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Error bounds for a class of history-dependent variational inequalities controlled by $\mathcal{D}$-gap functions

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Abstract

In the present paper, we are concerned with investigating error bounds for history-dependent variational inequalities controlled by the difference gap (for brevity, $\mathcal{D}$-gap) functions. First, we recall a class of elliptic variational inequalities involving the history-dependent operators (for brevity, HDVI). Then, we introduce a new concept of gap functions to the HDVI and propose the regularized gap function for the HDVI via the optimality condition for the concerning minimization problem. Consequently, the $\mathcal{D}$-gap function for the HDVI depends on these regularized gap functions is established. Finally, error bounds for the HDVI controlled by the regularized gap function and the $\mathcal{D}$-gap function are derived under suitable conditions.

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1. Introduction

In 1997, Peng [24] introduced the notion of the $\mathcal{D}$-gap (where $\mathcal{D}$ stands for "difference") function which provides a formulation of the variational inequality to the corresponding unconstrained optimization. Establishing $\mathcal{D}$-gap functions is based on the difference of two regularized gap functions studied in Yamashita-Fukushima [7, 36]. Peng-Fukushima [25] provided a global error bound result of $\mathcal{D}$-gap functions for variational inequalities. The theory of error bounds not only explores the upper estimates of the distance between an arbitrary feasible point and the solution set of a certain inequality problem, but also provides the convergence rate of iterative algorithms for solving optimization problems; see, e.g., [19, 33, 35]. Since then, the $\mathcal{D}$-gap functions and their error bounds have been studied for various kinds of variational inequalities and equilibrium problems. Li-Ng [16] established some error bounds for generalized $\mathcal{D}$-gap functions to a class of nonmonotone and nonsmooth variational inequalities and considered a derivative-free descent method. More recently, Bigi-Passacantando [1] developed $\mathcal{D}$-gap functions and introduced descent

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techniques for solving equilibrium problems. Hung-Tam [10] investigated a class of generalized $\mathcal{D}$-gap functions and global error bounds for a class of elliptic variational inequalities and applied those abstract results to a frictional contact mechanic problem. For more details on this topic, we refer to [4, 9–14, 17, 26, 32] and the references therein.

On the other hand, history-dependent operators represent a significant class of operators with definitions in vector-valued function spaces. They arise in functional analysis, theory of differential equations, and partial differential equations. Some simple examples of history-dependent operators in analysis are the Volterra-type operators and the integral operators. Especially, in Contact Mechanics history-dependent operators are useful to analysis the models involving both quasistatic frictional and frictionless contact conditions using elastic or viscoelastic materials. For all these reasons, various authors have developed of theory and applications for history-dependent operators to variational and hemivariational inequalities. For instance, see [20, 23, 27–31, 41] and the references therein. Besides, some recently important contributions on the topic of variational and hemivariational inequalities with applications to mixed boundary problems have been provided in [2, 18, 21, 37–40]. Very recently, Cen-Nguyen-Zeng [3] is the first time to introduce and study error bounds for a class of generalized time-dependent variational-hemivariational inequalities with history-dependent operators which implicitly depends on the regularized gap function. However to the best of our knowledge, up to now, there has not been any paper on $\mathcal{D}$-gap functions and their error bounds for variational inequalities or variational-hemivariational inequalities with history-dependent operators discussed in the literature.

Inspired by the works above, in this work, we continue the investigate of error bounds for a class of elliptic variational inequalities involving the history-dependent operators (for brevity, HDVI), which controlled by $\mathcal{D}$-gap functions. The aim of this manuscript is two folds. The first one is to introduce a new concept of gap functions to the HDVI and provide the regularized gap function for the HDVI. The proof is based on arguments on the optimality condition for the concerning minimization problem and consequently, the $\mathcal{D}$-gap function for the HDVI via these regularized gap functions is studied. The second aim is to derive error bounds for the HDVI controlled by the regularized gap function and the $\mathcal{D}$-gap function under suitable conditions.

The rest of the paper is organized as follows. In Section 2 we recall some preliminary material on nonlinear analysis and consider a class of history-dependent variational inequalities HDVI. Then, we list the hypotheses on the problem data and provide an existence and uniqueness result of the HDVI. Next, in Section 3, we introduce a new concept of gap functions to the HDVI and establish the regularized gap function and the $\mathcal{D}$-gap function for the HDVI. Finally, we derive error bounds for the HDVI controlled by the regularized gap function and the $\mathcal{D}$-gap function under suitable conditions.

2. Preliminaries and formulations

Throughout the paper, we adopt the following notation. Let V be a Banach space with the norm $\|\cdot\|_V$, V^* denote its dual space and $\langle \cdot, \cdot \rangle_{V^* \times V}$ be the duality brackets between V^* and V. For simplicity, we skip the subscripts. The symbols "$\to$" and "$\xrightarrow{\mathbf{w}}$" denote the strong and the weak convergence, respectively. A space V endowed with the weak topology is denoted by $V_{\mathbf{w}}$. Given $\mathbb{T} := [0, T]$ with $0 < T < \infty$ and a subset $P \subset V$, we denote by $L^2(\mathbb{T}; P)$ the set (equivalence classes) of functions in $L^2(\mathbb{T}; V)$ that for almost everywhere $s \in \mathbb{T}$ have values in P, i.e.,

$$L^2(\mathbb{T}; P) := \{z \in L^2(\mathbb{T}; V) : z(s) \in P \text{ for a.e. } s \in \mathbb{T}\}.$$

We denote $C(\mathbb{T};P)$ by the set of continuous functions on $\mathbb{T}$ with values in P. We recall some fundamental concepts that will be useful in the sequel. For more details, we refer to [5,6].

Definition 2.1. A function $\omega\colon V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be

(a) proper, if $\omega \not\equiv +\infty$;

(b) convex, if $\omega(tz + (1-t)v) \le t\omega(z) + (1-t)\omega(v)$ for all z, $v \in V$ and $t \in [0,1]$;

(c) lower semicontinuous at $z_0 \in V$, if for any sequence $\{z_n\} \subset V$ such that $z_n \to z_0$, it holds $\omega(z_0) \le \liminf \omega(z_n)$;

(d) upper semicontinuous at $z_0 \in V$, if for any sequence $\{z_n\} \subset V$ such that $z_n \to z_0$, it holds $\limsup \omega(z_n) \le \omega(z_0)$;

(e) lower semicontinuous (resp., upper semicontinuous) on V, if ω is lower semicontinuous (resp., upper semicontinuous) at every $z_0 \in V$.

Definition 2.2. Let $\psi\colon V \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. The convex subdifferential $\partial\psi\colon V \rightrightarrows V^*$ of ψ is defined by

$$\partial\psi(z) = \big\{w^* \in V^* \mid \langle w^*, v - z\rangle_{V^* \times V} \le \psi(v) - \psi(z) \ \text{ for all } \ v \in V\big\} \ \text{ for all } \ z \in V.$$

An element $w^* \in \partial\psi(z)$ is called a subgradient of ψ at $z \in V$.

Next, we recall the existence and uniqueness result of solutions for convex optimization problems.

Definition 2.3 (see [15]). A function $\phi\colon V \to \mathbb{R}$ is said to be uniformly convex if there exists a continuously increasing function $\pi\colon \mathbb{R} \to \mathbb{R}$ such that $\pi(0) = 0$ and that for all $z, v \in V$ and for each $t \in [0,1]$, we have

$$\phi(tz + (1-t)v) \le t\phi(z) + (1-t)\phi(v) - t(1-t)\pi(\|z - v\|)\|z - v\|.$$

If $\pi(r) = kr$ for $k > 0$, then ϕ is called a strongly convex function.

Throughout the paper, unless otherwise specified, let W, X and Z be separable Banach spaces, and E be a separable and reflexive Banach space. The norm in E and the duality brackets between E^* and E are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. We now consider the following elliptic mixed variational inequality with history-dependent operators:

Problem 2.4. Find $z \in L^2(\mathbb{T};E)$ such that $z(s) \in P$ for a.e. $s \in \mathbb{T}$ and

$$\begin{aligned}\langle A(s,(\mathcal{H}_1 z)(s), z(s)), v - z(s)\rangle + &\Psi(s,(\mathcal{H}_2 z)(s), z(s), v) - \Psi(s,(\mathcal{H}_2 z)(s), z(s), z(s)) \\ &\ge \langle f(s,(\mathcal{H}_3 z)(s)), v - z(s)\rangle\end{aligned}$$

for all $v \in P$ and a.e. $s \in \mathbb{T}$.

Note that Problem (2.4) is a special case of the history-dependent quasi variational-hemivariational inequalities introduced in [20, Problem 1] without the generalized directional derivative. The main feature of Problem (2.4) is the explicit dependence of the data A, Ψ and f on both the time parameter s and the history-dependent operators $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. Problem (2.4) without history-dependent operators is called a time-dependent variational inequality.

