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Research Article

Algebraic theory of degenerate general bivariate Appell polynomials and related interpolation hints

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Abstract

The algebraic study of polynomials based on determinant representations is important in many fields of mathematics, ranging from algebraic geometry to optimization. The motivation to introduce determinant expressions of special polynomials comes from the fact that they are useful in scientific computing in solving systems of equations effectively. It is critical for this application to have determinant representations not just for single valued polynomials but also for bivariate polynomials. In this article, a family of degenerate general bivariate Appell polynomials is introduced. Several different explicit representations, recurrence relations, and addition theorems are established for this family. With the aid of different recurrence relations, we establish the determinant expressions for the degenerate general bivariate Appell polynomials. We also establish determinant definitions for degenerate general polynomials. Several examples are framed as the applications of this family and their graphical representations are shown. As concluding remarks, we propose a linear interpolation problem for these polynomials and some hints are provided.

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1. Introduction and preliminaries

Many mathematicians, physicists, and combinatorial specialists have recently rekindled their interest in the study of degenerate forms of special polynomials, numbers and functions. The importance of degenerate versions goes beyond their combinatorial and mathematical features, to their applications in differential equations, probability theory, and symmetric identities. The idea extends beyond special polynomials and numbers to transcendental functions. As a consequence of Carlitz's study of degenerate versions of special polynomials and numbers several degenerate analogues of special polynomials and numbers came into existence [2, 4–8, 14, 15, 18]. From here, unless stated, λ is arbitrary

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but a fixed non zero real number. We recall the following definition of degenerate Appell polynomials [15]:

Definition 1.1. The degenerate Appell polynomial sequences $\{A_{n,\lambda}(\zeta)\}_{n\in\mathbb{N}}$ are defined by

$$A_\lambda(\tau)e_\lambda^\zeta(\tau) = \sum_{n=0}^{\infty} A_{n,\lambda}(\zeta)\frac{\tau^n}{n!}, \tag{1.1}$$

where $A_\lambda(\tau)$ is of the form:

$$A_\lambda(\tau) = \sum_{r=0}^{\infty} \rho_{r,\lambda}\frac{\tau^r}{r!}. \tag{1.2}$$

Remark 1.2. Note that

$$\lim_{\lambda\to 0} A_{n,\lambda}(\zeta) = A_n(\zeta),$$

where $A_n(\zeta)$ are the classical Appell polynomials [1].

We recall two particular members belonging to the degenerate Appell family:

Case I. Degenerate Bernoulli polynomial sequences $\{B_{n,\lambda}(\zeta)\}_{n\in\mathbb{N}}$ [5].
Taking $A_\lambda(\tau) = \frac{\tau}{e_\lambda(\tau)-1}$ in equation (1.1), we obtain the degenerate Bernoulli sequences $\{B_{n,\lambda}(\zeta)\}_{n\in\mathbb{N}}$, which possess the following generating function and series definition:

$$\frac{\tau}{e_\lambda(\tau)-1}e_\lambda^\zeta(\tau) = \sum_{n=0}^{\infty} B_{n,\lambda}(\zeta)\frac{\tau^n}{n!}, \qquad |\tau| < 2\pi$$

and

$$B_{n,\lambda}(\zeta) = \sum_{r=0}^{n}\binom{n}{r} B_{n-r,\lambda}\,(\zeta)_{r,\lambda},$$

respectively, where $B_{n,\lambda} := B_{n,\lambda}(0)$ are the degenerate Bernoulli numbers.

Case II. Degenerate Euler polynomial sequences $\{E_{n,\lambda}(\zeta)\}_{n\in\mathbb{N}}$ [5].
For $A_\lambda(\tau) = \frac{2}{e_\lambda(\tau)+1}$ in (1.1), we obtain the degenerate Euler sequences $\{E_{n,\lambda}(\zeta)\}_{n\in\mathbb{N}}$, which possess the following generating function and series definition:

$$\frac{2}{e_\lambda(\tau)+1}e_\lambda^\zeta(\tau) = \sum_{n=0}^{\infty} E_{n,\lambda}(\zeta)\frac{\tau^n}{n!}, \qquad |\tau| < \pi$$

and

$$E_{n,\lambda}(\zeta) = \sum_{r=0}^{n}\binom{n}{r} E_{n-r,\lambda}\,(\zeta)_{r,\lambda},$$

respectively, where $E_{n,\lambda} := E_{n,\lambda}(0)$ are the degenerate Euler numbers.

Multivariable special polynomials, or more specifically, special polynomials in two variables, hold a necessary position in terms of their importance in several sectors of research and engineering. Using multivariable special polynomials, various identities can be deduced in a simple, efficient, and transparent manner. Exploring multivariable special polynomials also resulted in the establishment of several new families of special polynomials in the literature [3, 13]. We recall the following degenerate general class of bivariate polynomials [17]:

Definition 1.3. The degenerate bivariate general polynomials $p_{n,\lambda}(\zeta,\eta)$ are defined by the generating function of the following form:

$$e_\lambda^\zeta(\tau)\phi_\lambda(\eta,\tau) = \sum_{n=0}^{\infty} p_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}, \tag{1.3}$$

where $\phi_\lambda(\eta,\tau)$ is given as

$$\phi_\lambda(\eta,\tau)=\sum_{r=0}^{\infty}\phi_{r,\lambda}(\eta)\frac{\tau^r}{r!}. \tag{1.4}$$

Remark 1.4. Note that

$$\lim_{\lambda\to 0} p_{n,\lambda}(\zeta,\eta)=p_n(\zeta,\eta),$$

where $p_n(\zeta,\eta)$ are the bivariate general polynomials [3].

By using the polynomials $p_n(\zeta,\eta)$, a bivariate family of general Appell polynomials is introduced and investigated in details, see [13]. The bivariate general family is important as it includes a variety of different polynomial sequences. Also, introducing degenerate version of any polynomial sequence may result in deriving many different forms of identities. Motivated by these, the authors aim to introduce a degenerate version of the general bivariate Appell polynomials and establish their several properties.

The article is structured as follows: In Section 2, a degenerate family of general bivariate Appell polynomials is introduced via generating function. Section 3 is followed by several series representations, recurrence formulas, differential identities, and other identities of these polynomials. The corresponding conjugate forms are also investigated which results in establishing determinant expressions for degenerate general bivariate Appell polynomials and degenerate general polynomials. In Section 4, an example of polynomial from this family, namely degenerate bivariate Laguerre-Appell is constructed and its special cases are also discussed. Section 5 deals with the shapes, surface plots, and zeros for degenerate bivariate Laguerre-Bernoulli polynomials via graphical approach. Finally, the degenerate general bivariate Appell interpolation problem is discussed in concluding remarks.

2. Degenerate general bivariate Appell polynomials

In this section, we introduce a degenerate version of the general bivariate Appell polynomials (dgbAp). A diverse range of important identities, recurrence relations and determinant expressions are derived for these polynomials. We give the following definitions:

Definition 2.1. The degenerate general bivariate Appell polynomial sequences $\{{}_pA_{n,\lambda}(\zeta,\eta)\}_{n\in\mathbb{N}}$ are defined by the following generating equation:

$$A_\lambda(\tau)e_\lambda^\zeta(\tau)\phi_\lambda(\eta,\tau)=\sum_{n=0}^{\infty}{}_pA_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}, \tag{2.1}$$

where $A_\lambda(\tau)$ and $\phi_\lambda(\eta,\tau)$ represent the series given by equations (1.2) and (1.4).

Next, we establish the connection between dgbAp ${}_pA_{n,\lambda}(\zeta,\eta)$ and degenerate Appell polynomials $A_{n,\lambda}(\zeta)$.

Definition 2.2. The degenerate general bivariate Appell polynomials ${}_pA_{n,\lambda}(\zeta,\eta)$ can be explicitly represented in terms of degenerate Appell polynomials $A_{n,\lambda}(\zeta)$ as

$${}_pA_{n,\lambda}(\zeta,\eta)=\sum_{r=0}^{n}\binom{n}{r}A_{n-r,\lambda}(\zeta)\ \phi_{r,\lambda}(\eta), \qquad n\geq 1.$$

Remark 2.3. For $A_{n,\lambda}(\zeta)=(\zeta)_{n,\lambda}$, we obtain the following explicit form for the degenerate bivariate general polynomials $p_{n,\lambda}(\zeta,\eta)$:

$$p_{n,\lambda}(\zeta,\eta)=\sum_{r=0}^{n}\binom{n}{r}(\zeta)_{n-r,\lambda}\ \phi_{r,\lambda}(\eta).$$

Definition 2.4. The degenerate general bivariate Appell polynomials ${}_pA_{n,\lambda}(\zeta,\eta)$ can be represented in terms of degenerate general polynomials $p_{n,\lambda}(\zeta,\eta)$ as:

$$ {}_pA_{n,\lambda}(\zeta,\eta) = \sum_{r=0}^{n} \binom{n}{r} \rho_{n-r,\lambda}\; p_{r,\lambda}(\zeta,\eta). \tag{2.2}$$

