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Research Article

Lower and upper stochastic bounds for the joint stationary distribution of a non-preemptive priority retrial queueing system

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Abstract

Consider a single-server retrial queueing system with non-preemptive priority service, where customers arrive in a Poisson process with a rate of λ_1 for high-priority customers (class 1) and λ_2 for low-priority customers (class 2). If a high-priority customer is blocked, they are queued, while a low-priority customer must leave the service area and return after some random period of time to try again. In contrast with existing literature, we assume different service time distributions for the two customer classes. This investigation proposes a stochastic comparison method based on the general theory of stochastic orders to obtain lower and upper bounds for the joint stationary distribution of the number of customers at departure epochs in the considered model. Specifically, we discuss the stochastic monotonicity of the embedded Markov queue-length process in terms of both the usual stochastic and convex orders. We also perform a numerical sensitivity analysis to study the effect of the arrival rate of high-priority customers on system performance measures.

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1. Introduction

Retrial queueing systems are a type of queueing models in which customers who are unable to enter service upon arrival are placed in a retrial group (orbit). Specifically, when the server is free, an arriving customer is immediately served. However, if the server is already occupied, the customer is unable to enter service and joins the retrial group, consisting of unsatisfied customers. Any customer in the retrial group generates a stream of repeated service requests, which is independent of the other customers in the

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retrial group. Customer's retrial phenomenon is common in various real-world systems, such as manufacturing systems, computer systems, wireless sensor networks, call centers, telecommunication networks, web access, switching networks, cognitive radio network, etc. [27,33,40,44,52]. Therefore, retrial queue analysis becomes a crucial problem in queueing theory. A comprehensive review on this topic can be found in [4–6,21–29,39,45,50,53].

Retrial queues with various types of customers have been extensively studied in the literature [21, 29, 39, 41, 42, 45]. Priority discipline, which involves preemptive and non-preemptive concepts, plays a crucial role in these queues [36]. Preemptive priority enables customers with higher priority to receive immediate service, even if a lower priority customer is currently being served. In this case, the lower priority service is preempted and restarted after the higher priority service is completed. However, non-preemptive priority allows higher priority customers to move to the front of the queue but must wait until the service of the lower priority customer is finished. Priority retrial queues have various applications, including real-time systems, operating systems, manufacturing systems, simulation and medical service systems, cognitive radio systems, and computer systems [22, 40, 46]. For a review of main results and methods, the reader is referred to [3, 6, 18–20, 28, 30–32, 36, 49, 51].

In this paper, we are interested in analyzing an $M_2/G_2/1$ queueing system with non-preemptive priority at which in the case of blocking, the high-priority of customers can be queued whereas the lower type of customers has to leave the service area and return after some random period of time to retry their service. The contributions and advantages of this paper are stated as follows:

1. **Queueing model :** The queueing model under consideration was proposed by [24] as a natural extension of the classic $M/G/1$ retrial queue. However, this extension introduces several complications. For example, the results obtained from the model are complex and involve integrals of Laplace transforms, solutions of functional equations, and so on. Unfortunately, these results are not easily exploitable from an application point of view.
2. **Methodology :** Due to the priority mechanism, finding analytic results for the system becomes more complicated. Such results are typically only available in the form of Laplace transforms and generating functions, which cannot be used in practice due to their complexity. To overcome this problem, we propose a monotonicity approach that can be investigated using the stochastic comparison method based on the general theory of stochastic orders. The stochastic comparison method allows us to produce approximations and bounds for various performance measures of the system, enabling us to study changes in performance due to parameter variations. The proposed approach is of interest as it allows for a trade-off between the role of qualitative bounds and the complexity involved in resolving intricate systems with some parameters that are not perfectly known. Specifically, our approach seeks to reveal the relationship between performance measures and the parameters of the system. Notably, the stochastic comparison method has been successfully applied in various queueing models, including multi-class queueing systems [48], risk models [38], and performance evaluation of mobile networks [17, 37]. Additionally, it has been employed in studies of partially ordered spaces [35] and applied probability [47]. Several previous works have also applied the stochastic comparison method to queueing models, including [1, 2, 7–16, 34].
3. **Motivation and main results :** Our study is motivated by the need to establish insensitive bounds on the performance measures of retrial queueing systems with non-preemptive priority service, where high-priority customers can be queued but low-priority customers must leave the service area and try again later. While previous research has focused on quantitative approaches to analyzing such systems (see [24]),

we propose a qualitative analysis in order to study the monotonicity properties of the embedded Markov chain relative to stochastic and convex orderings.