We now impose the following hypotheses on the data of Problem 2.4 (see Migórski-Bai-Zeng [20]).

$\underline{\mathfrak{a}(A)}$: For the operator $A\colon \mathbb{T} \times W \times E \to E^*$,

(i) $A(\cdot,\cdot,z)$ is continuous for all $z \in E$;

(ii) $\|A(s,w,z)\|_{E^*} \le a_0(s) + a_1\|w\|_W + a_2\|z\|$ for all $s \in \mathbb{T}$, $w \in W$, $z \in E$ with $a_0 \in C(\mathbb{T};\mathbb{R}_+)$, $a_1, a_2 \ge 0$;

(iii) $A(s,w,\cdot)$ is demicontinuous for all $(s,w) \in \mathbb{T} \times W$;

(iv) for all $s \in \mathbb{T}$, there exist $m_A > 0$ and $\bar{m}_A \geq 0$ such that

$$\langle A(s, w_1, z_1) - A(s, w_2, z_2), z_1 - z_2 \rangle_{E^* \times E} \geq m_A \|z_1 - z_2\|^2 - \bar{m}_A \|w_1 - w_2\|_W \|z_1 - z_2\|,$$

for all $w_1, w_2 \in W$, $z_1, z_2 \in E$.

$\underline{\mathfrak{a}(\mathcal{H})}$: For the operators $\mathcal{H}_1 \colon L^2(\mathbb{T}; E) \to L^2(\mathbb{T}; W)$, $\mathcal{H}_2 \colon L^2(\mathbb{T}; E) \to L^2(\mathbb{T}; X)$ and $\mathcal{H}_3 \colon L^2(\mathbb{T}; E) \to L^2(\mathbb{T}; Z)$, there exist constants $c_{\mathcal{H}_1} > 0$, $c_{\mathcal{H}_2} > 0$ and $c_{\mathcal{H}_3} > 0$ such that

(i) $\|(\mathcal{H}_1 z_1)(s) - (\mathcal{H}_1 z_2)(s)\|_W \leq c_{\mathcal{H}_1} \int_0^s \|z_1(t) - z_2(t)\| dt$ for all $z_1, z_2 \in L^2(\mathbb{T}; E)$ and for a.e. $s \in \mathbb{T}$;

(ii) $\|(\mathcal{H}_2 z_1)(s) - (\mathcal{H}_2 z_2)(s)\|_X \leq c_{\mathcal{H}_2} \int_0^s \|z_1(t) - z_2(t)\| dt$ for all $z_1, z_2 \in L^2(\mathbb{T}; E)$ and for a.e. $s \in \mathbb{T}$;

(iii) $\|(\mathcal{H}_3 z_1)(s) - (\mathcal{H}_3 z_2)(s)\|_Z \leq c_{\mathcal{H}_3} \int_0^s \|z_1(t) - z_2(t)\| dt$ for all $z_1, z_2 \in L^2(\mathbb{T}; E)$ and for a.e. $s \in \mathbb{T}$.

$\underline{\mathfrak{a}(f)}$: $f \colon \mathbb{T} \times Z \to E^*$ is such that

(i) $f(\cdot, \xi)$ is continuous for all $\xi \in Z$;

(ii) $\|f(s, \xi_1) - f(s, \xi_2)\|_E^* \leq L_f \|\xi_1 - \xi_2\|_Z$ for all $s \in \mathbb{T}$, $\xi_1, \xi_2 \in Z$ with $L_f > 0$.

$\underline{\mathfrak{a}(\Psi)}$: For the function $\Psi \colon \mathbb{T} \times X \times E \times E \to \mathbb{R}$,

(i) $\Psi(s, \zeta, w, \cdot)$ is convex, lower semicontinuous for all $s \in \mathbb{T}$, $\zeta \in X$, $w \in E$;

(ii) there exist $\alpha_\Psi, \beta_\Psi \geq 0$ such that

$$\Psi(s, \zeta_1, w_1, z_2) - \Psi(s, \zeta_1, w_1, z_1) + \Psi(s, \zeta_2, w_2, z_1) - \Psi(s, \zeta_2, w_2, z_2) \leq \alpha_\Psi \|w_1 - w_2\| \|z_1 - z_2\| + \beta_\Psi \|\zeta_1 - \zeta_2\|_X \|z_1 - z_2\|,$$

for all $s \in \mathbb{T}, \xi_1, \xi_2 \in X, w_1, w_2, z_1, z_2 \in E$.

(iii) $\Psi(s, \zeta, w, z_1) - \Psi(s, \zeta, w, z_2) \leq (c_{\Psi_1}(s) + c_{\Psi_1}(\|w\|) + c_3 \|\zeta\|_X) \|z_1 - z_2\|$ for all $(s, \zeta, w) \in \mathbb{T} \times X \times E$, $z_1, z_2 \in E$ where $c_{\Psi_1} \colon \mathbb{T} \to [0, \infty)$ and $c_{\Psi_2} \colon [0, \infty) \to [0, \infty)$ are continuous functions, and $c_3 > 0$.

(iv) $\liminf [\Psi(s_n, \zeta_n, w_n, w_n) - \Psi(s_n, \zeta_n, w_n, z)] \geq \Psi(s, \zeta, w, w) - \Psi(s, \zeta, w, z)$ for all $z \in E$, $s_n \to s$ in $\mathbb{T}$, $\zeta_n \to \zeta$ in X and $w_n \xrightarrow{\mathbf{w}} w$.

$\underline{\mathfrak{a}(P)}$: P is a nonempty, closed and convex subset of E with $\mathbf{0}_E \in P$.

$\underline{\mathfrak{H}(P_b)}$: P is a nonempty, bounded, closed and convex subset of E with $\mathbf{0}_E \in P$.

$\underline{\mathfrak{a}(0)}$: $m_A > \alpha_\Psi$.

The folowing existence and uniqueness result for Problem 2.4 can be obtained directly from [20, Theorem 5] without the generalized directional derivative.

Theorem 2.5. *Assume that the assumptions $\mathfrak{a}(A)$, $\mathfrak{a}(\mathcal{H})$, $\mathfrak{a}(P)$, $\mathfrak{a}(\Psi)$, $\mathfrak{a}(f)$ and $\mathfrak{a}(0)$ hold, then Problem 2.4 has a unique solution $z^* \in L^2(\mathbb{T}; P)$.*

3. Main results

In this section, we first establish a regularized gap function in the form of Yamashita-Fukushima [36] for Problem 2.4 involving the optimality condition for the concerning minimization problem. Furthermore, the $\mathcal{D}$-gap function for Problem 2.4 is formulated by using these regularized gap functions. Finally, we derive some error bounds for Problem 2.4 controlled by the regularized gap function and the $\mathcal{D}$-gap function under suitable conditions.

Let us introduce the exact definition of gap functions for Problem 2.4 as follows.

Definition 3.1. A real-valued function $\mathbf{n} \colon \mathbb{T} \times L^2(\mathbb{T}; P) \to \mathbb{R}$ is said to be a gap function for Problem 2.4, if it satisfies the following properties:

(a) $\mathbf{n}(s,z) \geq 0$ for all $z \in L^2(\mathbb{T};P)$ and $s \in \mathbb{T}$.
(b) $z^* \in L^2(\mathbb{T};P)$ is such that $\mathbf{n}(s,z^*) = 0$ for all $s \in \mathbb{T}$ if and only if z^* is a solution to Problem 2.4.

For each $\mu > 0$ fixed, let the function $\Theta_{\mu,f} \colon \mathbb{T} \times L^2(\mathbb{T};P) \times P \to \mathbb{R}$ be defined by

$$\begin{aligned} &\Theta_{\mu,f}(s,z,v) \\ &= \langle A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s)), v - z(s)\rangle \\ &\quad + \Psi(s,(\mathcal{H}_2 z)(s),z(s),v) - \Psi(s,(\mathcal{H}_2 z)(s),z(s),z(s)) + \frac{1}{2\mu}\|v - z(s)\|^2 \end{aligned} \tag{3.1}$$

for all $z \in L^2(\mathbb{T};P)$, $v \in P$, $s \in \mathbb{T}$.

Lemma 3.2. *Suppose that all the assumptions $\mathfrak{a}(\Psi)(i)$, $\mathfrak{a}(P)$ and $\mathfrak{a}(f)$ hold. Then, for each $z \in L^2(\mathbb{T};P)$, $s \in \mathbb{T}$ and $\mu > 0$ fixed, the optimization problem*

$$\min_{v\in P} \longrightarrow \Theta_{\mu,f}(s,z,v), \tag{3.2}$$

attains a unique solution $v_{\mu,f}(z) \in L^2(\mathbb{T};P)$.