Theorem 2.5. *The degenerate general bivariate Appell sequences ${}_pA_{n,\lambda}(\zeta,\eta)$ satisfy the following addition formula:*

$$ {}_pA_{n,\lambda}(\zeta_1+\zeta_2,\eta) = \sum_{r=0}^{n} \binom{n}{r} {}_pA_{n,\lambda}(\zeta_1,\eta)(\zeta_2)_{n-r,\lambda}. \tag{2.3}$$

Proof. Replacing ζ by $\zeta_1+\zeta_2$ in (2.1), using definition of degenerate exponential function and relation (2.1) in the resultant equation, simplifying and equating similar powers of τ, assertion (2.3) follows. □

The formal power series $A_\lambda(\tau)$ is an invertible series, given by

$$ A^*_\lambda(\tau) \equiv A^{-1}_\lambda(\tau) = \sum_{r=0}^{\infty} \gamma_{r,\lambda} \frac{\tau^r}{r!}, \tag{2.4}$$

such that the following holds:

$$ \sum_{r=0}^{n} \binom{n}{r} \rho_{n-r,\lambda}\gamma_{r,\lambda} = \delta_{n,0}, \tag{2.5}$$

where $\delta_{n,m}$ denotes the kronecker delta. Next, the conjugate forms of degenerate sequences, namely, the conjugate degenerate bivariate general polynomial sequences $p^*_{n,\lambda}(\zeta,\eta)$ and conjugate dgbAp ${}_pA^*_{n,\lambda}(\zeta,\eta)$ can be defined by making use of conjugate function $A^*_\lambda(\tau)$.

Definition 2.6. The conjugate degenerate general bivariate Appell polynomials ${}_pA^*_{n,\lambda}(\zeta,\eta)$ are defined as:

$$ A^*_\lambda(\tau) e^\zeta_\lambda(\tau)\phi_\lambda(\eta,\tau) = \sum_{n=0}^{\infty} {}_pA^*_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}. \tag{2.6}$$

Theorem 2.7. *The conjugate degenerate general bivariate Appell polynomials ${}_pA^*_{n,\lambda}(\zeta,\eta)$ can be explicitly represented as:*

$$ {}_pA^*_{n,\lambda}(\zeta,\eta) = \sum_{r=0}^{n} \binom{n}{r} \gamma_{n-r,\lambda}\; p_{r,\lambda}(\zeta,\eta). \tag{2.7}$$

Proof. Using relations (1.3) and (2.4) in equation (2.6), assertion (2.7) follows. □

Definition 2.8. The conjugate degenerate bivariate general polynomial sequences $p^*_{n,\lambda}(\zeta,\eta)$ are defined as:

$$ e^{-1}_\lambda(\tau) e^\zeta_\lambda(\tau)\phi_\lambda(\eta,\tau) = \sum_{n=0}^{\infty} p^*_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}. \tag{2.8}$$

Theorem 2.9. *The conjugate degenerate bivariate general polynomial sequences $p^*_{n,\lambda}(\zeta,\eta)$ can be explicitly represented as:*

$$ p^*_{n,\lambda}(\zeta,\eta) = \sum_{r=0}^{n} \binom{n}{r} (\zeta-1)_{r,\lambda}\phi_{n-r,\lambda}(\eta). \tag{2.9}$$

Proof. In view of equation (1.4) and series expansion of degenerate exponential function, equation (2.8) gives

$$ \sum_{n=0}^{\infty} p^*_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \phi_{n,\lambda}(\eta)(\zeta-1)_{r,\lambda}\frac{\tau^{n+r}}{n!\; r!}, $$

from which, applying Cauchy product rule we get assertion (2.9). □

Proposition 2.10. *The following identity holds for the degenerate bivariate general polynomial sequences* $\{p_{n,\lambda}(\zeta,\eta)\}_{n\in\mathbb{N}}$*:*

$$p_{n,\lambda}(\zeta,\eta)=\sum_{r=0}^{n}\binom{n}{r}\gamma_{n-r,\lambda}\ {}_{p}A_{r,\lambda}(\zeta,\eta). \tag{2.10}$$

Proof. Since the function $A_\lambda(\tau)$ is invertible, therefore, with the use of relation (1.3), equation (2.1) can be rewritten as:

$$\sum_{n=0}^{\infty}p_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}=A^*_\lambda(\tau)\sum_{n=0}^{\infty}{}_{p}A_{n,\lambda}\frac{\tau^n}{n!}.$$

Finally, using relation (2.4) in the above equation and applying Cauchy product rule, we get assertion (2.10). □

In the next section, we establish several recurrence relations for the dgbAp and degenerate bivariate general polynomials. Further, by using these recurrences, an algebraic approach for these polynomials is presented.

3. Recurrence relations and determinant expressions

Recently, Costabile et al. [9] established the determinant forms for the bivariate general Appell polynomials. In this section, we derive a variety of different recurrence formulas for the degenerate bivariate polynomials and their conjugate forms. Further, by using these recurrences, we establish different determinant expressions for the polynomials ${}_{p}A_{n,\lambda}(\zeta,\eta)$ and $p_{n,\lambda}(\zeta,\eta)$.

Corollary 3.1 (First recurrence formula)**.** *For the degenerate general bivariate Appell polynomials* ${}_{p}A_{n,\lambda}(\zeta,\eta)$*, the following recurrence formula holds:*

$${}_{p}A_{0,\lambda}(\zeta,\eta)=1,\qquad {}_{p}A_{n,\lambda}(\zeta,\eta)=p_{n,\lambda}(\zeta,\eta)-\sum_{r=0}^{n-1}\binom{n}{r}\gamma_{n-r,\lambda}\ {}_{p}A_{r,\lambda}(\zeta,\eta),\quad n\geq 1. \tag{3.1}$$

Lemma 3.2. *For the conjugate degenerate bivariate general polynomial sequences* $p^*_{n,\lambda}(\zeta,\eta)$ *and degenerate bivariate general polynomial sequences* $p_{n,\lambda}(\zeta,\eta)$*, the following inversion formulae hold:*

$$p^*_{n,\lambda}(\zeta,\eta)=\sum_{r=0}^{n}\binom{n}{r}(-1)_{n-r,\lambda}\ p_{r,\lambda}(\zeta,\eta) \tag{3.2}$$

and

$$p_{n,\lambda}(\zeta,\eta)=\sum_{r=0}^{n}\binom{n}{r}(1)_{n-r,\lambda}\ p^*_{r,\lambda}(\zeta,\eta), \tag{3.3}$$

respectively.

Corollary 3.3. *For the degenerate bivariate general polynomials* $p_{n,\lambda}(\zeta,\eta)$*, the following recurrence formula holds:*

$$p_{n,\lambda}(\zeta,\eta)=p^*_{n,\lambda}(\zeta,\eta)-\sum_{r=0}^{n-1}\binom{n}{r}(-1)_{n-r,\lambda}\ p_{r,\lambda}(\zeta,\eta). \tag{3.4}$$

Corollary 3.4. *For the degenerate bivariate general polynomials* $p_{n,\lambda}(\zeta,\eta)$*, the following identity holds:*

$$\sum_{r=0}^{n}\binom{n}{r}(-1)_{r,\lambda}\ p_{n-r,\lambda}(\zeta,\eta)=\sum_{r=0}^{n}\binom{n}{r}(\zeta-1)_{r,\lambda}\phi_{n-r,\lambda}(\eta). \tag{3.5}$$

Proof. The proof follows from (2.9) and (3.2). □

Lemma 3.5. *The following inversion formulae for the degenerate general bivariate Appell polynomial sequences* ${}_pA_{n,\lambda}(\zeta,\eta)$ *hold:*

$$ {}_pA_{n,\lambda}(\zeta,\eta)=\sum_{j=0}^{n}\Omega_{n,j;\lambda}\ p^*_{j,\lambda}(\zeta,\eta) \tag{3.6}$$

and

$$ p^*_{n,\lambda}(\zeta,\eta)=\sum_{j=0}^{n}\Omega^*_{n,j;\lambda}\ {}_pA_{j,\lambda}(\zeta,\eta), \tag{3.7}$$

where the coefficients $\Omega_{n,j;\lambda}$ and $\Omega^*_{n,j;\lambda}$ *are defined as follows:*

$$\Omega_{n,j;\lambda}=\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}(1)_{r-j,\lambda}\ \rho_{n-r,\lambda} \tag{3.8}$$

and

$$\Omega^*_{n,j;\lambda}=\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}(-1)_{n-r,\lambda}\ \gamma_{r-j,\lambda}, \tag{3.9}$$

respectively.