By examining the relationship between performance measures and system parameters, our approach offers insights that complement existing quantitative methods. Specifically, it provides a compromise between the role of qualitative bounds and the complexity of resolving complicated systems where some parameters may not be perfectly known. In this paper, we address two problems related to the embedded Markov chain in the suggested queuing system. Firstly, we establish conditions for the monotonicity of the transition operator associated with the embedded Markov chain. Secondly, we compare the transition operators and joint stationary distributions of two embedded Markov chains, both having the same structure but different parameters. We provide best insensitive stochastic bounds for the stationary distribution of the embedded Markov chain, using partial information on the ageing concepts of the service time distributions. To complement our theoretical study, we also use a discrete event simulation approach to analyze the impact of non-preemptive priority customer arrival rate on system performance measures.

The results given in our research article may be employed to improve the functioning of different real systems. Let us, in short, mention some of them.

- **Wireless sensor networks :** These systems have the capability to harvest energy from external sources such as solar, radio frequency, or wind, which is then stored in the form of energy units. To transmit one unit of information, multiple energy units are required. The sensor in this system transmits two types of information units to the central node: type 1 information units, which are of primary importance and correspond to events such as intrusions or critical equipment failures, and type 2 information units, which provide routine information about the object's parameters. If a type 1 information unit arrives, the sensor interrupts the transmission of routine information and switches to transmitting the important information. Thus, type 1 units have preemptive priority over type 2 units. Additionally, if a type 2 unit cannot be transmitted upon arrival, it can be regenerated later (cf. [43]).
- **Call centers :** These systems are modeled as retrial queues with two types of calls: customers can contact the call centers either over the phone (type 1 customers) or via email (type 2 customers). In this system, higher priority is given to more important customers or voice requests, while lower priority is attributed to less important customers or email requests (cf. [33]).
- **Cognitive radio systems:** Cognitive radio systems are a form of wireless communication where a transceiver can intelligently detect which communication channels are in use and which are not. In these systems, higher priority is attributed to primary users (licensed customers), while lower priority is given to secondary users (unlicensed customers). Arrival of primary users interrupts the service of secondary users (cf. [22, 40, 52]).

The remainder of this paper is structured as follows. In the next section, we briefly describe the queueing model, and the embedded Markov chain is given. In Section 3, we introduce some definitions of univariate and multivariate stochastic orders, as well as ageing notions. Notations and some preliminary results are presented in Section 4. Section 5 focuses on various monotonicity properties of the transition operator of the Markov chain. In Section 6, comparability conditions of two embedded Markov chains are given. Lower and upper stochastic bounds for the joint stationary distribution at a departure epoch are discussed in Section 7. In Section 8, numerical results are provided to illustrate the obtained theoretical results. Section 9 is devoted to the proof of the main results. Finally, in Section 10, the paper is concluded.

2. Queueing model description

Consider a non-preemptive priority retrial queue with a normal queue (high-priority) and an orbit (low-priority). The server offers service to two types of customers viz: high-priority (class 1) and low-priority (class 2) customers. Customers arrive in a Poisson process with arrival rate λ_1 for high-priority customers (class 1) and λ_2 for low-priority customers (class 2). We assume different service time distributions for the high and low-priority customers. The main assumptions required for the formulation of our queueing model are as follows :

- ▷ Customers are divided into two classes: high-priority (class 1) and ordinary (class 2). Upon arrival, high-priority customers are immediately served if a server is available; otherwise, they join the priority queue. Ordinary customers, on the other hand, are sent back to the retrial orbit if they find the server busy with priority customers. If a high-priority customer arrives and finds the server occupied with an ordinary customer, they must wait until the service of the latter is finished, as the service of ordinary customers cannot be interrupted by high-priority customers under the non-preemptive priority scheme. Only when all priority customers have been served may a new ordinary customer request service.
- ▷ Ordinary customers are treated as retrial customers. When a server is available, the first customer in the queue is served immediately, while any remaining customers move to a waiting area called the orbit. If a class 2 customer is not served upon their first attempt, they return to the orbit and wait for the server to become available again. This process is repeated until the customer is finally served. The retrial times are assumed to be independent and exponentially distributed with parameter θ ($\theta > 0$).
- ▷ Both priority and ordinary customers are served with a service process that has a general distribution. The service times of priority customers and ordinary customers are independent and identically distributed, with a general distribution function denoted as $B_1(x)$ and $B_2(x)$, respectively. The Laplace-Stieltjes transform (LST) of $B_1(x)$ is denoted as $\widetilde{B}_1(s)$, and the first moment of $B_1(x)$ is as β_1^1. The load of priority customers is $\rho_1 = \lambda_1\beta_1^1$. Similarly, the LST of $B_2(x)$ is denoted as $\widetilde{B}_2(s)$, the first moment of $B_2(x)$ is β_1^2, and the load of ordinary customers is given by $\rho_2 = \lambda_2\beta_1^2$.