Proof. By the condition $\mathfrak{a}(\Psi)(i)$, we get that function $v \mapsto \Psi(s,(\mathcal{H}_2 z)(s),z(s),v)$ is convex and lower semicontinuous for all $z \in L^2(\mathbb{T};P)$ and all $s \in \mathbb{T}$. Then, it is easy to show that the function $\Theta_{\mu,f}(s,z,\cdot)$ is a strongly convex function for all $z \in L^2(\mathbb{T};P)$ and all $s \in \mathbb{T}$. Furthermore, we also obtain that the function $\Theta_{\mu,f}(s,z,\cdot)$ is also lower semicontinuous for all $z \in L^2(\mathbb{T};P)$ and all $s \in \mathbb{T}$. Since P is a nonempty, closed and convex set, applying [34, Chapter 1, Section 3, Theorem 9], the convex minimization problem (3.2) attains a unique minimum $v_{\mu,f}(z) \in L^2(\mathbb{T};P)$, for any $z \in L^2(\mathbb{T};P)$ and $\mu > 0$ fixed. $\square$

The following result provides a formulation of optimality condition for the minimization problem (3.2).

Lemma 3.3. *Suppose that all the conditions of Lemma 3.2 hold. Then, for each $z \in L^2(\mathbb{T};P)$ and $\mu > 0$ fixed,*

$$\begin{aligned} &\left\langle A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s)) + \frac{1}{\mu}(v_{\mu,f}(z)(s) - z(s)), v - v_{\mu,f}(z)(s)\right\rangle \\ &\quad + \Psi(s,(\mathcal{H}_2 z)(s),z(s),v) - \Psi(s,(\mathcal{H}_2 z)(s),z(s),v_{\mu,f}(z)(s)) \geq 0, \end{aligned} \tag{3.3}$$

holds for all $v \in P$ and all $s \in \mathbb{T}$, where $v_{\mu,f}(z) \in L^2(\mathbb{T};P)$ is a unique solution of the problem (3.2).

Proof. For each $z \in L^2(\mathbb{T};P)$ and $\mu > 0$ fixed, let $v_{\mu,f}(z)$ be a unique solution of the problem (3.2). Applying the chain rule for generalized subgradient in [22, Proposition 3.35(ii) and Proposition 3.37(ii)] and the optimality condition for the problem (3.2) (see [8, Theorem 1.23]) leads to

$$\begin{aligned} 0 &\in \partial_3\Theta_{\mu,f}(s,z,v_{\mu,f}(z)(s)) \\ &\subset A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s) \\ &\qquad + \partial_4\Psi(s,(\mathcal{H}_2 z)(s),z(s),v_{\mu,f}(z)(s)) + \frac{1}{\mu}(v_{\mu,f}(z)(s) - z(s)) \end{aligned}$$

for all $s \in \mathbb{T}$. This implies that there exists $\xi(s) \in \partial_4\Psi(s,(\mathcal{H}_2 z)(s),z(s),v_{\mu,f}(z)(s))$ such that

$$-A(s,(\mathcal{H}_1 z)(s),z(s)) + f(s,(\mathcal{H}_3 z)(s)) - \frac{1}{\mu}(v_{\mu,f}(z)(s) - z(s)) = \xi(s)$$

for all $s \in \mathbb{T}$. Hence, for all $v \in P$ and all $s \in \mathbb{T}$, we have

$$\begin{aligned}&\left\langle -A(s,(\mathcal{H}_1 z)(s),z(s)) + f(s,(\mathcal{H}_3 z)(s)) - \frac{1}{\mu}(v_{\mu,f}(z)(s) - z(s)), v - v_{\mu,f}(z)(s)\right\rangle\\ &\qquad = \langle \xi(s), v - v_{\mu,f}(z)(s)\rangle\\ &\qquad \leq \Psi(s,(\mathcal{H}_2 z)(s),z(s),v) - \Psi(s,(\mathcal{H}_2 z)(s),z(s),v_{\mu,f}(z)(s)))\end{aligned}$$

that is,

$$\begin{aligned}&\left\langle A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s)) + \frac{1}{\mu}(v_{\mu,f}(z)(s) - z(s)), v - v_{\mu,f}(z)(s)\right\rangle\\ &\qquad + \Psi(s,(\mathcal{H}_2 z)(s),z(s),v) - \Psi(s,(\mathcal{H}_2 z)(s),z(s),v_{\mu,f}(z)(s)) \geq 0,\end{aligned}$$

Thus, for each $z \in L^2(\mathbb{T};P)$, the inequality (3.3) holds for all $v \in P$ and $s \in \mathbb{T}$. □

For each $\mu > 0$ fixed, we consider the function $\Pi_{\mu,f} \colon \mathbb{T} \times L^2(\mathbb{T};P) \to \mathbb{R}$ defined by

$$\Pi_{\mu,f}(s,z) = \sup_{v\in P}\left(-\Theta_{\mu,f}(s,z,v)\right), \tag{3.4}$$

for all $z \in L^2(\mathbb{T};P)$ and all $s \in \mathbb{T}$, where the function $\Theta_{\mu,f}$ is given by (3.1). Then, we can write

$$\begin{aligned}&\Pi_{\mu,f}(s,z)\\ &= \sup_{v\in P}\Big(\langle A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s)), z(s) - v\rangle\\ &\quad -\Psi(s,(\mathcal{H}_2 z)(s),z(s),v) + \Psi(s,(\mathcal{H}_2 z)(s),z(s),z(s)) - \frac{1}{2\mu}\|v - z(s)\|^2\Big).\end{aligned}$$

Next, we prove that $\Pi_{\mu,f}$ is a gap function of Problem 2.4 which is called to be a regularized gap function of Problem 2.4 in the form introduced by Yamashita-Fukushima [36].

Theorem 3.4. *Suppose the hypotheses of Theorem 2.5. Then, the function $\Pi_{\mu,f}$ defined by* (3.4) *for any parameter $\mu > 0$ is a gap function to Problem 2.4.*

Proof. For any fixed parameter $\mu > 0$, we shall verify that $\Pi_{\mu,f}$ satisfies the conditions of Definition 3.1. Indeed,

(a) Let $z \in L^2(\mathbb{T};P)$ be arbitrary. By the definition of $\Pi_{\mu,f}$, we have

$$\begin{aligned}&\Pi_{\mu,f}(s,z)\\ &= \sup_{v\in P}\left(-\Theta_{\mu,f}(s,z,v)\right)\\ &\geq -\Theta_{\mu,f}(s,z,z(s))\\ &= \langle A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s)), z(s) - z(s)\rangle\\ &\qquad - \Psi(s,(\mathcal{H}_2 z)(s),z(s),z(s)) + \Psi(s,(\mathcal{H}_2 z)(s),z(s),z(s)) - \frac{1}{2\mu}\|z(s) - z(s)\|^2\\ &= 0\end{aligned}$$

for all $s \in \mathbb{T}$. This means that $\Pi_{\mu,f}(s,z) \geq 0$ for all $s \in \mathbb{T}$ and all $z \in L^2(\mathbb{T};P)$.

(b) Suppose that $z^* \in L^2(\mathbb{T};P)$ is a solution of Problem 2.4. From (3.4), we have

$$\begin{aligned}\Pi_{\mu,f}(s,z^*) &= \sup_{v\in P}\left(-\Theta_{\mu,f}(s,z^*,v)\right)\\ &= -\inf_{v\in P}\Theta_{\mu,f}(s,z^*,v)\\ &= -\Theta_{\mu,f}(s,z^*,v_{\mu,f}(z^*)(s)),\end{aligned} \tag{3.5}$$

where $v_{\mu,f}(z^*) \in L^2(\mathbb{T};P)$ is a unique solution of the convex minimization problem

$$\min_{v\in P} \longrightarrow \Theta_{\mu,f}(s,z^*,v), \text{ for all } s \in \mathbb{T}.$$

Moreover, since $z^* \in L^2(\mathbb{T}; P)$ is a solution of Problem 2.4, for all $v \in P$ and all $s \in \mathbb{T}$, we obtain

$$\begin{aligned}&\langle A(s, (\mathcal{H}_1 z^*)(s), z^*(s)) - f(s, (\mathcal{H}_3 z^*)(s)), v_{\mu,f}(z^*)(s) - z^*(s)\rangle \\ &\quad + \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), v_{\mu,f}(z^*)(s)) - \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), z^*(s)) \geq 0. \end{aligned} \tag{3.6}$$

It follows from the result of Lemma 3.3 that

$$\begin{aligned}&\left\langle A(s, (\mathcal{H}_1 z^*)(s), z^*(s)) - f(s, (\mathcal{H}_3 z^*)(s)) + \frac{1}{\mu}(v_{\mu,f}(z^*)(s) - z^*(s)), z^*(s) - v_{\mu,f}(z^*)(s)\right\rangle \\ &\quad + \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), z^*(s)) - \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), v_{\mu,f}(z^*)(s)) \geq 0, \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$-\frac{1}{\mu}\|v_{\mu,f}(z^*)(s) - z^*(s)\|^2 \geq 0$$

for all $s \in \mathbb{T}$. This implies that

$$\|v_{\mu,f}(z^*)(s) - z^*(s)\|^2 \leq 0,$$

for all $s \in \mathbb{T}$ and so $z^* = v_{\mu,f}(z^*)$ in $L^2(\mathbb{T}; P)$. Therefore, it follows from (3.5) that $\Pi_{\mu,f}(s, z^*) = 0$ for all $s \in \mathbb{T}$.