Proof. In view of relations (2.2) and (3.3), we have

$$ {}_pA_{n,\lambda}(\zeta,\eta)=\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}(1)_{r-j,\lambda}\ \rho_{n-r,\lambda}\ p^*_{j,\lambda}(\zeta,\eta).$$

Using relation (3.8) in the above equation, assertion (3.6) follows. To prove relation (3.7), we use relation (2.10) in the r.h.s. of equation (3.2) so that we have

$$\begin{aligned} p^*_{n,\lambda}(\zeta,\eta)&=\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}(-1)_{n-r,\lambda}\ \gamma_{r-j,\lambda}\ {}_pA_{j,\lambda}(\zeta,\eta)\\ &=\sum_{j=0}^{n}\sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}(-1)_{n-r,\lambda}\ \gamma_{r-j,\lambda}\ {}_pA_{j,\lambda}(\zeta,\eta).\end{aligned}$$

In view of relation (3.9), the above equation yields assertion (3.7). □

Corollary 3.6 (Second recurrence formula)**.** *For the degenerate general bivariate Appell sequences* ${}_pA_{n,\lambda}(\zeta,\eta)$*, the following recurrence formula holds:*

$$ {}_pA_{0,\lambda}(\zeta,\eta)=1,\qquad {}_pA_{n,\lambda}(\zeta,\eta)=p^*_{n,\lambda}(\zeta,\eta)-\sum_{r=0}^{n-1}\Omega^*_{n,r;\lambda}\ {}_pA_{r,\lambda}(\zeta,\eta), \tag{3.10}$$

where the coefficients $\Omega^*_{n,r;\lambda}$ *are defined by equation* (3.7)*.*

Proof. The proof easily follows from relation (3.9). □

Theorem 3.7 (Third recurrence formula)**.** *For the degenerate general bivariate Appell polynomials* ${}_pA_{n,\lambda}(\zeta,\eta)$*, the following recurrence relation holds:*

$$ {}_pA_{n+1,\lambda}(\zeta,\eta)=\{\zeta+\chi_{0,\lambda}+\vartheta_{0,\lambda}(\eta)\}\,{}_pA_{n,\lambda}(\zeta,\eta)+\sum_{r=0}^{n-1}\binom{n}{r}\{\chi_{n-r,\lambda}+\vartheta_{n-r,\lambda}(\eta)\}\,{}_pA_{r,\lambda}(\zeta,\eta), \tag{3.11}$$

where $\chi_{r,\lambda}$ *and* $\vartheta_{r,\lambda}(\eta)$ *are given as:*

$$\frac{A'_\lambda(\tau)}{A_\lambda(\tau)}=\sum_{r=0}^{\infty}\chi_{r,\lambda}\frac{\tau^r}{r!},\qquad \frac{\partial_\tau\phi_\lambda(\eta,\tau)}{\phi_\lambda(\eta,\tau)}=\sum_{r=0}^{\infty}\vartheta_{r,\lambda}(\eta)\frac{\tau^r}{r!}. \tag{3.12}$$

Proof. On differentiating relation (2.1) w.r.t. τ, it follows that

$$\left(\zeta+\frac{A'_\lambda(\tau)}{A_\lambda(\tau)}+\frac{\partial_\tau\phi_\lambda(\eta,\tau)}{\phi_\lambda(\eta,\tau)}\right)A_\lambda(\tau)e_\lambda^\zeta(\tau)\phi_\lambda(\eta,\tau)=\sum_{n=0}^{\infty}{}_pA_{n+1,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}.$$

In view of relations (2.1) and (3.12), the above equation leads to

$$\sum_{n=0}^{\infty}{}_pA_{n+1,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}=\sum_{n=0}^{\infty}\left(\zeta+\sum_{r=0}^{\infty}\chi_{r,\lambda}\frac{\tau^r}{r!}+\sum_{r=0}^{\infty}\vartheta_{r,\lambda}(\eta)\frac{\tau^r}{r!}\right){}_pA_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}.$$

Combining the series in the r.h.s. of above equation, using Cauchy product rule and simplifying the resultant equation, we get assertion (3.11). □

Remark 3.8 (Condition for orthogonality)**.** If we take

$$\sum_{r=0}^{n-2}\binom{n}{r}\left\{\chi_{n-r,\lambda}+\vartheta_{n-r,\lambda}(\eta)\right\}{}_pA_{r,\lambda}(\zeta,\eta)=0,$$

then relation (3.11) becomes a three term recurrence relation given as:

$${}_pA_{n+1,\lambda}(\zeta,\eta)=\left\{\zeta+\chi_{0,\lambda}+\vartheta_{0,\lambda}(\eta)\right\}{}_pA_{n,\lambda}(\zeta,\eta)+n\left\{\chi_{1,\lambda}+\vartheta_{1,\lambda}(\eta)\right\}{}_pA_{n-1,\lambda}(\zeta,\eta),$$

which defines the orthogonality of the polynomials ${}_pA_{n,\lambda}(\zeta,\eta)$.

Theorem 3.9. *For $n,r>0$ and $r<n$, the degenerate general bivariate Appell sequences ${}_pA_{n,\lambda}(\zeta,\eta)$ satisfy the following differential relations:*

$$\frac{\gamma_{n,\lambda}}{n!}\frac{\partial^n}{\partial\zeta^n}{}_pA_{n,\lambda}(\zeta,\eta)+\frac{\gamma_{n-1,\lambda}}{(n-1)!}\frac{\partial^{n-1}}{\partial\zeta^{n-1}}{}_pA_{n,\lambda}(\zeta,\eta)+\cdots+{}_pA_{n,\lambda}(\zeta,\eta)$$
$$=\sum_{r=0}^{n}\binom{n}{r}(\zeta)_{r,\lambda}\phi_{n-r,\lambda}(\eta),\tag{3.13}$$

$$\Omega^*_{n,j;\lambda}\frac{\partial^n}{\partial\zeta^n}{}_pA_{n,\lambda}(\zeta,\eta)+\frac{n\Omega^*_{n-1,j;\lambda}}{(n-1)!}\frac{\partial^n}{\partial\zeta^n}{}_pA_{n,\lambda}(\zeta,\eta)+\cdots+{}_pA_{n,\lambda}(\zeta,\eta)$$
$$=\sum_{r=0}^{n}\binom{n}{r}(\zeta-1)_{r,\lambda}\phi_{n-r,\lambda}(\eta).\tag{3.14}$$

Proof. Differentiating generating function (2.1) r times w.r.t. ζ yields the following relation:

$${}_pA_{n-r,\lambda}(\zeta,\eta)=\frac{1}{n(n-1)...(n-r+1)}\frac{\partial^r}{\partial\zeta^r}{}_pA_{n,\lambda}(\zeta,\eta).\tag{3.15}$$

Now, by making use of relation (3.15) in equations (3.1) and (3.10), we obtain assertions (3.13) and (3.14) respectively. □

Theorem 3.10. *The degenerate general polynomials $p_{n,\lambda}(\zeta,\eta)$ satisfy the following differential relation:*

$$\frac{(-1)_{n,\lambda}}{n!}\frac{\partial^n}{\partial\zeta^n}p_{n,\lambda}(\zeta,\eta)+\frac{(-1)_{n-1,\lambda}}{(n-1)!}\frac{\partial^{n-1}}{\partial\zeta^{n-1}}p_{n,\lambda}(\zeta,\eta)+\cdots+p_{n,\lambda}(\zeta,\eta)$$
$$=\sum_{r=0}^{n}\binom{n}{r}(\zeta-1)_{r,\lambda}\phi_{n-r,\lambda}(\eta).\tag{3.16}$$

Proof. In view of equations (3.5) and (3.15), assertion (3.16) follows. □

Next, by using three different recurrence relations derived above, we establish three different determinant expressions for the dgbAp ${}_pA_{n,\lambda}(\zeta,\eta)$. In addition, we also establish determinant forms for the degenerate bivariate general polynomials $p_{n,\lambda}(\zeta,\eta)$. For this, we have the following result.

Theorem 3.11. *The elements of the sequence* $\{{}_pA_{n,\lambda}(\zeta,\eta)\}_{n\in\mathbb{N}}$ *can be expressed in the form of the following three determinant expressions:*

(i) ${}_pA_{0,\lambda}(\zeta,\eta)=1,$

$$
{}_pA_{n,\lambda}(\zeta,\eta)=(-1)^n\begin{vmatrix} p_{0,\lambda}(\zeta,\eta) & p_{1,\lambda}(\zeta,\eta) & p_{2,\lambda}(\zeta,\eta) & \cdots & p_{n-1,\lambda}(\zeta,\eta) & p_{n,\lambda}(\zeta,\eta) \\ \gamma_{0,\lambda} & \gamma_{1,\lambda} & \gamma_{2,\lambda} & \cdots & \gamma_{n-1,\lambda} & \gamma_{n,\lambda} \\ 0 & \gamma_{0,\lambda} & \binom{2}{1}\gamma_{1,\lambda} & \cdots & \binom{n-1}{1}\gamma_{n-2,\lambda} & \binom{n}{1}\gamma_{n-1,\lambda} \\ 0 & 0 & \gamma_{0,\lambda} & \cdots & \binom{n-1}{2}\gamma_{n-3,\lambda} & \binom{n}{2}\gamma_{n-2,\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \gamma_{0,\lambda} & \binom{n}{n-1}\gamma_{1,\lambda} \end{vmatrix}, \tag{3.17}
$$

$n=1,2,...,$ *where* $p_{n,\lambda}(\zeta,\eta)$ *are the degenerate bivariate general sequences and the coefficients* $\gamma_{n,\lambda}$ *are given by equation* (2.4)*.*

