r\psi^s\right).$$

From (14) and (25), we also obtain that

$$\begin{aligned} LHS &= \left(\mathbb{BL}_{e_s}-\mathbb{BL}_{e_n}+\mathbb{BL}_{e_r}-\mathbb{BL}_{e_m}\right)C \\ &\quad +\frac{12}{5}\left(2j+ij\right)\left((-1)^m L_{n-m}-(-1)^r L_{s-r}\right). \end{aligned}$$

This completes the proof. □

Theorem 10. *For r and s positive integers with $r\geq 1$, $s\geq 1$, then we have*

$$\begin{aligned} \mathbb{BL}_{e_{s+1}}\mathbb{BL}_{e_{r+1}}-\mathbb{BL}_{e_{s-1}}\mathbb{BL}_{e_{r-1}} &= 4\left[\begin{array}{c} -5F_{s+r+3}+5F_{s+r+7}+2i(F_{s+r+3}-F_{s+r+7}) \\ -2jF_{s+r+5}+4ijF_{s+r+5} \end{array}\right] \\ &\quad -(\mathbb{BL}_{e_s}+\mathbb{BL}_{e_r}+2C)C. \end{aligned}$$

Proof. By using (22) to left hand side (LHS), we get

$$\begin{aligned} LHS &= \left(2\frac{\underline{\Phi}\phi^{s+2}-\underline{\Psi}\psi^{s+2}}{\phi-\psi}-C\right)\left(2\frac{\underline{\Phi}\phi^{r+2}-\underline{\Psi}\psi^{r+2}}{\phi-\psi}-C\right) \\ &\quad -\left(2\frac{\underline{\Phi}\phi^{s}-\underline{\Psi}\psi^{s}}{\phi-\psi}-C\right)\left(2\frac{\underline{\Phi}\phi^{r}-\underline{\Psi}\psi^{r}}{\phi-\psi}-C\right) \\ &= -(\mathbb{BL}_{e_s}+\mathbb{BL}_{e_r}+2C)C \\ &\quad +\frac{4}{(\phi-\psi)^2}\left(\underline{\Phi}^2\phi^{s+r}\left(\phi^4-1\right)+\underline{\Psi}^2\psi^{s+r}\left(\psi^4-1\right)\right). \end{aligned}$$

Here, if we use the Binet formula for the Lucas numbers (4) and make the necessary calculations, we obtain

$$\begin{aligned} LHS &= -(\mathbb{BL}_{e_s}+\mathbb{BL}_{e_r}+2C)C \\ &\quad +\left(-2L_{s+r+5}+L_{s+r+1}+L_{s+r+9}\right)+i\left(4L_{s+r+5}-2L_{s+r+1}-L_{s+r+9}\right) \\ &\quad +j\left(-2L_{s+r+7}+2L_{s+r+3}\right)+k\left(4L_{s+r+7}-4L_{s+r+3}\right) \end{aligned}$$

Also by using the equation (14), we have

$$\begin{aligned} LHS &= -(\mathbb{BL}_{e_s}+\mathbb{BL}_{e_r}+2C)C \\ &\quad +4\left[\begin{array}{c} -5F_{s+r+3}+5F_{s+r+7}+2i(F_{s+r+3}-F_{s+r+7}) \\ -2jF_{s+r+5}+4ijF_{s+r+5} \end{array}\right]. \end{aligned}$$

This completes the proof. □

3. Conclusion

In the present paper, bicomplex Leonardo numbers with coefficients of basis of Leonardo numbers have been introduced. First of all the recurrence relation and generating function for these numbers have been obtained. Then summation formulas for these numbers have been provided. Furthermore, Catalan's and Cassini's identities and some interesting properties have been given.

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