Note that inter-arrival times of primary customers, intervals between repeated trials, and service times are assumed to be mutually independent.

At time t, the state of the system can be described by the continuous time stochastic process

$$X(t) = (A(t), C(t), N(t), \xi(t))_{t\geq 0},$$

where $A(t)$ is the type of the customers in service, $C(t)$ represents the number of customers in queue (excluding the customer in service), $N(t)$ denotes the number of customers in orbit, and $\xi(t)$ is the corresponding elapsed time. The transition diagram is illustrated in Figure 1.

Remark 2.1. Note that $A(t) = 0$ when no customer is in service at time t, in this case we have $C(t) = 0$.

Let η_d be the time of the d^{th} departure. Clearly, a sequence of random vectors $X_d = (A(\eta_d - 0), C(\eta_d - 0), N(\eta_d - 0))$ forms a Markov chain, which is the embedded Markov chain for our queueing system. Its state space is $\{1, 2\} \times \mathbf{Z}_+^2$ and its one-step transition probabilities

$$r_{(k,n,m)(l,i,j)} = P\{X_{d+1} = (l, i, j)|X_d = (k, n, m)\}.$$

The embedded Markov chain X_d is ergodic if and only if $\rho = \rho_1 + \rho_2 < 1$.

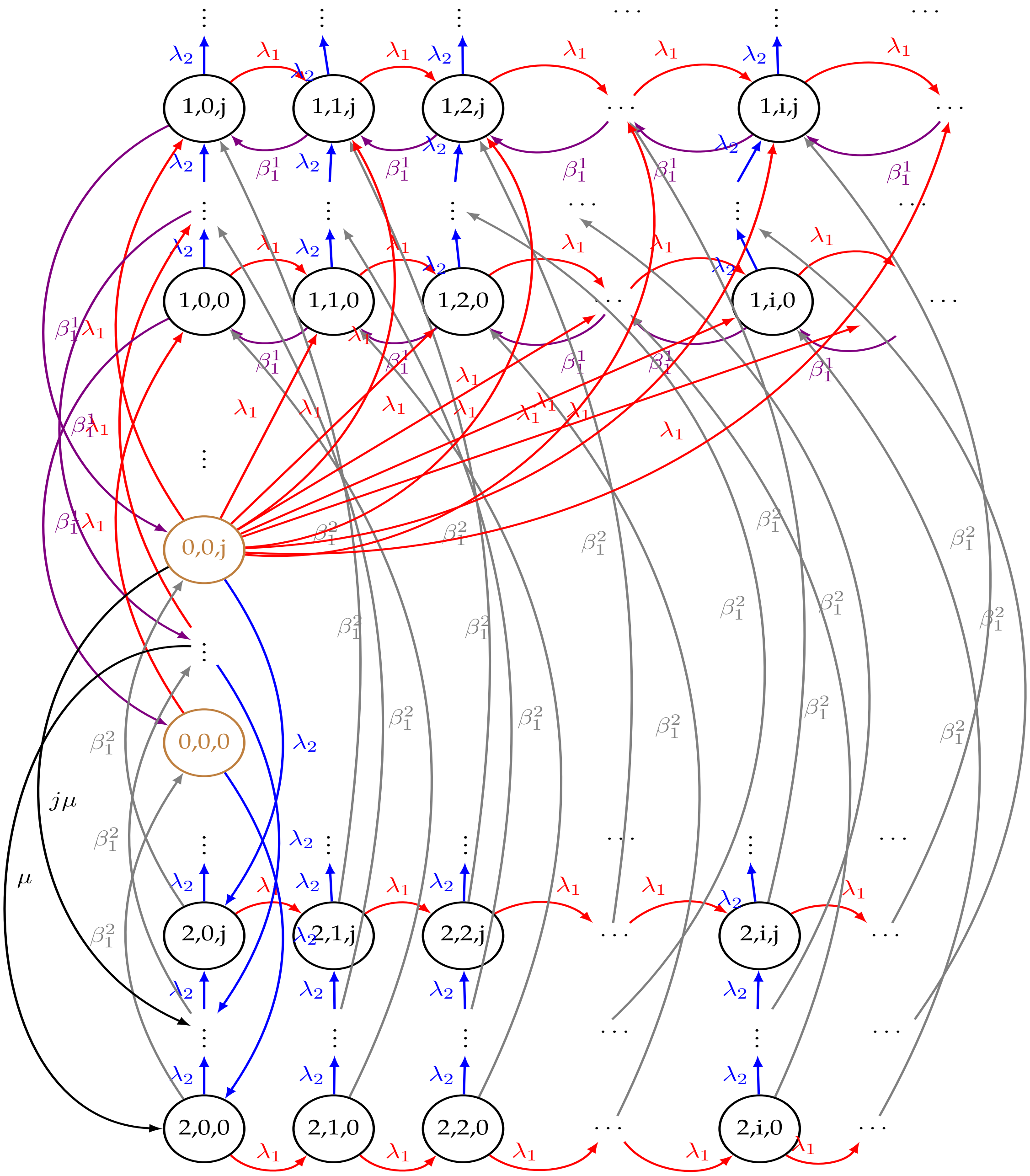

Figure 1. The state transition diagram

3. Some stochastic orders

Stochastic ordering is a valuable tool in analyzing changes in performance resulting from variations in system parameters, comparing different systems, approximating complex systems with simpler ones, and obtaining bounds for key performance measures. This paper aims to explore several stochastic orders and ageing concepts that are particularly relevant to the main results presented. The following sections provide a brief overview of these key concepts, which will be used extensively throughout the paper. For further reading, see [35, 38, 47], and the references therein.

Definition 3.1. Let X and Y be two non-negative random variables with distribution functions F and G, respectively. X is said to be smaller than Y with respect to

