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Catalan Numbers in Terms of $(\alpha, k)-$Gamma Function and $(\alpha, k)-$Beta Function

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Abstract. In the paper, the authors discuss some extended results involving the Catalan numbers and establish an integral representation of the Catalan numbers in terms of the (α, k)-gamma and (α, k)-beta function. We refer to the results available in the literature by giving special values to the parameters in the obtained theorems.

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1. Introduction

The first few Catalan numbers C_n for $0 \leq n \leq 14$ are

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440.

The Catalan numbers are defined by means of the following generating functions

$$\begin{aligned} \frac{1-\sqrt{1-4x}}{2x} &= \sum_{n=0}^{\infty} C_n x^n \\ &= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 \ldots \end{aligned}$$

One of explicit formulas of C_n for $n \geq 0$ reads that

$$C_n = \frac{4^n \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+2)} = \frac{1}{n+1}\binom{2n}{n}.$$

For more information on the Catalan numbers C_n, please see ([4, 9, 12]).

In [8], the classical gamma function is given by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \ x > 0,$$

and the classical beta function is defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \ x, \ y > 0.$$

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Relationship between beta function and gamma function in [2] is given by

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ x,\ y>0.$$

The rising factorial, denoted by $(x)_n$ or $x^{(n)}$, is defined by [3]

$$x^{(n)}=x(x+1)\ldots(x+n-1).$$

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in ([1,5,6]).

Definition 1.1. (**Conformable fractional derivative**) Given a function $f:[0,\infty)\to\mathbb{R}$. Then, the "conformable fractional derivative" of f of order α is defined by

$$D_\alpha(f)(t)=\lim_{\epsilon\to 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon}$$

for all $t>0,\ \alpha\in(0,1)$. If f is $\alpha-$differentiable in some $(0,a)$, $\alpha>0$, $\lim\limits_{t\to 0^+}f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\to 0^+}f^{(\alpha)}(t).$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable. For $2\le n\in\mathbb{N}$, we denote $D_\alpha^n(f)(t)=D_\alpha D_\alpha^{n-1}(f)(t)(t)$.

Theorem 1.2. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then,*

i. $D_\alpha(af+bg)=aD_\alpha(f)+bD_\alpha(g)$, for all $a,b\in\mathbb{R}$,

ii. $D_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$,

iii. $D_\alpha(fg)=fD_\alpha(g)+gD_\alpha(f)$,

iv. $D_\alpha\left(\frac{f}{g}\right)=\frac{fD_\alpha(g)-gD_\alpha(f)}{g^2}$.

If f is differentiable, then

$$D_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t).$$

Definition 1.3 (Conformable fractional integral)**.** Let $\alpha\in(0,1]$ and $0\le a<b$. A function $f:[a,b]\to\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral

$$\int_a^b f(x)\,d_\alpha x:=\int_a^b f(x)\,x^{\alpha-1}dx$$

exists and is finite.

Remark 1.4.

$$I_\alpha^a(f)(t)=I_1^a\left(t^{\alpha-1}f\right)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

In [11], Sarıkaya et al. introduced Pochhammer $(p)_{n,k}^\alpha$-symbol as follows

$$(p)_{n,k}^\alpha=(p+\alpha-1)(p+\alpha-1+\alpha k)\ldots(p+\alpha-1+(n-1)\alpha k).$$

Setting $\alpha=1$ and $k\to 1$ one obtains the usual Pochhammer symbol $(x)_n$.