(ii) ${}_pA_{0,\lambda}(\zeta,\eta)=1,$

$$
{}_pA_{n,\lambda}(\zeta,\eta)=(-1)^n\begin{vmatrix} p^*_{0,\lambda}(\zeta,\eta) & p^*_{1,\lambda}(\zeta,\eta) & p^*_{2,\lambda}(\zeta,\eta) & \cdots & p^*_{n-1,\lambda}(\zeta,\eta) & p^*_{n,\lambda}(\zeta,\eta) \\ \Omega^*_{0,\lambda} & \Omega^*_{1,\lambda} & \Omega^*_{2,\lambda} & \cdots & \Omega^*_{n-1,\lambda} & \Omega^*_{n,\lambda} \\ 0 & \Omega^*_{0,\lambda} & \binom{2}{1}\Omega^*_{1,\lambda} & \cdots & \binom{n-1}{1}\Omega^*_{n-2,\lambda} & \binom{n}{1}\Omega^*_{n-1,\lambda} \\ 0 & 0 & \Omega^*_{0,\lambda} & \cdots & \binom{n-1}{2}\Omega^*_{n-3,\lambda} & \binom{n}{2}\Omega^*_{n-2,\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \Omega^*_{0,\lambda} & \binom{n}{n-1}\Omega^*_{1,\lambda} \end{vmatrix}, \tag{3.18}
$$

$n=1,2,...,$ $p^*_{n,\lambda}(\zeta,\eta)$ *are the conjugate degenerate bivariate general sequences and the coefficients* $\Omega^*_{n,\lambda}\in\mathbb{C}$ *are defined by* (3.9)*.*

(iii) ${}_pA_{0,\lambda}(\zeta,\eta)=1,$

$$
{}_pA_{n,\lambda}(\zeta,\eta)=\begin{vmatrix} \zeta+\chi_{0,\lambda}+\vartheta_{0,\lambda}(\eta) & -1 & \cdots & 0 \\ \chi_{1,\lambda}+\vartheta_{1,\lambda}(\eta) & \zeta+\chi_{0,\lambda}+\vartheta_{0,\lambda}(\eta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{n,\lambda}+\vartheta_{n,\lambda}(\eta) & \binom{n}{1}\chi_{n-1,\lambda}+\vartheta_{n-1,\lambda}(\eta) & \cdots & \zeta+\chi_{0,\lambda}+\vartheta_{0,\lambda}(\eta) \end{vmatrix}, \tag{3.19}
$$

$n=1,2,...,$ *where* $\chi_{n,\lambda}$ *and* $\vartheta_{n,\lambda}(\eta)$ *are defined by* (3.12)*.*

Proof. The first, second, and third recurrence formulae given by equations (3.1), (3.10), and (3.11) respectively can be regarded as infinite lower triangular systems in unknowns ${}_pA_{n,\lambda}(\zeta,\eta)$. Therefore, after performing elementary determinant operations on the given equations and then solving the first $(n+1)$ equations using Cramer's rule, we obtain three different determinant expressions for the dgbAp ${}_pA_{n,\lambda}(\zeta,\eta)$ given by equations (3.17), (3.18), and (3.19). □

Further, we establish determinant expressions for the degenerate bivariate general polynomials $p_{n,\lambda}(\zeta,\eta)$. In view of recurrences (3.4) and (3.11), we obtain the following determinant forms for degenerate bivariate general polynomials:

Theorem 3.12. *The members of the degenerate bivariate general polynomial sequence* $\{p_{n,\lambda}(\zeta,\eta)\}_{n\in\mathbb{N}}$ *can be represented by means of the following determinant expressions:*

(i) $p_{0,\lambda}(\zeta,\eta)=1$,

$$p_{n,\lambda}(\zeta,\eta)=(-1)^n\begin{vmatrix} p^*_{0,\lambda}(\zeta,\eta) & p^*_{1,\lambda}(\zeta,\eta) & p^*_{2,\lambda}(\zeta,\eta) & \cdots & p^*_{n-1,\lambda}(\zeta,\eta) & p^*_{n,\lambda}(\zeta,\eta) \\ 1 & (-1)_{1,\lambda} & (-1)_{2,\lambda} & \cdots & (-1)_{n-1,\lambda} & (-1)_{n,\lambda} \\ 0 & 1 & 2(-1)_{1,\lambda} & \cdots & (n-1)(-1)_{n-2,\lambda} & n(-1)_{n-1,\lambda} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}(-1)_{n-3,\lambda} & \binom{n}{2}(-1)_{n-2,\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & n(-1)_{1,\lambda} \end{vmatrix},$$

$n=1,2,...,$ *where* $p^*_{n,\lambda}(\zeta,\eta)$ *are the conjugate degenerate bivariate general polynomials.*

(ii) $p_{0,\lambda}(\zeta,\eta)=1$,

$$p_{n,\lambda}(\zeta,\eta)=\begin{vmatrix} \zeta+\vartheta_{0,\lambda}(\eta) & -1 & 0 & \cdots & 0 \\ \vartheta_{1,\lambda}(\eta) & \zeta+\vartheta_{0,\lambda}(\eta) & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ \vartheta_{n,\lambda}(\eta) & \binom{n}{1}\vartheta_{n-1,\lambda}(\eta) & \cdots & \binom{n}{n-1}\vartheta_{1,\lambda}(\eta) & \zeta+\vartheta_{0,\lambda}(\eta) \end{vmatrix},$$

$n=1,2,...,$ *where the coefficients* $\vartheta_{n,\lambda}(\eta)$ *are defined by* (3.12).

In the next section, we consider a particular form of degenerate bivariate polynomial and derive results for its hybrid form.

4. Degenerate bivariate Laguerre-Appell sequences and their special cases

The family of degenerate bivariate general polynomials $p_{n,\lambda}(\zeta,\eta)$ includes in itself a variety of polynomials, some of which are explored but many are yet to be discovered. For the suitable choices of the differentiable function $\phi_\lambda(\eta,\tau)$, different degenerate bivariate general polynomials can be obtained. We discuss some of the important members of the degenerate bivariate general polynomials family by making different choices of the function $\phi_\lambda(\eta,\tau)$.

Case I. Degenerate Gould-Hopper polynomials $H^{(m)}_{n,\lambda}(\zeta,\eta)$
For the choice of $\phi_\lambda(\eta,\tau)=e^\eta_\lambda(\tau^m)$, equation (1.3) yields the degenerate Gould-Hopper

polynomial sequences $H^{(m)}_{n,\lambda}(\zeta,\eta)$ [17], whose generating relation and series representation are given as follows:

$$e^{\zeta}_{\lambda}(\tau)e^{\eta}_{\lambda}(\tau^m)=\sum_{n=0}^{\infty}H^{(m)}_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!} \tag{4.1}$$

and

$$H^{(m)}_{n,\lambda}(\zeta,\eta)=n!\sum_{r=0}^{[n/m]}\frac{(\eta)_{r,\lambda}(\zeta)_{n-mr,\lambda}}{(n-mr)!\,r!},$$

respectively.

For $m=2$, the degenerate Gould-Hopper sequences becomes the degenerate Hermite polynomials $H_{n,\lambda}(\zeta,\eta)$ [16, 17].

Case II. Degenerate bivariate Laguerre polynomials $L_{n,\lambda}(\zeta,\eta)$
For the choice of $\phi_{\lambda}(\eta,\tau)=C_{0,\lambda}(\eta\tau)$, equation (1.3) yields the degenerate bivariate Laguerre polynomials $L_{n,\lambda}(\zeta,\eta)$, which are defined by the generating equation of the form:

$$e^{\eta}_{\lambda}(\tau)C_{0,\lambda}(\zeta\tau)=\sum_{n=0}^{\infty}L_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!},$$

where $C_{0,\lambda}(\eta\tau)$ are degenerate Tricomi functions of order 0 defined by the following generating function and series representation:

$$e_{\lambda}(\tau)e^{-\zeta}_{\lambda}\left(\frac{1}{\tau}\right)=\sum_{n=0}^{\infty}C_{n,\lambda}(\zeta)\frac{\tau^n}{n!}$$

and

$$C_{n,\lambda}(\zeta)=\sum_{r=0}^{\infty}\frac{(-1)_{r,\lambda}(\zeta)_{r,\lambda}}{r!(n+r)!},$$

respectively.
The degenerate bivariate Laguerre polynomials $L_{n,\lambda}(\zeta,\eta)$ can be explicitly represented by

$$L_{n,\lambda}(\zeta,\eta)=\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)_{r,\lambda}(\zeta)_{r,\lambda}(\eta)_{n-r,\lambda}}{r!} \tag{4.2}$$

and its conjugate generating function is given by:

$$e^{-1}_{\lambda}(\tau)e^{\zeta}_{\lambda}(\tau)C_{0,\lambda}(\eta\tau)=\sum_{n=0}^{\infty}L^{*}_{n,\lambda}(\eta,\zeta)\frac{\tau^n}{n!}.$$

Case III. Degenerate bivariate Legendre polynomials $S_{n,\lambda}(\zeta,\eta)$
For the choice of $\phi_{\lambda}(\eta,\tau)=C_{0,\lambda}(-\eta\tau^2)$, equation (1.3) yields the degenerate bivariate Legendre polynomials $S_{n,\lambda}(\zeta,\eta)$, which are defined by the following generating equation and series definition:

$$e^{\eta}_{\lambda}(\tau)C_{0,\lambda}(-\zeta\tau^2)=\sum_{n=0}^{\infty}S_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}.$$

For the degenerate bivariate Legendre polynomials $S_{n,\lambda}(\zeta,\eta)$, the following series holds:

$$S_{n,\lambda}(\zeta,\eta)=n!\sum_{r=0}^{[n/2]}\frac{(\eta)_{n-2r,\lambda}(-\zeta)_{r,\lambda}(-1)_{r,\lambda}}{(n-2r)!(r!)^2}.$$

Now, by taking one of the members of the degenerate bivariate general family, we frame one example for the degenerate general bivariate Appell family.