Conversely, assume that $z^* \in L^2(\mathbb{T}; P)$ is such that $\Pi_{\mu,f}(s, z^*) = 0$ for all $s \in \mathbb{T}$, that is, $-\Theta_{\mu,f}(s, z^*, v) \leq 0$, i.e., $\Theta_{\mu,f}(s, z^*, v) \geq 0$ for all $v \in P$ and all $s \in \mathbb{T}$. Since $\Theta_{\mu,f}(s, z^*, z^*(s)) = 0$ for all $s \in \mathbb{T}$, $z^*(s)$ solves the following convex minimization problem

$$\min_{v \in P} \longrightarrow \Theta_{\mu,f}(s, z^*, v).$$

Using the optimality condition for this problem, we get

$$0 \in \partial_3 \Theta_{\mu,f}(s, z^*, z^*(s)).$$

By the similar arguments of the proof of Lemma 3.3 with fixed first argument of the function $\Theta_{\mu,f}$, we obtain

$$-A(s, (\mathcal{H}_1 z^*)(s), z^*(s)) + f(s, (\mathcal{H}_3 z^*)(s)) = \xi^*(s)$$

where $\xi^*(s) \in \partial_4 \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), z^*(s))$ for a.e. $s \in \mathbb{T}$. Then for all $v \in P$ and a.e. $s \in \mathbb{T}$,

$$\begin{aligned}&\langle -A(s, (\mathcal{H}_1 z^*)(s), z^*(s)) + f(s, (\mathcal{H}_3 z^*)(s)), v - z^*(s)\rangle \\ &= \langle \xi^*(s), v - z^*(s)\rangle \\ &\leq \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), v) - \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), z^*(s))\end{aligned}$$

that is,

$$\begin{aligned}&\langle A(s, (\mathcal{H}_1 z^*)(s), z^*(s)), v - z^*(s)\rangle + \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), v) \\ &\qquad - \Psi(s, (\mathcal{H}_2 z^*)(s), z^*(s), z^*(s)) \geq \langle f(s, (\mathcal{H}_3 z^*)(s)), v - z^*(s)\rangle\end{aligned}$$

which implies that z^* is a solution of Problem 2.4. Therefore, $\Pi_{\mu,f}$ is a gap function for Problem 2.4. ☐

Using the regularized gap functions of Yamashita-Fukushima [36] in the form of $\Pi_{\mu,f}$, we now provide $\mathcal{D}$-gap function for Problem 2.4.

For $\mu > \eta > 0$ fixed, let the regularized gap functions $\Pi_{\mu,f}$ and $\Pi_{\eta,f}$ be defined by the form of (3.4). We consider the function $\mho^f_{\mu,\eta} \colon \mathbb{T} \times L^2(\mathbb{T}; P) \to \mathbb{R}$ defined by

$$\mho^f_{\mu,\eta}(s, z) = \Pi_{\mu,f}(s, z) - \Pi_{\eta,f}(s, z) \tag{3.8}$$

for all $z \in L^2(\mathbb{T}; P)$ and all $s \in \mathbb{T}$. Then we obtain the following property of $\mho^f_{\mu,\eta}$.

Lemma 3.5. *Keep the hypotheses of Theorem 2.5. Then for any $\mu > \eta > 0$, we have*

$$\|z(s) - v_{\eta,f}(z)(s)\|^2 \leq \frac{2\mu\eta}{\mu - \eta}\mho^f_{\mu,\eta}(s,z), \tag{3.9}$$

where

$$v_{\eta,f}(z)(s) = \arg\min_{v\in P} \Theta_{\eta,f}(s,z,v),$$

for all $z \in L^2(\mathbb{T};P)$ and all $s \in \mathbb{T}$.

Proof. By the definitions of the gap functions $\Pi_{\mu,f}, \Pi_{\eta,f}$ and the function $\mho^f_{\mu,\eta}$, we obtain

$$\begin{aligned}\mho^f_{\mu,\eta}(s,z) &= \sup_{v\in P}\{-\Theta_{\mu,f}(s,z,v)\} - \sup_{v\in P}\{-\Theta_{\eta,f}(s,z,v)\}\\ &\geq -\Theta_{\mu,f}(s,z,v_{\eta,f}(z)(s)) + \Theta_{\eta,f}(s,z,v_{\eta,f}(z)(s))\\ &= \left(\frac{1}{2\eta} - \frac{1}{2\mu}\right)\|z(s) - v_{\eta,f}(z)(s)\|^2.\end{aligned}$$

Therefore, the inequality in (3.9) holds. □

Theorem 3.6. *Keep the hypotheses of Theorem 2.5. Then, the function $\mho^f_{\mu,\eta}$ defined by* (3.8) *for any parameters $\mu > \eta > 0$ is a gap function to Problem 2.4.*

Proof. For any fixed parameters $\mu > \eta > 0$, we shall show that $\mho^f_{\mu,\eta}$ satisfies the conditions of Definition 3.1.

(a) It is clearly follows from (3.9) that $\mho^f_{\mu,\eta}(s,z) \geq 0$, for all $z \in L^2(\mathbb{T};P)$ and all $s \in \mathbb{T}$.

(b) Suppose that $z^* \in L^2(\mathbb{T};P)$ is a solution of Problem 2.4. It follows from Theorem 3.4 that $\Pi_{\mu,f}(s,z^*) = \Pi_{\eta,f}(s,z^*) = 0$ and so $\mho^f_{\mu,\eta}(s,z^*) = 0$ for all $s \in \mathbb{T}$.

Conversely, assume that $z^* \in L^2(\mathbb{T};P)$ is such that $\mho^f_{\mu,\eta}(s,z^*) = 0$ for all $s \in \mathbb{T}$. From (3.9), we have $z^* = v_{\eta,f}(z^*)$ in $L^2(\mathbb{T};P)$. Applying Lemma 3.3 with $z = z^*$ and $\mu = \eta$, we have

$$\begin{aligned}&\langle A(s,(\mathcal{H}_1z^*)(s),z^*(s)), v - z^*(s)\rangle + \Psi(s,(\mathcal{H}_2z^*)(s),z^*(s),v)\\ &\quad - \Psi(s,(\mathcal{H}_2z^*)(s),z^*(s),z^*(s)) \geq \langle f(s,(\mathcal{H}_3z^*)(s)), v - z^*(s)\rangle\end{aligned}$$

for all $v \in P$ and a.e. $s \in \mathbb{T}$, which implies that z^* is a solution of Problem 2.4. Thus, $\mho^f_{\mu,\eta}$ is a gap function of Problem 2.4. □

To establish error bounds for Problem 2.4 controlled by the regularized gap function $\Pi_{\mu,f}$ and the $\mathcal{D}$-gap function $\mho^f_{\mu,\eta}$, we need the following assumption in the sequel.

$\underline{\mathfrak{a}'(A)}$: For the operator $A\colon \mathbb{T}\times W\times E \to E^*$,

(i) there exist $\bar{L}_A, L_A, L'_A > 0$,

$$\|A(s_1,w_1,z_1) - A(s_2,w_2,z_2)\|_{E^*} \leq \bar{L}_A|s_1 - s_2| + L_A\|w_1 - w_2\|_W + L'_A\|z_1 - z_2\|$$

for all $s \in \mathbb{T}$, $w_1, w_2 \in W$, $z_1, z_2 \in E$, $s_1, s_2 \in \mathbb{T}$;

(ii) for all $s \in \mathbb{T}$, there exists $m_A > 0$ such that

$$\langle A(s,w,z_1) - A(s,w,z_2), z_1 - z_2\rangle_{E^*\times E} \geq m_A\|z_1 - z_2\|^2, \text{ for all } w \in W, z_1, z_2 \in E.$$

Remark 3.7. (i) It is obvious that the condition $\mathfrak{a}'(A)$(i) implies the conditions $\mathfrak{a}(A)$(i,ii,iii).

(ii) Using [20, Remark 2], it follows from the assumptions $\mathfrak{a}'(A)$(i,ii) that

$$\langle A(s,w_1,z_1) - A(s,w_2,z_2), z_1 - z_2\rangle_{E^*\times E} \geq m_A\|z_1 - z_2\|^2 - L_A\|w_1 - w_2\|_W\|z_1 - z_2\|,$$

for all $w_1, w_2 \in W$, $z_1, z_2 \in E$. Thus, the condition $\mathfrak{a}(A)$(iv) holds.