The (α,k)-gamma functions is defined by [11]

$$\Gamma_k^\alpha(p)=\int_0^\infty t^{p-1}e^{-\frac{t^{\alpha k}}{\alpha k}}d_\alpha t=\lim_{n\to\infty}\frac{n!\alpha^n k^n(n\alpha k)^{\frac{p+\alpha-1}{\alpha k}-1}}{(p)_{n,k}^\alpha}.$$

The (α,k)-Gamma function $\Gamma_k^\alpha(p)$ satisfies the following identities

(1) $\Gamma_k^{\alpha}(p+\alpha k)=(p+\alpha-1)\,\Gamma_k^{\alpha}(p)$,
(2) $\Gamma_k^{\alpha}(p+n\alpha k)=(p)_{n,k}^{\alpha}\,\Gamma_k^{\alpha}(p)$,
(3) $\Gamma_k^{\alpha}(p)=(\alpha k)^{\frac{p+\alpha-1}{\alpha k}-1}\,\Gamma\left(\frac{p+\alpha-1}{\alpha k}\right)$,
(4) $\Gamma_k^{\alpha}(p)=\alpha^{\frac{p+\alpha-1}{\alpha k}-1}\Gamma_k\left(\frac{p+\alpha-1}{\alpha}\right)$,
(5) $\Gamma_k^{\alpha}(\alpha k+1-\alpha)=1$,
(6) $\Gamma_k^{\alpha}(p)=a^{\frac{p+\alpha-1}{\alpha k}}\int_0^{\infty}t^{p-1}e^{-a\frac{t^{\alpha k}}{\alpha k}}\,d_{\alpha}t.$

This gives rise to (α,k)-beta function defined by [11]

$$B_k^{\alpha}(p,q)=\frac{1}{\alpha k}\int_0^1 t^{\frac{p}{\alpha k}-1}(1-t)^{\frac{q}{\alpha k}-1}\,d_{\alpha}t,\ p,\ q,\ k>0,$$

also

$$B_k\left(p+\alpha k(1-\alpha),q\right)=\frac{\Gamma_k^{\alpha}(p)\,\Gamma_k^{\alpha}(q)}{\Gamma_k^{\alpha}(p+q+1-\alpha)}. \tag{1.1}$$

2. Main Results

In this section, we will give some generalized results for the C_n numbers known as Catalan numbers in the literature. We will give some new formulas for Catalan numbers, their integral representation, and a parametric integral notation such as the $(\alpha,k)-$gamma and $(\alpha,k)-$beta functions. We will show that the obtained results with their special selection give the current results in the literature.

Theorem 2.1. *Let $k>0,\ \alpha\in(0,1]$ and $n\in\mathbb{N}^{+}$,*

$$\Gamma_k^{\alpha}\left(\frac{(2n+1)\,\alpha k}{2}+1-\alpha\right)=(\alpha k)^{\frac{2n-1}{2}}\frac{(2n)!\,\sqrt{\pi}}{4^n n!},$$

and

$$C_{n,k}^{\alpha}=\frac{4^n(\alpha k)^{\frac{3}{2}}}{\sqrt{\pi}}\frac{\Gamma_k^{\alpha}\left(\frac{(2n+1)\alpha k}{2}+1-\alpha\right)}{\Gamma_k^{\alpha}\left((n+2)\,\alpha k+1-\alpha\right)}. \tag{2.1}$$

Proof. From $\Gamma_k^{\alpha}(p)=(\alpha k)^{\frac{p+\alpha-1}{\alpha k}-1}\,\Gamma\left(\frac{p+\alpha-1}{\alpha k}\right)$, we have

$$\begin{aligned}\Gamma_k^{\alpha}\left(\frac{(2n+1)\,\alpha k}{2}+1-\alpha\right) &= (\alpha k)^{\frac{(2n+1)\alpha k+1-\alpha+\alpha-1}{2\alpha k}-1}\Gamma\left(\frac{(2n+1)\,\alpha k+1-\alpha+\alpha-1}{2\alpha k}\right)\\ &= (\alpha k)^{\frac{2n-1}{2}}\Gamma\left(n+\frac{1}{2}\right)\\ &= (\alpha k)^{\frac{2n-1}{2}}\frac{(2n)!\,\sqrt{\pi}}{4^n n!}.\end{aligned}$$