(1) Stochastic order ($\leq_{st}$):

$$X \leq_{st} Y \Leftrightarrow F(x) \geq G(x), \forall x \geq 0.$$

(2) Convex order ($\leq_v$):

$$X \leq_v Y \Leftrightarrow \int_x^\infty (1-F(t))dt \leq \int_x^\infty (1-G(t))dt, \forall x \geq 0.$$

(3) Laplace ordering ($\leq_L$):

$$X \leq_L Y \Leftrightarrow \int_0^{+\infty} e^{-sx} dF(x) \geq \int_0^{+\infty} e^{-sx} dG(x), \forall s \geq 0.$$

If the random variables of interest are discrete and $\omega = (\omega_n)_{n\geq 0}$, $\nu = (\nu_n)_{n\geq 0}$ are their corresponding distributions. Then

(1) $\omega \leq_{st} \nu$ iff $\overline{\omega}_m = \sum_{n\geq m} \omega_n \leq \overline{\nu}_m = \sum_{n\geq m} \nu_n$, $\forall m$.
(2) $\omega \leq_v \nu$ iff $\overline{\overline{\omega}}_m = \sum_{n\geq m} \sum_{k\geq n} \omega_k \leq \overline{\overline{\nu}}_m = \sum_{n\geq m} \sum_{k\geq n} \nu_k$, $\forall m$.
(3) $\omega \leq_L \nu$ iff $\sum_{n\geq 0} \omega_n z^n \geq \sum_{n\geq 0} \nu_n z^n$, $\forall z \in [0,1]$.

3.1. Some multivariate extensions

In this section, we recall some multivariate extensions of the stochastic orders that were discussed in the previous section.

Definition 3.2. Given two random vectors X and Y, we say that X is less than Y in:

(1) Multivariate stochastic order iff $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$,
(2) Multivariate increasing convex order iff $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$,
(3) Multivariate Laplace ordering iff $\mathbb{E}[\exp\{-S^T X\}] \geq \mathbb{E}[\exp\{-S^T Y\}]$,

for all $S \in \mathbb{R}^n_+$ and for all increasing function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, such that the previous expectations exist.

Proposition 3.3. *Let X be a random variable with distribution function F and finite mean m.*

(a) F is New Better than Used in Expectation (NBUE) iff $F \leq_v F^*$,
(b) F is New Worse than Used in Expectation (NWUE) iff $F^* \leq_v F$,

where F^* is the exponential distribution function with the same mean as F.