Example 4.1 (Degenerate bivariate Laguerre-Appell polynomial sequences ${}_LA_{n,\lambda}(\zeta,\eta)$)**.** Taking $\phi_\lambda(\eta,\tau) = C_{0,\lambda}(\eta,\tau)$ in generating equation (2.1), we obtain the degenerate bivariate Laguerre-Appell polynomials (dbLAp) ${}_LA_{n,\lambda}(\zeta,\eta)$ which are defined by the generating relation of the form:

$$A_\lambda(\tau)e_\lambda^\zeta(\tau)C_{0,\lambda}(\eta\tau) = \sum_{n=0}^{\infty} {}_LA_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!} \tag{4.3}$$

and its conjugate form is given as follows:

$$A_\lambda^*(\tau)e_\lambda^\zeta(\tau)C_{0,\lambda}(\eta\tau) = \sum_{n=0}^{\infty} {}_LA^*_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}.$$

The series representation for the dbLAp ${}_LA_{n,\lambda}(\zeta,\eta)$ in terms of degenerate bivariate Laguerre polynomials is defined as follows:

$${}_LA_{n,\lambda}(\zeta,\eta) = \sum_{r=0}^{n}\binom{n}{r}\rho_{n-r,\lambda}L_{r,\lambda}(\eta,\zeta).$$

In view of equation (4.2), the above equation reduces to

$${}_LA_{n,\lambda}(\zeta,\eta) = n!\sum_{k=0}^{n}\sum_{r=0}^{n}\frac{\rho_{n-k-r,\lambda}\ (\zeta)_{k,\lambda}\ (-1)_{r,\lambda}\ (\eta)_{r,\lambda}}{(r!)^2\ (n-k-r)!\ k!}.$$

The following connection formula between the dbLAp ${}_LA_{n,\lambda}(\zeta,\eta)$ and degenerate Appell polynomials $A_{n,\lambda}$ holds:

$${}_LA_{n,\lambda}(\zeta,\eta) = n!\sum_{r=0}^{n}\frac{A_{n-r,\lambda}(\zeta)(-1)_{r,\lambda}(\eta)_{r,\lambda}}{(r!)^2(n-r)!}.$$

Now, we present some of the recurrence formulae for the dbLAp ${}_LA_{n,\lambda}(\zeta,\eta)$.

Theorem 4.2. *The degenerate bivariate Laguerre-Appell polynomials* ${}_LA_{n,\lambda}(\zeta,\eta)$ *satisfy the following recurrence relations:*

(i) ${}_LA_{n,\lambda}(\zeta,\eta) = L_{n,\lambda}(\eta,\zeta) - \sum_{r=0}^{n-1}\binom{n}{r}\gamma_{n-r,\lambda}\ {}_LA_{r,\lambda}(\zeta,\eta), \quad n \geq 1,$
where $L_{n,\lambda}(\eta,\zeta)$ *are the degenerate bivariate Laguerre polynomials.*

(ii) ${}_LA_{n,\lambda}(\zeta,\eta) = L^*_{n,\lambda}(\eta,\zeta) - \sum_{r=0}^{n-1}\Omega^*_{n,r;\lambda}\ {}_LA_{r,\lambda}(\zeta,\eta),$
where $L^*_{n,\lambda}(\eta,\zeta)$ *are the conjugate degenerate bivariate Laguerre polynomials.*

(iii) ${}_LA_{n+1,\lambda}(\zeta,\eta)$
$= \{\zeta + \chi_{0,\lambda} + \vartheta_{0,\lambda}(\eta)\}\,{}_LA_{n,\lambda}(\zeta,\eta) + \sum_{r=0}^{n-1}\binom{n}{r}\{\chi_{n-r,\lambda} + \vartheta_{n-r,\lambda}(\eta)\}\,{}_LA_{r,\lambda}(\zeta,\eta),$
where the coefficients $\Omega^*_{n,r;\lambda}$ *and* $\chi_{r,\lambda}$ *are defined by equations* (3.7) *and* (3.12) *respectively and* $\vartheta_{r,\lambda}(\eta)$ *are given by*

$$\frac{\partial_\tau C_{0,\lambda}(\eta\tau)}{C_{0,\lambda}(\eta\tau)} = \sum_{r=0}^{\infty}\vartheta_{r,\lambda}(\eta)\frac{\tau^r}{r!}.$$

For the choice of $p_{n,\lambda}(\zeta,\eta) = L_{n,\lambda}(\eta,\zeta)$ in determinant (3.17), we find the following determinant representation for dbLAp ${}_LA_{n,\lambda}(\zeta,\eta)$:

$$\begin{aligned}&{}_LA_{0,\lambda}(\zeta,\eta) = 1,\\&{}_LA_{n,\lambda}(\zeta,\eta)\\&= (-1)^n \begin{vmatrix} L_{0,\lambda}(\eta,\zeta) & L_{1,\lambda}(\eta,\zeta) & L_{2,\lambda}(\eta,\zeta) & \cdots & L_{n-1,\lambda}(\eta,\zeta) & L_{n,\lambda}(\eta,\zeta) \\ \gamma_{0,\lambda} & \gamma_{1,\lambda} & \gamma_{2,\lambda} & \cdots & \gamma_{n-1,\lambda} & \gamma_{n,\lambda} \\ 0 & \gamma_{0,\lambda} & \binom{2}{1}\gamma_{1,\lambda} & \cdots & \binom{n-1}{1}\gamma_{n-2,\lambda} & \binom{n}{1}\gamma_{n-1,\lambda} \\ 0 & 0 & \gamma_{0,\lambda} & \cdots & \binom{n-1}{2}\gamma_{n-3,\lambda} & \binom{n}{2}\gamma_{n-2,\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \gamma_{0,\lambda} & \binom{n}{n-1}\gamma_{1,\lambda} \end{vmatrix}, \qquad (4.4)\end{aligned}$$

for $n = 1, 2, \ldots$

Likewise taking $p^*_{n,\lambda}(\zeta,\eta) = L^*_{n,\lambda}(\eta,\zeta)$ in determinant (3.18), we obtain the second determinant expression for dbLAp ${}_LA_{n,\lambda}(\zeta,\eta)$:

$$\begin{aligned}&{}_LA_{0,\lambda}(\zeta,\eta) = 1,\\&{}_LA_{n,\lambda}(\zeta,\eta)\\&= (-1)^n \begin{vmatrix} L^*_{0,\lambda}(\eta,\zeta) & L^*_{1,\lambda}(\eta,\zeta) & L^*_{2,\lambda}(\eta,\zeta) & \cdots & L^*_{n-1,\lambda}(\eta,\zeta) & L^*_{n,\lambda}(\eta,\zeta) \\ \Omega^*_{0,\lambda} & \Omega^*_{1,\lambda} & \Omega^*_{2,\lambda} & \cdots & \Omega^*_{n-1,\lambda} & \Omega^*_{n,\lambda} \\ 0 & \Omega^*_{0,\lambda} & \binom{2}{1}\Omega^*_{1,\lambda} & \cdots & \binom{n-1}{1}\Omega^*_{n-2,\lambda} & \binom{n}{1}\Omega^*_{n-1,\lambda} \\ 0 & 0 & \Omega^*_{0,\lambda} & \cdots & \binom{n-1}{2}\Omega^*_{n-3,\lambda} & \binom{n}{2}\Omega^*_{n-2,\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \Omega^*_{0,\lambda} & \binom{n}{n-1}\Omega^*_{1,\lambda} \end{vmatrix}, \qquad (4.5)\end{aligned}$$

for $n = 1, 2, \ldots$

It is well known that the polynomials $B_{n,\lambda}(\zeta)$ and $E_{n,\lambda}(\zeta)$ are the particular members of the degenerate Appell family. Therefore, by making particular choices for the function $A_\lambda(\tau)$ in dbLAp, we obtain the following special cases of the dbLAp family:

Case I. Taking $A_\lambda(\tau) = \frac{\tau}{e_\lambda(\tau)-1}$ of the degenerate Bernoulli polynomials in equation (4.3), we obtain the degenerate bivariate Laguerre-Bernoulli polynomials (dbLBp) ${}_LB_{n,\lambda}(\zeta,\eta)$ which are defined by the generating function and series definition of the following form:

$$\left(\frac{\tau}{e_\lambda(\tau)-1}\right)e^\zeta_\lambda(\tau)C_{0,\lambda}(\eta\tau) = \sum_{n=0}^{\infty} {}_LB_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}$$

and

$${}_LB_{n,\lambda}(\zeta,\eta) = n!\sum_{r=0}^{n} \frac{(-1)_{r,\lambda}(\eta)_{r,\lambda}B_{n-r,\lambda}(\zeta)}{(r!)^2(n-r)!},$$

respectively.
Taking

$$\gamma_{n,\lambda} = \frac{1}{n+1}(1)_{n+1,\lambda},$$

in determinant (4.4), we obtain the following determinant expression for the dbLBp ${}_LB_{n,\lambda}(\zeta,\eta)$:

$$\begin{aligned}
&{}_LB_{0,\lambda}(\zeta,\eta) = 1,\\
&{}_LB_{n,\lambda}(\zeta,\eta)\\
&= (-1)^n \begin{vmatrix}
L_{0,\lambda}(\eta,\zeta) & L_{1,\lambda}(\eta,\zeta) & L_{2,\lambda}(\eta,\zeta) & \cdots & L_{n-1,\lambda}(\eta,\zeta) & L_{n,\lambda}(\eta,\zeta)\\
(1)_{1,\lambda} & \frac{1}{2}(1)_{2,\lambda} & \frac{1}{3}(1)_{3,\lambda} & \cdots & \frac{1}{n}(1)_{n,\lambda} & \frac{1}{n+1}(1)_{n+1,\lambda}\\
0 & (1)_{1,\lambda} & \binom{2}{1}\frac{1}{2}(1)_{2,\lambda} & \cdots & \binom{n-1}{1}\frac{1}{(n-1)}(1)_{n-1,\lambda} & \binom{n}{1}\frac{1}{n}(1)_{n,\lambda}\\
0 & 0 & (1)_{1,\lambda} & \cdots & \binom{n-1}{2}\frac{1}{n-2}(1)_{n-2,\lambda} & \binom{n}{2}\frac{1}{n-1}(1)_{n-1,\lambda}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \cdots & \cdots & 0 & (1)_{1,\lambda} & \binom{n}{n-1}\frac{1}{2}(1)_{2,\lambda}
\end{vmatrix},
\end{aligned}$$

for $n = 1, 2, \ldots$
Also, taking

$$\Omega^*_{n,j;\lambda} = \binom{n}{j}\Omega^*_{n-j,\lambda} = \sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}(-1)_{n-r,\lambda}\ \gamma_{r-j,\lambda},$$

in determinant (4.5), we obtain the following second determinant expression for the dbLBp ${}_LB_{n,\lambda}(\zeta,\eta)$:

$$\begin{aligned}
&{}_LB_{0,\lambda}(\zeta,\eta) = 1,\\
&{}_LB_{n,\lambda}(\zeta,\eta)\\
&= (-1)^n \begin{vmatrix}
L^*_{0,\lambda}(\eta,\zeta) & L^*_{1,\lambda}(\eta,\zeta) & L^*_{2,\lambda}(\eta,\zeta) & \cdots & L^*_{n,\lambda}(\eta,\zeta)\\
(1)_{1,\lambda} & \frac{1}{2}\big[2(-1)_{1,\lambda}(1)_{1,\lambda}+(1)_{2,\lambda}\big] & \binom{n}{2}\big[(-1)_{2,\lambda}(1)_{1,\lambda}+\frac{(-1)_{1,\lambda}(1)_{2,\lambda}}{n}+\frac{(1)_{3,\lambda}}{3}\big] & \cdots & \sum\limits_{r=0}^{n}\binom{n}{r}\frac{(-1)_{n-r,\lambda}(1)_{r+1,\lambda}}{r+1}\\
0 & (1)_{1,\lambda} & \big[2(-1)_{1,\lambda}(1)_{1,\lambda}+(1)_{2,\lambda}\big] & \cdots & \sum\limits_{r=1}^{n}\binom{n}{r}(-1)_{n-r,\lambda}(1)_{r,\lambda}\\
0 & 0 & (1)_{1,\lambda} & \cdots & \sum\limits_{r=2}^{n}\binom{r}{2}\binom{n}{r}\frac{(-1)_{n-r,\lambda}(1)_{r-1,\lambda}}{\binom{n}{2}(r-1)}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & \cdots & \frac{n}{2}\big[2(-1)_{1,\lambda}(1)_{1,\lambda}+(1)_{2,\lambda}\big]
\end{vmatrix},
\end{aligned}$$

for $n = 1, 2, \ldots$
For the dbLBp ${}_LB_{n,\lambda}(\zeta,\eta)$, the following recurrence relation and addition formula hold:

$$ {}_LB_{0,\lambda}(\zeta,\eta) = 1, \quad {}_LB_{n,\lambda}(\zeta,\eta) = L_{n,\lambda}(\eta,\zeta) - \sum_{r=0}^{n-1}\binom{n}{r}\frac{{}_LB_{r,\lambda}(\zeta,\eta)(1)_{n-r+1,\lambda}}{(n-r+1)}, \quad n \geq 1 $$

and

$$ {}_LB_{n,\lambda}(\zeta_1+\zeta_2,\eta) = \sum_{r=0}^{n}\binom{n}{r}{}_LB_{n,\lambda}(\zeta_1,\eta)(\zeta_2)_{n-r,\lambda}, $$

respectively.

Case II. Taking $A_\lambda(\tau) = \frac{2}{e_\lambda(\tau)+1}$ of the degenerate Euler polynomials in equation (4.3), we find the degenerate bivariate Laguerre-Euler polynomials (dbLEp) ${}_LE_{n,\lambda}(\zeta,\eta)$ which are defined by the generating function and series definition of the following form:

$$ \left(\frac{2}{e_\lambda(\tau)+1}\right)e_\lambda^\zeta(\tau)C_{0,\lambda}(\eta\tau) = \sum_{n=0}^{\infty}{}_LE_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!} $$

and

$$ {}_LE_{n,\lambda}(\zeta,\eta) = n!\sum_{r=0}^{n}\frac{(-1)_{r,\lambda}(\eta)_{r,\lambda}E_{n-r,\lambda}(\zeta)}{(r!)^2(n-r)!}, $$

respectively.
Taking

$$ \gamma_{n,\lambda} = \frac{1}{2}(1)_{n,\lambda} $$

in determinant (4.4), we obtain the following determinant definition for the dbLEp ${}_LE_{n,\lambda}(\zeta,\eta)$:

$$ {}_LE_{0,\lambda}(\zeta,\eta) = 1, $$

$$ {}_LE_{n,\lambda}(\zeta,\eta) $$

$$ = (-1)^n \begin{vmatrix} L_{0,\lambda}(\eta,\zeta) & L_{1,\lambda}(\eta,\zeta) & L_{2,\lambda}(\eta,\zeta) & \cdots & L_{n-1,\lambda}(\eta,\zeta) & L_{n,\lambda}(\eta,\zeta) \\ \frac{1}{2} & \frac{1}{2}(1)_{1,\lambda} & \frac{1}{2}(1)_{2,\lambda} & \cdots & \frac{1}{2}(1)_{n-1,\lambda} & \frac{1}{2}(1)_{n,\lambda} \\ 0 & \frac{1}{2} & \binom{2}{1}\frac{1}{2}(1)_{1,\lambda} & \cdots & \binom{n-1}{1}\frac{1}{2}(1)_{n-2,\lambda} & \binom{n}{1}\frac{1}{2}(1)_{n-1,\lambda} \\ 0 & 0 & \frac{1}{2} & \cdots & \binom{n-1}{2}\frac{1}{2}(1)_{n-3,\lambda} & \binom{n}{2}\frac{1}{2}(1)_{n-2,\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \frac{1}{2} & \binom{n}{n-1}\frac{1}{2}(1)_{1,\lambda} \end{vmatrix}, $$

for $n = 1, 2, \ldots$
For the dbLEp ${}_LE_{n,\lambda}(\zeta,\eta)$, the following recurrence relation and addition formula holds:

$$ {}_LE_{0,\lambda}(\zeta,\eta) = 1, \quad {}_LE_{n,\lambda}(\zeta,\eta) = L_{n,\lambda}(\eta,\zeta) - \frac{1}{2}\sum_{r=0}^{n-1}\binom{n}{r}{}_LE_{r,\lambda}(\zeta,\eta)(1)_{n-r,\lambda}, \quad n \geq 1 $$

and

$$ {}_LE_{n,\lambda}(\zeta_1+\zeta_2,\eta) = \sum_{r=0}^{n}\binom{n}{r}{}_LE_{n,\lambda}(\zeta_1,\eta)(\zeta_2)_{n-r,\lambda}, $$

respectively.
Again, taking

$$ \Omega^*_{n,j;\lambda} = \binom{n}{j}\Omega^*_{n-j,\lambda} = \sum_{r=j}^{n}\binom{n}{r}\binom{r}{j}(-1)_{n-r,\lambda}\ \gamma_{r-j,\lambda}, $$

in (4.5), we obtain the following second determinant expression for the dbLEp ${}_LE_{n,\lambda}(\zeta,\eta)$:

$${}_LE_{0,\lambda}(\zeta,\eta)=1,$$

$${}_LE_{n,\lambda}(\zeta,\eta)$$

$$=(-1)^n\begin{vmatrix} L^*_{0,\lambda}(\eta,\zeta) & L^*_{1,\lambda}(\eta,\zeta) & L^*_{2,\lambda}(\eta,\zeta) & \cdots & L^*_{n,\lambda}(\eta,\zeta) \\ \frac{1}{2} & \frac{1}{2}\left[(-1)_{1,\lambda}+(1)_{1,\lambda}\right] & \frac{1}{2}\left[(-1)_{2,\lambda}+(-1)_{1,\lambda}(1)_{1,\lambda}+(1)_{2,\lambda}\right] & \cdots & \frac{1}{2}\sum\limits_{r=0}^{n}(-1)_{n-r,\lambda}(1)_{r,\lambda} \\ 0 & \frac{1}{2} & (-1)_{1,\lambda}+(1)_{1,\lambda} & \cdots & \frac{1}{2}\sum\limits_{r=1}^{n}\binom{n}{r}r(-1)_{n-r,\lambda}(1)_{r-1,\lambda} \\ 0 & 0 & \frac{1}{2} & \cdots & \sum\limits_{r=2}^{n}\frac{n!(-1)_{n-r,\lambda}(1)_{r-2,\lambda}}{4(n-r)!(r-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \frac{n}{2}\left[(-1)_{1,\lambda}+(1)_{1,\lambda}\right] \end{vmatrix},$$

for $n=1,2,\ldots$

By varying the values of $\phi_\lambda(\eta,\tau)$, we can obtain distinct members of the dgbAp family. These members are listed in Table 1.