And, so

$$\begin{aligned}\frac{\Gamma_k^{\alpha}\left(\frac{(2n+1)\alpha k}{2}+1-\alpha\right)}{\Gamma_k^{\alpha}\left((n+2)\,\alpha k+1-\alpha\right)} &= \frac{(\alpha k)^{\frac{2n-1}{2}}\frac{(2n)!\,\sqrt{\pi}}{4^n n!}}{(\alpha k)^{n+1}(n+1)!}\\ &= (\alpha k)^{-\frac{3}{2}}\frac{(2n)!\,\sqrt{\pi}}{4^n n!\,(n+1)!}\\ &= (\alpha k)^{-\frac{3}{2}}\frac{\sqrt{\pi}C_{n,k}^{\alpha}}{4^n},\end{aligned}$$

in this way

$$C_{n,k}^{\alpha}=\frac{4^n(\alpha k)^{\frac{3}{2}}}{\sqrt{\pi}}\frac{\Gamma_k^{\alpha}\left(\frac{(2n+1)\alpha k}{2}+1-\alpha\right)}{\Gamma_k^{\alpha}\left((n+2)\,\alpha k+1-\alpha\right)}.$$

The proof is completed. □

Remark 2.2. If we choose $\alpha = 1$ in (2.1), we have the following equality [10],

$$C^1_{n,k} = \frac{4^n k^{\frac{3}{2}}}{\sqrt{\pi}} \frac{\Gamma_k\left(\frac{(2n+1)k}{2}\right)}{\Gamma_k\left((n+2)k\right)}.$$

Remark 2.3. If we choose $\alpha = k = 1$ in (2.1), we have the following equality [7],

$$C^1_{n,1} = \frac{4^n}{\sqrt{\pi}} \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(n+2\right)}.$$

Theorem 2.4. *Let $a,\ k > 0,\ \alpha \in (0,1]$ and $n \geq 0$, the following equality holds;*

$$\begin{aligned}
I_{n,k}(a) &= \int_0^a x^{(n+1)\alpha k-\alpha}\left(a^{2\alpha k} - x^{2\alpha k}\right)^{\frac{1}{2}} d_\alpha x \\
&= \frac{a^{(n+2)\alpha k}}{2} B^\alpha_k\left(\frac{(n+1)\alpha k}{2} + (1-\alpha)\alpha k, \frac{3\alpha k}{2}\right) \\
&= \frac{a^{(n+2)\alpha k}}{2} \frac{\Gamma^\alpha_k\left(\frac{(n+1)\alpha k}{2}\right)\Gamma^\alpha_k\left(\frac{3\alpha k}{2}\right)}{\Gamma^\alpha_k\left(\frac{(n+4)\alpha k}{2} + 1 - \alpha\right)}.
\end{aligned}$$

Proof. We use the substitution $x = at^{\frac{1}{2\alpha k}}$, so this transforms the integral:

$$\begin{aligned}
I^\alpha_{n,k}(a) &= \int_0^1 \left(at^{\frac{1}{2\alpha k}}\right)^{(n+1)\alpha k-1}\left(a^{2\alpha k} - \left(at^{\frac{1}{2\alpha k}}\right)^{2\alpha k}\right)^{\frac{1}{2}} \frac{a}{2\alpha k} t^{\frac{1}{2\alpha k}-1} dt \\
&= \frac{a^{(n+2)\alpha k}}{2\alpha k} \int_0^1 t^{\frac{(n+1)\alpha k}{2\alpha k}-1}(1-t)^{\frac{1}{2}+1-1} dt \\
&= \frac{a^{(n+2)\alpha k}}{2\alpha k} \int_0^1 t^{\frac{\frac{(n+1)\alpha k}{2}+(1-\alpha)\alpha k}{\alpha k}-1}(1-t)^{\frac{3k}{2k}-1} d_\alpha t \\
&= \frac{a^{(n+2)\alpha k}}{2} B^\alpha_k\left(\frac{(n+1)\alpha k}{2} + (1-\alpha)\alpha k, \frac{3\alpha k}{2}\right).
\end{aligned}$$