4. Preliminary comparison results

In this section, we introduce useful lemmas in establishing the main results of the paper. Consider two $M_2/G_2/1$ retrial queueing systems with non-preemptive priority service with the parameters $\lambda_1^{(p)}$, $\lambda_2^{(p)}$, $\theta^{(p)}$, $B_1^{(p)}(x)$, and $B_2^{(p)}(x)$, where $p = 1,2$. Let $a_{ij}^{k,(1)}$ and $a_{ij}^{k,(2)}$ denote the probability that i priority and j ordinary customers arrive at the system during a service interval of type k, $(k = 1,2)$, where

$$a_{ij}^{k,(p)} = \int_0^\infty \frac{(\lambda_1^{(p)} x)^i}{i!} e^{-\lambda_1^{(p)} x} \frac{(\lambda_2^{(p)} x)^j}{j!} e^{-\lambda_2^{(p)} x} dB_k^{(p)}(x), \quad k = 1,2, \quad p = 1,2.$$

Here, we employ the general theory of stochastic orderings in order to study the monotonicity properties for our system, relative to stochastic, convex, and Laplace orderings, respectively.

Lemma 4.1. *If $\lambda_1^{(1)} \leq \lambda_1^{(2)}$, $\lambda_2^{(1)} \leq \lambda_2^{(2)}$ and $B_k^{(1)} \leq_{st} B_k^{(2)}$, then $\left\{a_{ij}^{k,(1)}\right\} \leq_{st} \left\{a_{ij}^{k,(2)}\right\}$.*

Lemma 4.2. *If $\lambda_1^{(1)} \leq \lambda_1^{(2)}$, $\lambda_2^{(1)} \leq \lambda_2^{(2)}$ and $B_k^{(1)} \leq_v B_k^{(2)}$, then $\left\{a_{ij}^{k,(1)}\right\} \leq_v \left\{a_{ij}^{k,(2)}\right\}$.*

Lemma 4.3. *If $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$, $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$, $B_k^{(1)} \leqslant_L B_k^{(2)}$, then $\left\{a_{ij}^{k,(1)}\right\} \leqslant_L \left\{a_{ij}^{k,(2)}\right\}$.*

5. Monotonicity properties of the Markov chain $\{X_d\}$

In this section, we study the monotonicity properties of the transition operator of the embedded Markov chain relative to the stochastic ordering and the convex ordering.

Let Θ be the transition operator of our embedded Markov chain $\{X_d\}$, which associates to every distribution $\phi = (\phi_{(k,n,m)})$, a distribution $\Theta\phi = \delta_{(l,i,j)}, l = 1, 2$, such that

$$\delta_{(l,i,j)} = \sum_{i\geq 0}\sum_{j\geq 0} \phi_{(k,n,m)} r_{(k,n,m)(l,i,j)},$$

where $r_{(k,n,m)(l,i,j)} = P\{X_{d+1} = (l,i,j)|X_d = (k,n,m)\}$ are one-step transition probabilities of the considered Markov chain given by the formulas:

$$r_{(k,n,m)(1,i,j)} = \begin{cases} \frac{\lambda_1}{\lambda_1+\lambda_2+m\theta} a^{(1)}_{i,j-m}, & \text{if } n = 0, \\ a^{(1)}_{i-n+1,j-m}, & n \geq 1, \end{cases}$$

$$r_{(k,n,m)(2,i,j)} = \begin{cases} \frac{\lambda_2}{\lambda_1+\lambda_2+m\theta} a^{(2)}_{i,j-m} + \frac{m\theta}{\lambda_1+\lambda_2+m\theta} a^{(2)}_{i,j-m+1}, & \text{if } n = 0, \\ 0, & n \geq 1. \end{cases}$$

- The transition operator Θ is monotone with respect to $\leq_{st}$ if and only if

$$\overline{r}_{(k,n-1,m-1)(l,i,j)} \leq \overline{r}_{(k,n,m)(l,i,j)} \text{ for all } i,\ j > 0 \text{ and } l = 1, 2. \tag{5.1}$$

- The transition operator Θ is monotone with respect to $\leq_v$ if and only if

$$2\overline{\overline{r}}_{(k,n,m)(l,i,j)} \leq \overline{\overline{r}}_{(k,n-1,m-1)(l,i,j)} + \overline{\overline{r}}_{(k,n+1,m+1)(l,i,j)}, \forall i, j, \text{ and } l = 1, 2. \tag{5.2}$$

Here, we defined

$$\overline{r}_{(k,n,m)(l,i,j)} = \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} r_{(k,n,m)(l,s,r)}.$$

$$\overline{\overline{r}}_{(k,n,m)(l,i,j)} = \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{r}_{(k,n,m)(l,s,r)}.$$

The following Theorem presents the monotonicity condition of the transition operator Θ with respect to stochastic orderings.