An important property to gap functions $\Pi_{\mu,f}$ and $\mho^f_{\mu,\eta}$ is presented in the following lemma:

Lemma 3.8. Assume that all the hypotheses $\mathfrak{a}'(A)$, $\mathfrak{a}(\mathcal{H})$, $\mathfrak{a}(\Psi)$, $\mathfrak{a}(P)$, $\mathfrak{a}(f)$ and $\mathfrak{a}(0)$ hold. If, in addition, P is bounded, then, for any parameters $\mu > \eta > 0$ fixed and for each fixed $z \in L^2(\mathbb{T}; P)$, the gap functions $s \mapsto \Pi_{\mu,f}(s, z)$ and $s \mapsto \mho^f_{\mu,\eta}(s, z)$ belong to $L^\infty_+(0, T)$.

Proof. For any fixed $z \in L^2(\mathbb{T}; P)$, we verify that the function $s \mapsto \Pi_{\mu,f}(s, z)$ is measurable and essentially bounded. In fact, if we can show that, for each $r \in \mathbb{R}$, the set

$$\mathcal{K}_r := \{s \in \mathbb{T} : \Pi_{\mu,f}(s, z) \leq r\} \neq \emptyset$$

is closed, then $s \mapsto \Pi_{\mu,f}(s, z)$ is measurable. Let sequence $\{s_n\} \subset \mathcal{K}_r$ be such that $s_n \to s$ in $\mathbb{T}$ as $n \to \infty$ for some $s \in \mathbb{T}$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned} r &\geq \Pi_{\mu,f}(s_n, z) \\ &\geq \langle A(s_n, (\mathcal{H}_1 z)(s_n), z(s_n)) - f(s_n, (\mathcal{H}_3 z)(s_n)), z(s_n) - v\rangle \\ &\quad - \Psi(s_n, (\mathcal{H}_2 z)(s_n), z(s_n), v) + \Psi(s_n, (\mathcal{H}_2 z)(s_n), z(s_n), z(s_n)) - \frac{1}{2\mu}\|z(s_n) - v\|^2 \end{aligned}$$

for all $v \in P$.

Passing to the lower limit as $n \to \infty$ for the inequality above and employing the continuity of $z\colon \mathbb{T} \to P$, $s \mapsto \mathcal{H}_1 z(s)$, $s \mapsto \mathcal{H}_2 z(s)$, $s \mapsto \mathcal{H}_3 z(s)$, $(s, \xi) \mapsto f(s, \xi)$, $(s, w, z) \mapsto A(s, w, z)$ and the condition $\mathfrak{a}(\Psi)(iv)$, we have

$$\begin{aligned} r &\geq \Pi_{\mu,f}(s_n, z) \\ &\geq \liminf \Big(\langle A(s_n, (\mathcal{H}_1 z)(s_n), z(s_n)) - f(s_n, (\mathcal{H}_3 z)(s_n)), z(s_n) - v\rangle \\ &\quad - \Psi(s_n, (\mathcal{H}_2 z)(s_n), z(s_n), v) + \Psi(s_n, (\mathcal{H}_2 z)(s_n), z(s_n), z(s_n)) - \frac{1}{2\mu}\|z(s_n) - v\|^2\Big) \\ &\geq \liminf \langle A(s_n, (\mathcal{H}_1 z)(s_n), z(s_n)) - f(s_n, (\mathcal{H}_3 z)(s_n)), z(s_n) - v\rangle \\ &\quad + \liminf (\Psi(s_n, (\mathcal{H}_2 z)(s_n), z(s_n), z(s_n)) - \Psi(s_n, (\mathcal{H}_2 z)(s_n), z(s_n), v)) \\ &\quad - \limsup \frac{1}{2\mu}\|z(s_n) - v\|^2 \\ &\geq \langle A(s, (\mathcal{H}_1 z)(s), z(s)) - f(s, (\mathcal{H}_3 z)(s)), z(s) - v\rangle \\ &\quad - \Psi(s, (\mathcal{H}_2 z)(s), z(s), v) + \Psi(s, (\mathcal{H}_2 z)(s), z(s), z(s)) - \frac{1}{2\mu}\|z(s) - v\|^2 \end{aligned}$$

for all $v \in P$. Taking the supremum in the above inequality with $v \in P$ leads to

$$\begin{aligned} r &\geq \sup_{v \in P} \Big(\langle A(s, (\mathcal{H}_1 z)(s), z(s)) - f(s, (\mathcal{H}_3 z)(s)), z(s) - v\rangle \\ &\quad - \Psi(s, (\mathcal{H}_2 z)(s), z(s), v) + \Psi(s_n, (\mathcal{H}_2 z)(s), z(s), z(s)) - \frac{1}{2\mu}\|z(s) - v\|^2\Big) \\ &= \Pi_{\mu,f}(s, z). \end{aligned}$$

This implies that $s \in \mathcal{K}_r$, i.e., $\mathcal{K}_r$ is closed. Therefore, the function $s \mapsto \Pi_{\mu,f}(s, z)$ is measurable on $\mathbb{T}$.

Let $z \in L^2(\mathbb{T}; P)$ be fixed. Next, we show that the function $s \mapsto \Pi_{\mu,f}(s,z)$ is uniformly bounded. By virtue of hypotheses $\mathfrak{a}'(A)$(i) and $\mathfrak{a}(\mathcal{H})$(i), we get

$$\begin{aligned}
&\langle A(s,(\mathcal{H}_1 z)(s),z(s)), z(s)-v\rangle \\
&= \langle A(s,(\mathcal{H}_1 z)(s),z(s)) - A(0,(\mathcal{H}_1 0)(s),0), z(s)-v\rangle \\
&\quad + \langle A(0,(\mathcal{H}_1 0)(s),0), z(s)-v\rangle \\
&\le \|A(s,(\mathcal{H}_1 z)(s),z(s)) - A(0,(\mathcal{H}_1 0)(s),0)\|_{E^*}\|z(s)-v\| \\
&\quad + \|A(0,(\mathcal{H}_1 0)(s),0)\|_{E^*}\|z(s)-v\| \\
&\le \left(\bar{L}_A T + L_A\|(\mathcal{H}_1 z)(s) - (\mathcal{H}_1 0)(s)\|_W + L'_A\|z(s)\|\right)\|z(s)-v\| \\
&\quad + \|A(0,(\mathcal{H}_1 0)(s),0)\|_{E^*}\|z(s)-v\| \\
&\le \left(\bar{L}_A T + L_A c_{\mathcal{H}_1}\int_0^s \|z(t)\|dt + L'_A\|z(s)\|\right)\|z(s)-v\| \\
&\quad + \|A(0,(\mathcal{H}_1 0)(s),0)\|_{E^*}\|z(s)-v\| \\
&\le \left(\bar{L}_A T + L_A c_{\mathcal{H}_1} T\|z\|_{L^2(\mathbb{T};E)} + L'_A\|z(s)\|\right)(\|z(s)\|+\|v\|) \\
&\quad + \|A(0,(\mathcal{H}_1 0)(s),0)\|_{E^*}(\|z(s)\|+\|v\|).
\end{aligned} \tag{3.10}$$

Using the condition $\mathfrak{a}(\Psi)$ and $\mathfrak{a}(\mathcal{H})$(ii), one has

$$\begin{aligned}
&\Psi(s,(\mathcal{H}_2 z)(s),z(s),z(s)) - \Psi(s,(\mathcal{H}_2 z)(s),z(s),v) \\
&\le \left(c_{\Psi_1}(s) + c_{\Psi_1}(\|z(s)\|) + c_3\|(\mathcal{H}_2 z)(s)\|_X\right)\|v-z(s)\| \\
&\le \left(c_{\Psi_1}(s) + c_{\Psi_1}(\|z(s)\|) + c_3\|(\mathcal{H}_2 0)(s)\|_X + c_3 c_{\mathcal{H}_2} T\|z\|_{L^2(\mathbb{T};E)}\right)(\|z(s)\|+\|v\|).
\end{aligned} \tag{3.11}$$

It follows from the conditions $\mathfrak{a}(f)$ and $\mathfrak{a}(\mathcal{H})$(iii) that

$$\begin{aligned}
&\langle f(s,(\mathcal{H}_3 z)(s)), v-z(s)\rangle \\
&= \langle f(s,(\mathcal{H}_3 z)(s)) - f(s,(\mathcal{H}_3 0)(s)), v-z(s)\rangle + \langle f(s,(\mathcal{H}_3 0)(s)), v-z(s)\rangle \\
&\le \|f(s,(\mathcal{H}_3 z)(s)) - f(s,(\mathcal{H}_3 0)(s))\|_{E^*}\|z(s)-v\| \\
&\quad + \|f(s,(\mathcal{H}_3 0)(s))\|_{E^*}\|z(s)-v\| \\
&\le L_f\|(\mathcal{H}_3 z)(s) - (\mathcal{H}_3 0)(s)\|_Z\|z(s)-v\| + \|f(s,(\mathcal{H}_3 0)(s))\|_{E^*}\|z(s)-v\| \\
&\le \left(L_f c_{\mathcal{H}_3} T\|z\|_{L^2(\mathbb{T};E)} + \|f(s,(\mathcal{H}_3 0)(s))\|_{E^*}\|\right)(\|z(s)\|+\|v\|).
\end{aligned} \tag{3.12}$$