So, from (1.1) we obtain

$$I^\alpha_{n,k}(a) = \frac{a^{(n+2)\alpha k}}{2} \frac{\Gamma^\alpha_k\left(\frac{(n+1)\alpha k}{2}\right)\Gamma^\alpha_k\left(\frac{3\alpha k}{2}\right)}{\Gamma^\alpha_k\left(\frac{(n+4)\alpha k}{2} + 1 - \alpha\right)}.$$

This is the proof of Theorem 2.4. □

Remark 2.5. If we choose $\alpha = 1$ in Theorem 2.4, we get Theorem 1.2 in [10]. This theorem;

$$\begin{aligned}
I^1_{n,k}(a) &= \int_0^a x^{(n+1)k-1}\left(a^{2k} - x^{2k}\right)^{\frac{1}{2}} dx \\
&= \frac{a^{(n+2)k}}{2} \frac{\Gamma_k\left(\frac{(n+1)k}{2}\right)\Gamma_k\left(\frac{3k}{2}\right)}{\Gamma_k\left(\frac{(n+4)k}{2}\right)}.
\end{aligned}$$

Remark 2.6. If we choose $k = 1$ in Theorem 2.4, we get Theorem 2.1 in [9]. This theorem;

$$\begin{aligned} I_n(a) &= \int_0^a x^n \left(a^2 - x^2\right)^{\frac{1}{2}} dx \\ &= a^{n+2} \frac{\sqrt{\pi}\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{4\Gamma\left(\frac{n}{2}+2\right)}. \end{aligned}$$

Theorem 2.7. *Let $a, k > 0$, $\alpha \in (0, 1]$ and r, $s > -1$, then*

$$\begin{aligned} I^{\alpha}_{r,s,k}(a) &= \int_0^a x^{(r+1)\alpha k-\alpha} \left(a^{2\alpha k} - x^{2\alpha k}\right)^s d_\alpha x \\ &= \frac{a^{(r+2s+1)\alpha k}}{2} B^{\alpha}_k \left(\frac{r+1}{2}\alpha k + (1-\alpha)\alpha k, (s+1)\alpha k\right). \end{aligned}$$

Proof. To prove this theorem, by changing variable $x = a \sin^{\frac{1}{\alpha k}} \theta$ for $\theta \in \left[0, \frac{\pi}{2}\right]$;

$$\begin{aligned} I^{\alpha}_{r,s,k}(a) &= \int_0^a x^{(r+1)\alpha k-\alpha} \left(a^{2\alpha k} - x^{2\alpha k}\right)^s d_\alpha x \\ &= \int_0^{\frac{\pi}{2}} \left(a (\sin\theta)^{\frac{1}{\alpha k}}\right)^{(r+1)\alpha k-1} \left(a^{2\alpha k} - a^{2\alpha k}\sin^2\theta\right)^s \frac{a}{\alpha k} (\sin\theta)^{\frac{1}{\alpha k}-1} \cos\theta d\theta \\ &= a^{(r+2s+1)\alpha k} \alpha k \int_0^{\frac{\pi}{2}} (\sin\theta)^r (\cos\theta)^{2s+1} d\theta \qquad (2.2) \\ &= \frac{a^{(r+2s+1)\alpha k}}{\alpha k} \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{2\alpha k(r+1)}{2\alpha k}-1} (\cos\theta)^{\frac{2\alpha k(s+1)}{\alpha k}-1} d\theta \\ &= \frac{a^{(r+2s+1)\alpha k}}{2} B^{\alpha}_k \left(\frac{r+1}{2}\alpha k + (1-\alpha)\alpha k, (s+1)\alpha k\right). \end{aligned}$$

To get (2.2), we used

$$\begin{aligned} B^{\alpha}_k(p,q) &= \frac{1}{\alpha k} \int_0^1 x^{\frac{p}{\alpha k}-1} (1-x)^{\frac{q}{\alpha k}-1} x^{\alpha-1} dx \\ &= \frac{1}{\alpha k} \int_0^{\frac{\pi}{2}} \left(\sin^2\theta\right)^{\frac{p}{\alpha k}-1} \left(1-\sin^2\theta\right)^{\frac{q}{\alpha k}-1} (\sin^2\theta)^{\alpha-1} (2\sin\theta\cos\theta) d\theta \\ &= \frac{2}{\alpha k} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\frac{p+(\alpha-1)\alpha k}{\alpha k}-1} (\cos\theta)^{2\frac{q}{\alpha k}-1} d\theta. \end{aligned}$$