Theorem 5.1. *The transition operator of the embedded Markov chain $\{X_d\}$ is monotone with respect to the order $\leq_{so}$, that is, for any two distributions $\phi^{(1)}$ and $\phi^{(2)}$, the inequality $\phi^{(1)} \leq_{so} \phi^{(2)}$ implies that $\Theta\phi^{(1)} \leq_{so} \Theta\phi^{(2)}$, where $\leq_{so}=\leq_{st}$ or $\leq_v$.*

6. Comparability bounds of two embedded Markov chains

In this section, we incorporate the transition operators $\Theta^{(1)}$ and $\Theta^{(2)}$ into models $\Sigma^{(1)}$ and $\Sigma^{(2)}$, respectively. The following theorem provides comparability conditions for two embedded Markov chains.

Theorem 6.1. *If $\lambda_1^{(1)} \leq \lambda_1^{(2)}$, $\lambda_2^{(1)} \leq \lambda_2^{(2)}$, $\theta^{(1)} \geq \theta^{(2)}$, $B_1^{(1)} \leq_{so} B_1^{(2)}$ and $B_2^{(1)} \leq_{so} B_2^{(2)}$ then $\Theta^{(1)} \leq_{so} \Theta^{(2)}$, that is, for any distribution ϕ, one has $\Theta^{(1)}\phi \leq_{so} \Theta^{(2)}\phi$, where $\leq_{so}$ is either $\leq_{st}$ or $\leq_v$.*

7. Lower and upper stochastic bounds for the joint stationary distribution

Now, we establish the comparability conditions of the joint stationary distributions of the number of customers for two $M_2/G_2/1$ retrial queueing systems with non-preemptive priority service and different parameters, with respect to stochastic orders.

Theorem 7.1. *If $\lambda_1^{(1)} \leq \lambda_1^{(2)}$, $\lambda_2^{(1)} \leq \lambda_2^{(2)}$, $\theta^{(1)} \geq \theta^{(2)}$, $B_1^{(1)} \leq_{so} B_1^{(2)}$ and $B_2^{(1)} \leq_{so} B_2^{(2)}$ then joint stationary distribution $\{\pi_{ij}^{(1)}\} \leq_{so} \{\pi_{ij}^{(2)}\}$, where $\leq_{so}=\leq_{st}$ or $\leq_v$.*

From Theorem 7.1, we can establish insensitive stochastic inequalities for the joint stationary distribution of the number of customers in the system at a departure epochs.

Theorem 7.2. *Assume that for a single server retrial queue with non-preemptive priority customers, the service time distribution of the priority customers (class 1) $B_1(x)$ and the service time distribution of ordinary customers (class 2) $B_2(x)$ are NBUE (or NWUE), then the joint stationary distribution π_n of the number of customers in the system at a departure epoch is less (resp. greater), with respect to the convex ordering, than the joint stationary distribution $\pi_n^{\exp}$ of the number of customers at a departure epochs in the $M_2/M_2/1$ retrial queue with non-preemptive priority customers.*

8. Numerical illustration

In this section, we provide numerical examples to validate the results obtained in Theorem 7.2 using tables and graphs. We develop a simulator, under Matlab environment, based on discrete event simulation, to investigate the performance measures of the $M_2/G_2/1$ retrial queue with non-preemptive priority customers and to analyze the impact of the arrival rate of priority customers (λ_1) on the system's performance measures. The simulator is capable of estimating the joint stationary probabilities π_n of the system when the service time distributions of the priority customers ($B_1(x)$) and the ordinary customers ($B_2(x)$) are either NBUE (New Better than Used in Expectation) or NWUE (New Worse than Used in Expectation). We compare the outcomes with those of an $M_2/M_2/1$ retrial queue with non-preemptive priority customers relative to the convex ordering.

Table 1. Different situations taken into consideration during the simulation study.

	π_{lower}^{NBUE}		$\pi_n^{\exp}$	π_{upper}^{NWUE}	
λ_1	E_2	$Wbl(\alpha,\beta)$	$\exp(\lambda)$	H_2	$\Gamma(\alpha,\beta)$
	0.6543	0.7027	0.7941	0.8017	0.8777
	0.1772	0.1046	0.1813	0.1867	0.3273
0.1	0.0156	0.0173	0.0181	0.0962	0.0233
	0.0003	0.0004	0.0014	0.0141	0.0055
	0.4821	0.5019	0.5365	0.5967	0.6996
	0.2554	0.2665	0.3334	0.3682	0.3862
0.3	0.0418	0.0766	0.0863	0.1003	0.1482
	0.0033	0.0169	0.0243	0.0288	0.0325
	0.3472	0.3684	0.4196	0.4870	0.5443
	0.3377	0.3355	0.3993	0.4123	0.4122
0.5	0.1036	0.0928	0.1302	0.2135	0.1789
	0.0136	0.0185	0.0307	0.0630	0.0540
	0.1732	0.1855	0.1876	0.2122	0.2258
0.7	0.0732	0.0780	0.0932	0.1555	0.1283
	0.0174	0.0327	0.0314	0.0731	0.0553
	0.0041	0.0085	0.0109	0.0141	0.0184
	0.2144	0.2151	0.2407	0.2413	0.2931
0.9	0.1648	0.1890	0.1950	0.1962	0.2002
	0.0930	0.0854	0.1152	0.1324	0.1395
	0.0432	0.0265	0.0403	0.0612	0.0633