Table 1. Certain other members of the dgbAp family.

$\phi_\lambda(\eta,\tau)$	**Generating Function**	**Series Definition**	**Name of Polynomial**
$C_{0,\lambda}(-\eta\tau^2)$	$A_\lambda(\tau)e^{\zeta}_{\lambda}(\tau)C_{0,\lambda}(-\eta\tau^2)=\sum\limits_{n=0}^{\infty}{}_SA_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}$	${}_SA_{n,\lambda}(\zeta,\eta)=\sum\limits_{r=0}^{n}\binom{n}{r}\rho_{n-r,\lambda}\ S_{r,\lambda}(\zeta,\eta)$	Degenerate bivariate Legendre-Appell polynomials ${}_SA_{n,\lambda}(\zeta,\eta)$
$e^{\eta}_{\lambda}(\tau^m)$	$A_\lambda(\tau)e^{\zeta}_{\lambda}(\tau)e^{\eta}_{\lambda}(\tau^m)=\sum\limits_{n=0}^{\infty}{}_HA^{(m)}_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}$	${}_HA^{(m)}_{n,\lambda}(\zeta,\eta)=\sum\limits_{r=0}^{n}\binom{n}{r}\rho_{n-r,\lambda}\ H^{(m)}_{r,\lambda}(\zeta,\eta)$	Degenerate bivariate Gould-Hopper- Appell polynomials ${}_HA^{(m)}_{n,\lambda}(\zeta,\eta)$
$e^{\eta}_{\lambda}(\tau^2)$	$A_\lambda(\tau)e^{\zeta}_{\lambda}(\tau)e^{\eta}_{\lambda}(\tau^2)=\sum\limits_{n=0}^{\infty}{}_HA_{n,\lambda}(\zeta,\eta)\frac{\tau^n}{n!}$	${}_HA_{n,\lambda}(\zeta,\eta)=\sum\limits_{r=0}^{n}\binom{n}{r}\rho_{n-r,\lambda}\ H_{r,\lambda}(\zeta,\eta)$	Degenerate bivariate Hermite-Appell polynomials ${}_HA_{n,\lambda}(\zeta,\eta)$

Remark 4.3. We remark that the results established in Section 4 can be correspondingly obtained for the members listed in Table 1 and can be recast into its special cases.

Next, we will look at the dbLBp ${}_LB_{n,\lambda}(\zeta,\eta)$ from a graphical standpoint. We present graphs, contour plots and zero distribution plots for the dbLBp ${}_LB_{n,\lambda}(\zeta,\eta)$ for various values of index n and parameter λ.

Figures 1-4 depict the behaviour of single variable ${}_LB_{n,\lambda}(\zeta,11)$ for even $n=4$ and odd $n=7$ with $\lambda>1$ and $\lambda<1$. We interpret that graphs with even index show convex behavior and graphs with odd index show concave behavior. We also know that the polynomials of degree n has at most $n-1$ turning points.

Figures 5-8 depict the pattern for zero distribution for odd $n=23$ and even $n=40$ with $\lambda=5,0.7$. We interpret that the distribution of zeros do not show any pattern of

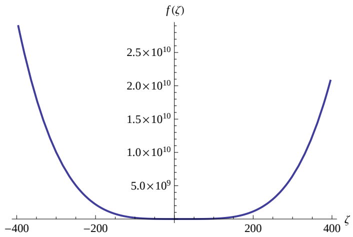

Figure 1. Graph of ${}_LB_{4,5}(\zeta, 11)$

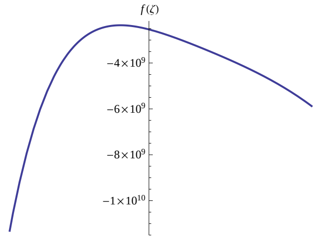

Figure 2. Graph of ${}_LB_{7,5}(\zeta, 11)$

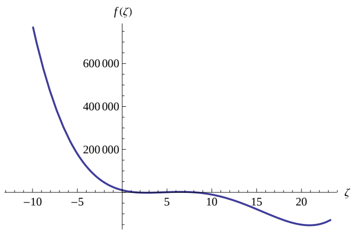

Figure 3. Graph of ${}_LB_{4,0.7}(\zeta, 11)$

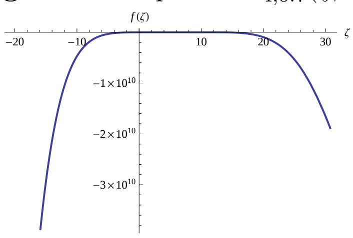

Figure 4. Graph of ${}_LB_{7,0.7}(\zeta, 11)$

symmetry. For odd n one real zero lies on Im(v) $= 0$ and complex zeros are scattered along a line while for even n all complex zeros occur in pair and are scattered along a line.

Figures 9-12 show contour plots for ${}_LB_{n,\lambda}(\zeta, \eta)$ with even $n = 4$ and odd $n = 7$ with $\lambda = 5, 0.7$. We interpret that for even and odd index with $\lambda = 5$, the contour plots of ${}_LB_{n,\lambda}(\zeta, \eta)$ show minima at $\eta = -5$ while the contour plots of ${}_LB_{n,\lambda}(\zeta, \eta)$ for even and odd index with $\lambda = 0.7$ show maxima at $\zeta = 5$.

5. Concluding remarks

The fundamentals of interpolation and polynomial approximation are well established. This topic may be traced back to the pre calculus era, but it has witnessed the majority of its growth and development since the end of the 19th century and is still a vital and growing component of mathematics. Some univariate interpolation and approximation problems are studied in literature, see [11,12]. Recently works are done on multivariate interpolation problem [9,10]. Here, we pose a degenerate general bivariate Appell interpolation problem and try to find its solution in terms of $\{{}_pA_{n,\lambda}(\zeta, \eta)\}_{n\in\mathbb{N}}$. Let $\mathfrak{X}$ be the linear space of real

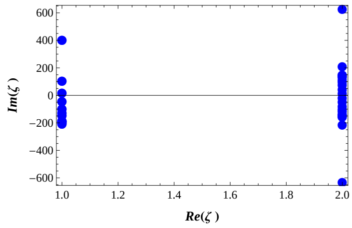

Figure 5. Zeros of ${}_LB_{23,5}(\zeta, 11)$

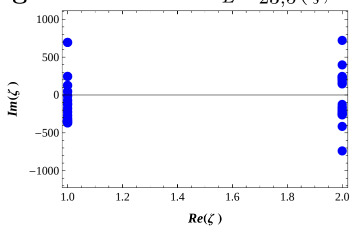

Figure 6. Zeros of ${}_LB_{40,5}(\zeta, 11)$

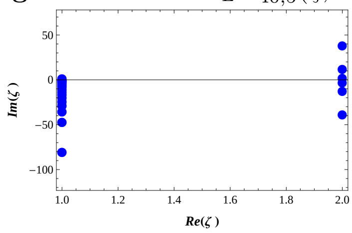

Figure 7. Zeros of ${}_LB_{23,0.7}(\zeta, 11)$

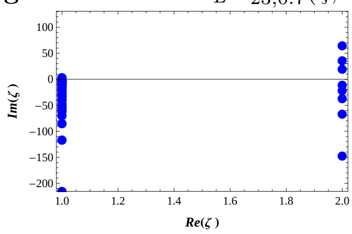

Figure 8. Zeros of ${}_LB_{40,0.7}(\zeta, 11)$

bivariate continuous functions defined in a domain D and belonging to $\mathfrak{C}^n(D)$. We define

$$\mathfrak{S}_n := span\{p_{0,\lambda}, p_{1,\lambda}, \cdots, p_{n,\lambda}\}, \qquad n \in \mathbb{N},$$

where $p_{n,\lambda}$ are defined by equation (1.3). Observe that $\mathfrak{S}_n \subset \mathfrak{X}$. Next, we define a linear functional $\mathfrak{L}$ on $\mathfrak{X}$ such that $\mathfrak{L}(1) \neq 0$. Set

$$\mathfrak{L}(p_{r,\lambda}) := \gamma_{r,\lambda}, \quad r = 0, 1, ...n, \quad \gamma_{0,\lambda} = 1, \quad r \geq 1, \quad \forall\, p_{r,\lambda} \in \mathfrak{S}, \tag{5.1}$$

and consider the degenerate general bivariate Appell polynomials defined by (3.17) and we call as the degenerate bivariate polynomial sequence asociated to the linear functional $\mathfrak{L}$ denoted by $\{{}_pA^{\mathfrak{L}}_{n,\lambda}\}_{n\in\mathbb{N}}$.