Because P is bound, combining (3.10)–(3.12), we have

$$\begin{aligned}
&-\Theta_{\mu,f}(s,z,v) \\
&= \langle A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s)), z(s)-v\rangle \\
&\quad - \Psi(s,(\mathcal{H}_2 z)(s),z(s),v) + \Psi(s_n,(\mathcal{H}_2 z)(s),z(s),z(s)) - \frac{1}{2\mu}\|z(s)-v\|^2 \\
&\le \Big(\bar{L}_A T + L_A c_{\mathcal{H}_1} T\|z\|_{L^2(\mathbb{T};E)} + L'_A\|z(s)\| + \|A(0,(\mathcal{H}_1 0)(s),0)\|_{E^*} \\
&\quad + c_{\Psi_1}(s) + c_{\Psi_1}(\|z(s)\|) + c_3\|(\mathcal{H}_2 0)(s)\|_X + c_3 c_{\mathcal{H}_2} T\|z\|_{L^2(\mathbb{T};E)} \\
&\quad + L_f c_{\mathcal{H}_3} T\|z\|_{L^2(\mathbb{T};E)} + \|f(s,(\mathcal{H}_3 0)(s))\|_{E^*}\Big)(\|z(s)\|+\|v\|) \\
&\le \Big(\bar{L}_A T + L_A c_{\mathcal{H}_1} T\|z\|_{L^2(\mathbb{T};E)} + L'_A\|z\|_{L^2(\mathbb{T};E)} + \|A(0,(\mathcal{H}_1 0)(\cdot),0)\|_{L^2(\mathbb{T};E^*)} \\
&\quad + \|c_{\Psi_1}\|_{C(\mathbb{T};\mathbb{R}_+)} + c_{\Psi_1}(\|z\|_{L^2(\mathbb{T};E)}) + c_3\|(\mathcal{H}_2 0)(\cdot)\|_{L^2(\mathbb{T};X)} + c_3 c_{\mathcal{H}_2} T\|z\|_{L^2(\mathbb{T};E)} \\
&\quad + L_f c_{\mathcal{H}_3} T\|z\|_{L^2(\mathbb{T};E)} + \|f(\cdot,(\mathcal{H}_3 0)(\cdot))\|_{L^2(\mathbb{T};E^*)}\Big)(\|z\|_{L^2(\mathbb{T};E)}+\|v\|) \\
&\le \mathbf{M},
\end{aligned}$$

for all $v \in P$, where $\mathbf{M} > 0$ is independent of $s \in \mathbb{T}$ and $v \in P$. Hence, it follows from the above estimates

$$0 \leq \Pi_{\mu,f}(s,z) = \sup_{v\in P}(-\Theta_{\mu,f}(s,z,v)) \leq \mathbf{M}, \text{ for all } s \in \mathbb{T}$$

which implies that $s \mapsto \Pi_{\mu,f}(s,z)$ is essentially bounded. Hence, $s \mapsto \Pi_{\mu,f}(s,z)$ is uniformly bounded. Therefore, for any parameter $\mu > 0$ fixed, the function $s \mapsto \Pi_{\mu,f}(s,z)$ belongs to $L^\infty_+(0,T)$ for each fixed $z \in L^2(\mathbb{T};P)$. This implies that for any parameters $\mu > \eta > 0$ fixed, the function $s \mapsto \mho^f_{\mu,\eta}(s,z) := \Pi_{\mu,f}(s,z) - \Pi_{\eta,f}(s,z)$ also belongs to $L^\infty_+(0,T)$ for each fixed $z \in L^2(\mathbb{T};P)$. □

Lemma 3.9. *Assume that the hypotheses $\mathfrak{a}'(A)$, $\mathfrak{a}(\mathcal{H})$, $\mathfrak{a}(\Psi)$, $\mathfrak{a}(P_b)$, $\mathfrak{a}(f)$ and $\mathfrak{a}(0)$ hold. Let $z^* \in L^2(\mathbb{T};P)$ be the unique solution to Problem 2.4. Then, for each $z \in L^2(\mathbb{T};P)$ and $\eta > 0$, we have*

$$\mathbf{c}_0\|z(s)-z^*(s)\|^2 \leq \mathbf{c}_1\|z(s)-v_{\eta,f}(z)(s)\|^2 + \mathbf{c}_2\int_0^s \|z(t)-z^*(t)\|^2 dt, \tag{3.13}$$

for all $s \in \mathbb{T}$, where

$$\begin{cases} \mathbf{c}_0 := m_A - \frac{1}{2}\left(L'_A + \frac{1}{\eta} + 3\alpha_\Psi + \mathbf{c}_2\right); \\ \mathbf{c}_1 := \frac{1}{2}\left(L'_A + \frac{1}{\eta} + \alpha_\Psi + \mathbf{c}_2\right); \\ \mathbf{c}_2 := L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3}, \end{cases} \tag{3.14}$$

and

$$v_{\eta,f}(z)(s) = \arg\min_{v\in P} \Theta_{\eta,f}(s,z,v),$$

for all $z \in L^2(\mathbb{T};P)$ and $s \in \mathbb{T}$.

Proof. For each $z \in L^2(\mathbb{T};P)$, since $z^* \in L^2(\mathbb{T};P)$ is a solution of Problem 2.4 and $v_{\eta,f}(z) \in L^2(\mathbb{T};P)$, one has

$$\begin{aligned} &\langle A(s,(\mathcal{H}_1z^*)(s),z^*(s)) - f(s,(\mathcal{H}_3z^*)(s)), v_{\eta,f}(z)(s) - z^*(s)\rangle \\ &\quad + \Psi(s,(\mathcal{H}_2z^*)(s),z^*(s),v_{\eta,f}(z)(s)) - \Psi(s,(\mathcal{H}_2z^*)(s),z^*(s),z^*(s)) \geq 0 \end{aligned} \tag{3.15}$$

for all $s \in \mathbb{T}$.

Moreover, we add (3.3) with $\mu = \eta, v = z^*(s)$ and obtain

$$\begin{aligned} &\left\langle A(s,(\mathcal{H}_1z^*)(s),z^*(s)) - f(s,(\mathcal{H}_3z)(s)) + \frac{1}{\eta}(v_{\eta,f}(z)(s) - z(s)), z^*(s) - v_{\eta,f}(z)(s)\right\rangle \\ &\qquad + \Psi(s,(\mathcal{H}_2z)(s),z(s),z^*(s)) - \Psi(s,(\mathcal{H}_2z)(s),z(s),v_{\eta,f}(z)(s)) \geq 0 \end{aligned} \tag{3.16}$$

for all $s \in \mathbb{T}$.

Combining (3.15) and (3.16), we get

$$\begin{aligned} 0 \leq\ &\langle A(s,(\mathcal{H}_1z^*)(s),z^*(s)) - A(s,(\mathcal{H}_1z)(s),z(s)), v_{\eta,f}(z)(s) - z^*(s)\rangle \\ &+ \langle f(s,(\mathcal{H}_3z)(s)) - f(s,(\mathcal{H}_3z^*)(s)), v_{\eta,f}(z)(s) - z^*(s)\rangle \\ &+ \Psi(s,(\mathcal{H}_2z^*)(s),z^*(s),v_{\eta,f}(z)(s)) - \Psi(s,(\mathcal{H}_2z^*)(s),z^*(s),z^*(s)) \\ &+ \Psi(s,(\mathcal{H}_2z)(s),z(s),z^*(s)) - \Psi(s,(\mathcal{H}_2z)(s),z(s),v_{\eta,f}(z)(s)) \\ &+ \frac{1}{\eta}\left\langle v_{\eta,f}(z)(s) - z(s), z^*(s) - v_{\eta,f}(z)(s)\right\rangle. \end{aligned} \tag{3.17}$$