To this end, we choose two probability laws of NBUE-type to calculate the lower bound of the joint stationary distribution π_{lower}^{NBUE}, namely, a Weibull distribution ($Wbl(\alpha,\beta)$; $\alpha > 1$) and a two-step Erlang distribution (E_2). Furthermore, we select two other probability laws of NWUE-type to compute the lower bound of the joint stationary distribution

π_{upper}^{NWUE}, namely a Two-Stage Hyper-Exponential distribution (H_2) and a Gamma distribution ($\Gamma(\alpha,\beta);\;\; 0<\alpha<1$).

We provide a numerical example for a wide range of parameter values that satisfy the ergodicity condition $\rho=\rho_1+\rho_2<1$. Moreover, we assume that the arrival rate of the ordinary customers is fixed at $\lambda_2=0.5$, and the retrial rate is $\theta=0.5$. Table 1 summarizes the considered scenarios in the simulation study.

Discussion of the results :

According to Table 1 and Figures 2–5, for different values of λ_1, we have:

- The joint stationary distribution π_n of the number of customers in the system at a departure epochs in the $M_2/G_2/1$ retrial queue with non-preemptive priority customers is greater (resp. lower) than the stationary distribution of the number of customers in the $M_2/M_2/1$ retrial queue with non-preemptive priority customers, where the service time distribution is NBUE (resp. NWUE). Succinctly, the following inequality holds:

$$\left\{\pi_{lower}^{NBUE}\right\}\leq_v\left\{\pi_n^{\exp}\right\}\leq_v\left\{\pi_{upper}^{NWUE}\right\}.$$

  The obtained results perfectly match with those given in Theorem 7.2. In other words, these results give insensitive bounds for the joint stationary distribution of the considered embedded Markov chain.
- The impact of the arrival rate of priority customers (λ_1) on the joint stationary distribution of the number of customers at departure epochs is illustrated through Figures 2–5. The results show that λ_1 has a significant influence on the behavior of the bounds. Specifically, an increase in λ_1 results in an increase in the bounds. It should be noted that the lower and upper stochastic bounds presented in Theorem 7.2 are dependent on the value of λ_1.
- The behavior of the $M_2/G_2/1$ retrial queueing system with priority customers is compared to an $M_2/M_2/1$ retrial queue with non-preemptive priority customers in Figures 4 and 5. The results show that when the arrival rate of priority customers, λ_1, is large (i.e., λ_1 close to 1), the behavior of the $M_2/G_2/1$ retrial queueing system with priority customers is similar to that of an $M_2/M_2/1$ retrial queue with non-preemptive priority customers. On the other hand, when λ_1 tends to 0, the system moves away from an $M_2/M_2/1$ retrial queue with non-preemptive priority customers, as illustrated in Figures 2 and 3. Consequently, the stochastic bounds presented in this study can provide a good approximation of the joint stationary probabilities of the considered model, regardless of the distribution of service times (i.e., NBUE or NWUE).

Remark 8.1. The qualitative bounds obtained in this study have important implications for robustness analysis. Specifically, our results provide valuable information on the extent to which deviations from the nominal model can be expected in the presence of input uncertainty. Theorem 7.2, in particular, provides a uniform bound on the effect of model insecurity.

In the context of gradient estimation, it is necessary to control the growth of the cycle length as a function of changes in the model. The results established in this paper may facilitate the derivation of quantitative estimates for the stationary distribution using measure valued derivatives (MVD). Furthermore, these results can be translated into unbiased higher-order derivative estimators with respect to specific parameters, such as the arrival rate of priority customers (λ_1). However, it should be noted that the primary focus of this paper is not on the derivation of such estimators.