Degenerate general bivariate Appell interpolation problem

For every $f \in \mathfrak{X}$, we find, if there exists, the bivariate polynomial $\mathfrak{g}_n[f]$ such that the following hold:

$$\mathfrak{L}\left(\frac{\partial^r}{\partial \zeta^r}\mathfrak{g}_n[f]\right) = \mathfrak{L}\left(\frac{\partial^r f}{\partial \zeta^r}\right).$$

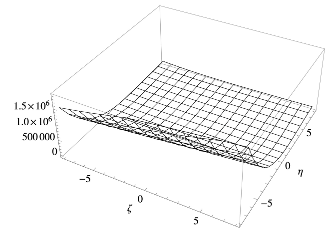

Figure 9. Contour plot of ${}_LB_{4,5}(\zeta,\eta)$

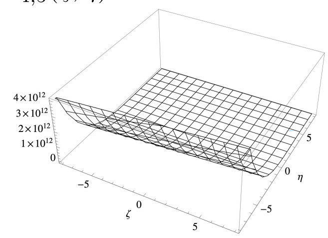

Figure 10. Contour plot of ${}_LB_{7,5}(\zeta,\eta)$

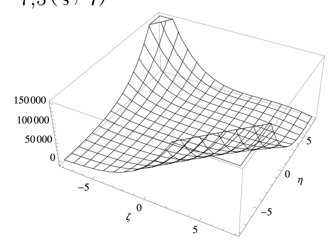

Figure 11. Contour plot of ${}_LB_{4,0.7}(\zeta,\eta)$

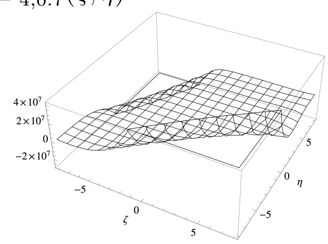

Figure 12. Contour plot of ${}_LB_{7,0.7}(\zeta,\eta)$

Let us define the $k+1\,(k=0,...,n)$ linear functionals as follows:

$$\mathfrak{L}_0(p_{r,\lambda})=\mathfrak{L}(p_{r,\lambda})=\gamma_{r,\lambda},\quad \mathfrak{L}_k(p_{r,\lambda})=\mathfrak{L}\left(p_{r,\lambda}^{(k,0)}\right)=k!\binom{r}{k}\gamma_{r-k,\lambda}, \tag{5.2}$$
$$k=1,...,r,\; r=0,...,n,$$

where

$$p_{r,\lambda}^{(m,n)}=\frac{\partial^{m+n}}{\partial\zeta^m\partial\eta^n},\qquad p_{r,\lambda}^{(0,0)}=p_{r,\lambda}.$$

The following conclusion shows how the linear functional $\mathfrak{L}$ is related to the generating function of the degenerate general bivariate Appell sequence $\{{}_pA^{\mathfrak{L}}_{n,\lambda}\}_{n\in\mathbb{N}}$:

Theorem 5.1. *The following identity holds for the generating function $\mathcal{G}(\zeta,\eta;\tau)$ of the degenerate general bivariate Appell polynomial sequence $\{{}_pA^{\mathfrak{L}}_{n,\lambda}\}_{n\in\mathbb{N}}$:*

$$\mathcal{G}(\zeta,\eta;\tau)=\frac{e^{\zeta}_{\lambda}(\tau)\phi_{\lambda}(\eta,\tau)}{\mathfrak{L}^{\zeta,\eta}\left\{e^{\zeta}_{\lambda}(\tau)\phi_{\lambda}(\eta,\tau)\right\}}, \tag{5.3}$$

where $\mathfrak{L}^{\zeta,\eta}$ denotes the action of the functional $\mathfrak{L}$ on $e^{\zeta}_{\lambda}(\tau)\phi_{\lambda}(\eta,\tau)$ w.r.t. ζ and η.

Proof. The generating function (2.1) can be written as

$$\mathcal{G}(\zeta,\eta;\tau)=\frac{e^{\zeta}_{\lambda}(\tau)\phi_{\lambda}(\eta,\tau)}{\frac{1}{A_{\lambda}(\tau)}}=\frac{e^{\zeta}_{\lambda}(\tau)\phi_{\lambda}(\eta,\tau)}{\sum\limits_{r=0}^{\infty}\gamma_{r,\lambda}\frac{\tau^r}{r!}}.$$

In view of equation (5.1), it follows that

$$\mathcal{G}(\zeta,\eta;\tau)=\frac{e^{\zeta}_{\lambda}(\tau)\phi_{\lambda}(\eta,\tau)}{\mathfrak{L}^{\zeta,\eta}\left\{\sum\limits_{r=0}^{\infty}p_{r,\lambda}(\zeta,\eta)\frac{\tau^r}{r!}\right\}}=\frac{e^{\zeta}_{\lambda}(\tau)\phi_{\lambda}(\eta,\tau)}{\mathfrak{L}^{\zeta,\eta}\left\{e^{\zeta}_{\lambda}(\tau)\phi_{\lambda}(\eta,\tau)\right\}}.$$

In view of equation (1.3), assertion (5.3) follows. □

Theorem 5.2. *The degenerate polynomial*

$$\mathfrak{g}_n[f](\zeta,\eta)=\sum_{r=0}^{n}\frac{\mathfrak{L}(f^{(r,0)})}{r!}{}_pA^{\mathfrak{L}}_{r,\lambda}(\zeta,\eta), \tag{5.4}$$

is the unique element of $\mathfrak{S}_n$ such that

$$L\left(\mathfrak{g}_n[f]^{(k,0)}\right)=L\left([f]^{(k,0)}\right),\qquad k=0,...,n.$$

Proof. In view of determinant expression (3.17), we find

$$\mathfrak{L}_k\left({}_pA^{\mathfrak{L}}_{r,\lambda}\right)=r!\,\delta_{r,k}\,,\qquad k=0,1,...r. \tag{5.5}$$

Now, the theorem follows from relations (5.2) and (5.5). Uniqueness is determined by linear independence of the functionals $\mathfrak{L}_k, k=0,...,n$. □

Examples of degenerate general bivariate Appell interpolation

Example 5.3. Let $\phi_{\lambda}(\eta,\tau)=e^{\eta}_{\lambda}(\tau)$
In this case the related bivariate general sequence is given by

$$p_{n,\lambda}(\zeta,\eta)=(\zeta+\eta)_{n,\lambda}.$$

Note that, in this case $p_{n,\lambda}(\zeta,\eta)=H^{1}_{n,\lambda}(\zeta,\eta)$, where $H^{(1)}_{n,\lambda}(\zeta,\eta)$ are the degenerate bivariate Hermite polynomials.
Next, we define a functional

$$\mathfrak{L}(f)=f(0,0),\qquad \forall f\in\mathfrak{X}.$$

Then

$$\mathfrak{L}(p_{n,\lambda}(\zeta,\eta))=\mathfrak{L}\left\{(\zeta+\eta)_{n,\lambda}\right\}=\gamma_{n,\lambda}=\begin{cases}1, & n=0,\\ 0, & n>0.\end{cases}$$

In view of equation (2.5), we obtain

$$\rho_{n,\lambda}=\begin{cases}1, & n=0,\\ 0, & n>0.\end{cases}$$

In this case, the degenerate general bivariate Appell interpolant, denoted by $\mathfrak{T}_n[f]$ is given as

$$\mathfrak{T}_n[f] = f(0,0) + \sum_{r=1}^{n} f^{(r,0)}(0,0)\frac{(\zeta+\eta)_{r,\lambda}}{r!}.$$

Example 5.4. Let $\phi_\lambda(\eta,\tau) = e_\lambda^\eta(\tau^2)$
In this case the related bivariate general sequence is given by

$$p_{n,\lambda}(\zeta,\eta) = H_{n,\lambda}^{(2)}(\zeta,\eta),$$

where $H_{n,\lambda}^{(2)}(\zeta,\eta)$ are the degenerate bivariate Hermite Kampé de Fériet polynomials defined by equation (4.1) for $m = 2$. Taking $\mathfrak{L}(f) = f(0,0)$, we have

$$\gamma_{n,\lambda} = \mathfrak{L}\left\{H_{n,\lambda}^{(2)}(\zeta,\eta)\right\} = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases}$$

Then, the degenerate general bivariate Appell interpolant is given as

$$\mathfrak{T}_n[f] = f(0,0) + \sum_{r=1}^{n} f^{(r,0)}(0,0)\frac{H_{n,\lambda}^{(2)}(\zeta,\eta)}{r!}.$$

Here, we give a brief viewpoint of the interpolation problem above. Some important properties likewise evaluating integral functional, arithmetic mean functional and numeric results are yet to be discussed in further investigation.

Statements and declarations

Conflict of Interests. The authors declare that they have no competing interests.

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