That is,

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2\frac{p+(\alpha-1)\alpha k}{\alpha k}-1} (\cos\theta)^{2\frac{q}{\alpha k}-1} d\theta = \frac{\alpha k}{2} B^{\alpha}_k(p,q).$$

Thus, the proof of Theorem 2.7 is completed. □

Remark 2.8. If we choose $\alpha = 1$ in Theorem 2.7, we get Theorem 1.2 in [10]. This theorem;

$$\begin{aligned} I^{1}_{r,s,k}(a) &= \int_0^a x^{(r+1)k-1}\left(a^{2k}-x^{2k}\right)^s dx \\ &= \frac{a^{(r+2s+1)k}}{2} B_k\left(\tfrac{r+1}{2}k, (s+1)k\right). \end{aligned}$$

Remark 2.9. If we choose $\alpha = 1$ and $k = 1$ in Theorem 2.7, we get Theorem 5.1 in [9]. This theorem;

$$\begin{aligned} I^{1}_{r,s,1}(a) &= \int_0^a x^{r}\left(a^{2}-x^{2}\right)^s dx \\ &= \frac{a^{r+2s+1}}{2} B\left(\tfrac{r+1}{2}, s+1\right). \end{aligned}$$

Remark 2.10. If we choose $k = 1,\ r = n$ and $s = \frac{1}{2}$ in Theorem 2.7, we get Remark 6.1 in [9]. This Remark;

$$I^{1}_{n,\frac{1}{2},1}(a) = \frac{a^{n+2}}{2} B\left(\tfrac{n+1}{2}, \tfrac{3}{2}\right).$$

Remark 2.11. If we choose $\alpha = 1,\ r = n$ and $s = \frac{1}{2}$ in Theorem 2.7, we obtain the results [10];

$$\begin{aligned} I^{1}_{n,\frac{1}{2},k}(a) &= \int_0^a x^{(n+1)k-1}\left(a^{2k}-x^{2k}\right)^{\frac{1}{2}} dx \\ &= \frac{a^{(n+2)k}}{2} B_k\left(\tfrac{(n+1)k}{2}, \tfrac{3}{2}k\right) \\ &= \frac{a^{(n+2)k}}{2} \frac{\Gamma_k\left(\frac{(n+1)k}{2}\right)\Gamma_k\left(\frac{3}{2}k\right)}{\Gamma_k\left(\frac{(n+4)k}{2}\right)}. \end{aligned}$$

Remark 2.12. If we choose $\alpha = k = 1,\ r = 2n$ and $s = \frac{1}{2}$ in Theorem 2.7, we get Remark 6.2 in [9]. This Remark;

$$C_n = \frac{1}{\pi} I^{1}_{2n,\frac{1}{2},1}(2) = \frac{2^{2n+1}}{\pi} B\left(\tfrac{2n+1}{2}, \tfrac{3}{2}\right).$$

Remark 2.13. If we choose $\alpha = 1,\ r = n$ and $s = -\frac{1}{2}$ in Theorem 2.7, we get Theorem 1.2 in [10]. This Theorem;

$$\int_0^a \frac{x^{(n+1)k}}{\sqrt{a^{2k}-x^{2k}}} dx = \frac{a^{nk}}{2} B_k\left(\frac{(n+1)k}{2}, \frac{k}{2}\right).$$

Remark 2.14. If we choose $\alpha = k = 1,\ r = n$ and $s = -\frac{1}{2}$ in Theorem 2.7, we get Remark 6.3 in [9]. This Remark;

$$\int_0^a \frac{x^{n}}{\sqrt{a^{2}-x^{2}}} dx = \frac{a^{n}}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right).$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

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