By the conditions $\mathfrak{a}'(A)$(ii,iii), $\mathfrak{a}(\mathcal{H})$(i) and Remark 3.7(ii), we have

$$\begin{aligned}
&\langle A(s,(\mathcal{H}_1z^*)(s),z^*(s))-A(s,(\mathcal{H}_1z)(s),z(s)),v_{\eta,f}(z)(s)-z^*(s)\rangle\\
=&\ \langle A(s,(\mathcal{H}_1z^*)(s),z^*(s))-A(s,(\mathcal{H}_1z)(s),z(s)),v_{\eta,f}(z)(s)-z(s)\rangle\\
&\ -\langle A(s,(\mathcal{H}_1z^*)(s),z^*(s))-A(s,(\mathcal{H}_1z)(s),z(s)),z^*(s)-z(s))\rangle\\
\le&\ \left(L_A\|(\mathcal{H}_1z^*)(s)-(\mathcal{H}_1z)(s))\|_W+L_A'\|z(s)-z^*(s)\|\right)\|v_{\eta,f}(z)(s)-z(s)\|\\
&\ +L_A\|(\mathcal{H}_1z^*)(s)-(\mathcal{H}_1z)(s))\|_W\|z(s)-z^*(s)\|-m_A\|z(s)-z^*(s)\|^2\\
\le&\ L_Ac_{\mathcal{H}_1}\int_0^s\|z(t)-z^*(t)\|dt\|v_{\eta,f}(z)(s)-z(s)\|\\
&\ +L_A'\|z(s)-z^*(s)\|\|v_{\eta,f}(z)(s)-z(s)\|\\
&\ +L_Ac_{\mathcal{H}_1}\int_0^s\|z(t)-z^*(t)\|dt\|z(s)-z^*(s)\|-m_A\|z(s)-z^*(s)\|^2.
\end{aligned}\tag{3.18}$$

Moreover, we also obtain

$$\begin{aligned}
&\frac{1}{\eta}\langle v_{\eta,f}(z)(s)-z(s),z^*(s)-v_{\eta,f}(z)(s)\rangle\\
=&\frac{1}{\eta}\langle v_{\eta,f}(z)(s)-z(s),z^*(s)-z(s)\rangle\\
&+\frac{1}{\eta}\langle v_{\eta,f}(z)(s)-z(s),z(s)-v_{\eta,f}(z)(s)\rangle\\
\le&\frac{1}{\eta}\|z(s)-z^*(s)\|\|z(s)-v_{\eta,f}(z)(s)\|-\frac{1}{\eta}\|z(s)-v_{\eta,f}(z)(s)\|^2\\
\le&\frac{1}{\eta}\|z(s)-z^*(s)\|\|z(s)-v_{\eta,f}(z)(s)\|.
\end{aligned}\tag{3.19}$$

Using the assumption $\mathfrak{a}(f)$ and $\mathfrak{a}(\mathcal{H})$(iii), one has

$$\begin{aligned}
&\langle f(s,(\mathcal{H}_3z)(s))-f(s,(\mathcal{H}_3z^*)(s)),v_{\eta,f}(z)(s)-z^*(s)\rangle\\
=&\ \langle f(s,(\mathcal{H}_3z)(s))-f(s,(\mathcal{H}_3z^*)(s)),v_{\eta,f}(z)(s)-z(s)\rangle\\
&\ +\langle f(s,(\mathcal{H}_3z)(s))-f(s,(\mathcal{H}_3z^*)(s)),z(s))-z^*(s)\rangle\\
\le&\ \|(\mathcal{H}_3z)(s)-(\mathcal{H}_3z^*)(s)\|_Z\|v_{\eta,f}(z)(s)-z(s)\|\\
&\ +\|(\mathcal{H}_3z)(s)-(\mathcal{H}_3z^*)(s)\|_Z\|z(s)-z^*(s)\|\\
\le&\ L_fc_{\mathcal{H}_3}\int_0^s\|z(t)-z^*(t)\|dt\|v_{\eta,f}(z)(s)-z(s)\|\\
&\ +L_fc_{\mathcal{H}_3}\int_0^s\|z(t)-z^*(t)\|dt\|z(s)-z^*(s)\|.
\end{aligned}\tag{3.20}$$

It follows from the hypotheses $\mathfrak{a}(\Psi)$ and $\mathfrak{a}(\mathcal{H})$(ii) that

$$\begin{aligned}
&\Psi(s,(\mathcal{H}_2z^*)(s),z^*(s),v_{\eta,f}(z)(s))-\Psi(s,(\mathcal{H}_2z^*)(s),z^*(s),z^*(s))\\
&\ +\Psi(s,(\mathcal{H}_2z)(s),z(s),z^*(s))-\Psi(s,(\mathcal{H}_2z)(s),z(s),v_{\eta,f}(z)(s))\\
\le&\ \alpha_\Psi\|z(s)-z^*(s)\|\|z^*(s)-v_{\eta,f}(z)(s)\|\\
&\ +\beta_\Psi\|(\mathcal{H}_2z^*)(s)-(\mathcal{H}_2z)(s))\|_X\|z^*(s)-v_{\eta,f}(z)(s)\|\\
\le&\ \alpha_\Psi\|z(s)-z^*(s)\|^2+\alpha_\Psi\|z(s)-z^*(s)\|\|z(s)-v_{\eta,f}(z)(s)\\
&\ +\beta_\Psi c_{\mathcal{H}_2}\int_0^s\|z(t)-z^*(t)\|dt\|z(s)-z^*(s)\|\\
&\ +\beta_\Psi c_{\mathcal{H}_2}\int_0^s\|z(t)-z^*(t)\|dt\|v_{\eta,f}(z)(s)-z(s)\|.
\end{aligned}\tag{3.21}$$

From (3.17)–(3.21), employing the inequality $ab \leq \dfrac{a^2+b^2}{2}$ for all $a, b \in \mathbb{R}_+$ and Hölders inequality gives

$$
\begin{aligned}
&(m_A - \alpha_\Psi)\,\|z(s) - z^*(s)\|^2 \\
&\leq \left(L'_A + \frac{1}{\eta} + \alpha_\Psi\right)\|z(s) - z^*(s)\|\|z(s) - v_{\eta,f}(z)(s)\| \\
&\quad + (L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3})\int_0^s \|z(t) - z^*(t)\|dt\|z(s) - v_{\eta,f}(z)(s)\| \\
&\quad + (L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3})\int_0^s \|z(t) - z^*(t)\|dt\|z(s) - z^*(s)\| \\
&\leq \frac{1}{2}\left(L'_A + \frac{1}{\eta} + \alpha_\Psi\right)(\|z(t) - z^*(t)\|^2 + \|z(s) - v_{\eta,f}(z)(s)\|^2) \\
&\quad + \frac{1}{2}(L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3})\left[\left(\int_0^s \|z(t) - z^*(t)\|dt\right)^2 + \|z(s) - v_{\eta,f}(z)(s)\|^2\right] \\
&\quad + \frac{1}{2}(L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3})\left[\left(\int_0^s \|z(t) - z^*(t)\|dt\right)^2 + \|z(s) - z^*(s)\|^2\right] \\
&\leq \frac{1}{2}\left(L'_A + \frac{1}{\eta} + \alpha_\Psi + L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3}\right)\|z(s) - z^*(s)\|^2 \\
&\quad + \frac{1}{2}\left(L'_A + \frac{1}{\eta} + \alpha_\Psi + L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3}\right)\|z(s) - v_{\eta,f}(z)(s)\|^2 \\
&\quad + (L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3})\int_0^s \|z(t) - z^*(t)\|^2 dt, \qquad (3.22)
\end{aligned}
$$

for all $s \in \mathbb{T}$. Set

$$
\begin{aligned}
\mathbf{c}_0 &:= m_A - \frac{1}{2}\left(L'_A + \frac{1}{\eta} + 3\alpha_\Psi + \mathbf{c}_2\right); \\
\mathbf{c}_1 &:= \frac{1}{2}\left(L'_A + \frac{1}{\eta} + \alpha_\Psi + \mathbf{c}_2\right); \\
\mathbf{c}_2 &:= L_A c_{\mathcal{H}_1} + \beta_\Psi c_{\mathcal{H}_2} + L_f c_{\mathcal{H}_3}.
\end{aligned}
$$

Then it follows from (3.22) that

$$
\mathbf{c}_0\|z(s) - z^*(s)\|^2 \leq \mathbf{c}_1\|z(s) - v_{\eta,f}(z)(s)\|^2 + \mathbf{c}_2\int_0^s \|z(t) - z^*(t)\|^2 dt,
$$

for all $s \in \mathbb{T}$. This implies that the inequality (3.13) holds. □

From Lemma 3.9, we obtain an error bound for Problem 2.4 controlled by the regularized gap function $\Pi_{\mu,f}$ as follows.

Theorem 3.10. *Assume that the hypotheses of Lemma 3.9 hold. Let $z^* \in L^2(\mathbb{T}; P)$ be the unique solution to Problem 2.4, $\mathbf{c}_0$, $\mathbf{c}_1$ and $\mathbf{c}_2$ be defined by (3.14). Assume furthermore that $\mathbf{c}_0 > 0$. Then, for each $z \in L^2(\mathbb{T}; P)$, we obtain*

$$
\|z(s) - z^*(s)\| \leq \mathbf{Q}_z^\Pi(s) \quad \text{for all } s \in \mathbb{T}, \qquad (3.23)
$$

where $\mathbf{Q}_z^\Pi \in L_+^\infty(0, T)$ is defined by

$$
\mathbf{Q}_z^\Pi := \sqrt{\frac{2\mu\mathbf{c}_1}{\mathbf{c}_0}\Pi_{\mu,f}(s,z) + \frac{2\mu\mathbf{c}_1\mathbf{c}_2}{\mathbf{c}_0^2}\int_0^s \Pi_{\mu,f}(t,z).exp\left\{\frac{\mathbf{c}_2}{\mathbf{c}_0}(s-t)\right\}dt} \qquad (3.24)
$$

for all $z \in L^2(\mathbb{T}; P)$ and all $s \in \mathbb{T}$.

Proof. Let $z^* \in L^2(\mathbb{T};P)$ be the unique solution to Problem 2.4. For any $z \in L^2(\mathbb{T};P)$, taking $v = z(s)$ in (3.3), we have

$$\Big\langle A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s)) + \frac{1}{\mu}(v_{\mu,f}(z)(s) - z(s)), z(s) - v_{\mu,f}(z)(s)\Big\rangle \\ + \Psi(s,(\mathcal{H}_2 z)(s),z(s),z(s)) - \Psi(s,(\mathcal{H}_2 z)(s),z(s),v_{\mu,f}(z)(s)) \geq 0,$$

that is,

$$\begin{aligned} & -\langle A(s,(\mathcal{H}_1 z)(s),z(s)) - f(s,(\mathcal{H}_3 z)(s)), v_{\mu,f}(z)(s) - z(s)\rangle \\ & \quad - \Psi(s,(\mathcal{H}_2 z)(s),z(s),v_{\mu,f}(z)(s)) + \Psi(s,(\mathcal{H}_2 z)(s),z(s),z(s)) \\ & \quad - \frac{1}{2\mu}\|z(s) - v_{\mu,f}(z)(s)\|^2 \\ & \geq \frac{1}{2\mu}\|z(s) - v_{\mu,f}(z)(s)\|^2, \end{aligned}$$

which implies that

$$-\Theta_{\mu,f}(s,z,v_{\mu,f}(z)) \geq \frac{1}{2\mu}\|z(s) - v_{\mu,f}(z)(s)\|^2. \tag{3.25}$$

It follows from (3.4) and (3.25) that

$$\|z(s) - v_{\mu,f}(z)(s)\|^2 \leq 2\mu \sup_{v\in P}(-\Theta_{\mu,f}(s,z,v)) = 2\mu\Pi_{\mu,f}(s,z), \tag{3.26}$$

for all $s \in \mathbb{T}$. Using (3.26) and taking $\eta = \mu$ in (3.13), we obtain

$$\|z(s) - z^*(s)\|^2 \leq \frac{2\mu\mathbf{c}_1}{\mathbf{c}_0}\Pi_{\mu,f}(s,z) + \frac{\mathbf{c}_2}{\mathbf{c}_0}\int_0^s \|z(t) - z^*(t)\|^2 dt$$

for all $s \in \mathbb{T}$. Invoking Gronwalls inequality for the above inequality yields

$$\|z(s) - z^*(s)\|^2 \leq \frac{2\mu\mathbf{c}_1}{\mathbf{c}_0}\Pi_{\mu,f}(s,z) + \frac{2\mu\mathbf{c}_1\mathbf{c}_2}{\mathbf{c}_0^2}\int_0^s \Pi_{\mu,f}(t,z).exp\left\{\frac{\mathbf{c}_2}{\mathbf{c}_0}(s-t)\right\}dt$$

for all $s \in \mathbb{T}$. In addition, Lemma 3.8 indicates that $s \mapsto \Pi_{\mu,f}(s,z)$ belongs to $L_+^\infty(0,T)$.

For each function $z \in L^2(\mathbb{T};P)$, let the function $\mathbf{Q}_z^\Pi\colon \mathbb{T} \to \mathbb{R}_+$ be defined by (3.24). Whereas, from Lemma 3.8, it is easy to see that $\mathbf{Q}_z^\Pi \in L_+^\infty(0,T)$. Then we can get

$$\|z(s) - z^*(s)\|_V \leq \mathbf{Q}_z^\Pi(s) \quad \text{for all } s \in \mathbb{T}.$$

Therefore, we conclude that inequality (3.23) is valid. □

The following important result in this paper is establishing the error bound for Problem 2.4 associated with the $\mathcal{D}$-gap function.

Theorem 3.11. *Assume that the hypotheses of Theorem 3.10 hold. Let $z^* \in L^2(\mathbb{T};P)$ be the unique solution to Problem 2.4, $\mathbf{c}_0$, $\mathbf{c}_1$ and $\mathbf{c}_2$ be defined by (3.14). Then, for each $z \in L^2(\mathbb{T};P)$ and $\mu > \eta > 0$, we can get the following error bound for Problem 2.4 controlled by the $\mathcal{D}$-gap function $\mho_{\mu,\eta}^f$:*

$$\|z(s) - z^*(s)\| \leq \mathbf{M}_z^\mho(s) \quad \textit{for all } s \in \mathbb{T}, \tag{3.27}$$

where $\mathbf{M}_z^\mho \in L_+^\infty(0,T)$ is defined by

$$\mathbf{M}_z^\mho(s) := \sqrt{\frac{2\mu\eta\mathbf{c}_1}{(\mu-\eta)\mathbf{c}_0}\mho_{\mu,\eta}^f(s,z) + \frac{2\mu\eta\mathbf{c}_1\mathbf{c}_2}{(\mu-\eta)\mathbf{c}_0^2}\int_0^s \mho_{\mu,\eta}^f(t,z).exp\left\{\frac{\mathbf{c}_2}{\mathbf{c}_0}(s-t)\right\}dt} \tag{3.28}$$

for all $z \in L^2(\mathbb{T};P)$ and all $s \in \mathbb{T}$.

Proof. Let $z^* \in L^2(\mathbb{T};P)$ be the unique solution to Problem 2.4. For any $z \in L^2(\mathbb{T};P)$, it follows from (3.9) and (3.13) that

$$\|z(s)-z^*(s)\|^2 \le \frac{2\mu\eta\mathbf{c}_1}{(\mu-\eta)\mathbf{c}_0}\mho^f_{\mu,\eta}(s,z) + \frac{\mathbf{c}_2}{\mathbf{c}_0}\int_0^s \|z(t)-z^*(t)\|^2 dt \tag{3.29}$$

for all $s \in \mathbb{T}$. Using Gronwalls inequality for the inequality (3.29), we have

$$\begin{aligned}&\|z(s)-z^*(s)\|^2\\ \le&\frac{2\mu\eta\mathbf{c}_1}{(\mu-\eta)\mathbf{c}_0}\mho^f_{\mu,\eta}(s,z) + \frac{2\mu\eta\mathbf{c}_1\mathbf{c}_2}{(\mu-\eta)\mathbf{c}_0^2}\int_0^s \mho^f_{\mu,\eta}(t,z).exp\left\{\frac{\mathbf{c}_2}{\mathbf{c}_0}(s-t)\right\}dt\end{aligned}$$

for all $s \in \mathbb{T}$.

For each function $z \in L^2(\mathbb{T};P)$, let the function $\mathbf{M}^{\mho}_z\colon \mathbb{T} \to \mathbb{R}_+$ be defined by (3.28). Since $s \mapsto \mho^f_{\mu,\eta}(s,z)$ belongs to $L^\infty_+(0,T)$ (Lemma 3.8), we can conclude that $\mathbf{M}^{\mho}_z \in L^\infty_+(0,T)$. Then we obtain that

$$\|z(s)-z^*(s)\| \le \mathbf{M}^{\mho}_z(s) \quad \text{for all } s \in \mathbb{T}.$$

Therefore, the inequality (3.27) is valid. □

4. Conclusions

In this paper, we have visited a class of elliptic variational inequalities involving the history-dependent operators (Problem 2.4). We have introduced a new concept of gap functions to Problem 2.4 and proposed the regularized gap function and the $\mathcal{D}$-gap function for Problem 2.4 via the optimality condition for the concerning minimization problem. Furthermore, we have derived error bounds for Problem 2.4 controlled by the regularized gap function and the $\mathcal{D}$-gap function under suitable conditions (Theorem 3.10 and Theorem 3.11). To the best of our knowledge, up to now, there is no paper devoted to $\mathcal{D}$-gap functions and their error bounds for variational inequalities involving the history-dependent operators. This paper is the first one to study $\mathcal{D}$-gap functions and their error bounds to history-dependent variational inequalities.

As future research, we intend to study of the following interesting problems: (i) descent techniques for solving Problem 2.4 based on $\mathcal{D}$-gap functions; (ii) applications to contact mechanics for Problem 2.4; (iii) developing $\mathcal{D}$-gap functions and their error bounds for history-dependent variational-hemivariational inequalities.

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