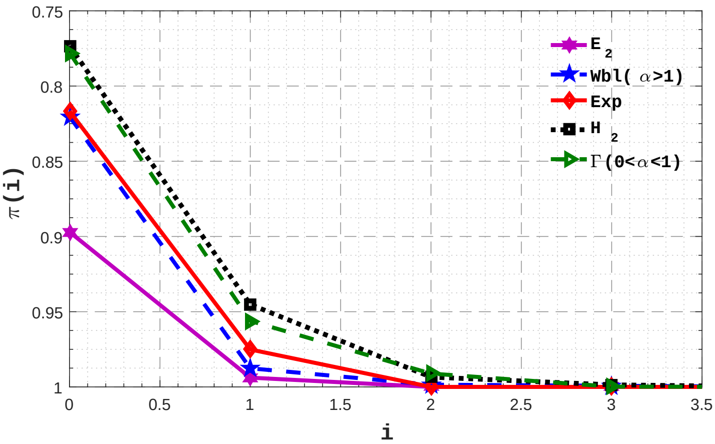

Figure 2. Lower and upper stochastic bounds for the joint stationary distribution when $\lambda_1 = 0.1$.

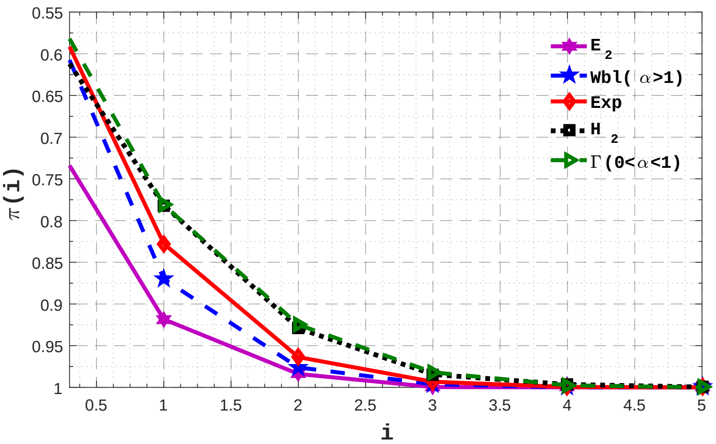

Figure 3. Lower and upper stochastic bounds for the joint stationary distribution when $\lambda_1 = 0.3$.

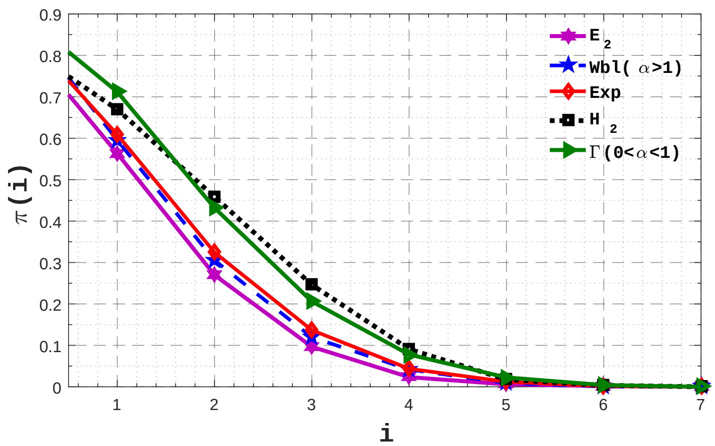

Figure 4. Lower and upper stochastic bounds for the joint stationary distribution when $\lambda_1 = 0.7$.

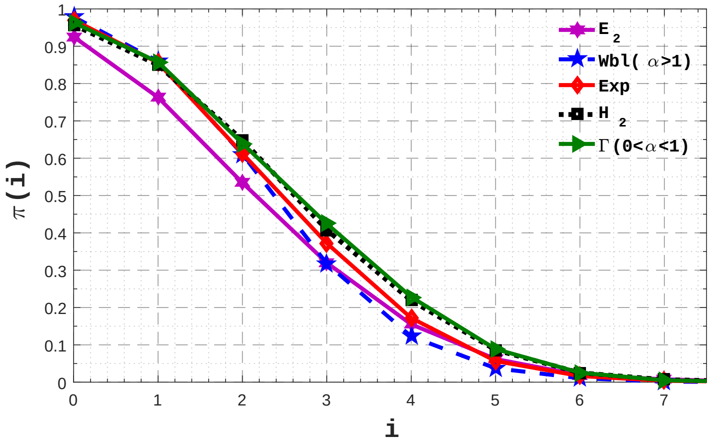

Figure 5. Lower and upper stochastic bounds for the joint stationary distribution when $\lambda_1 = 0.9$.

9. Technical proofs of the results

Proof of Lemma 4.1. By definition of the stochastic order $\leq_{st}$, we have for a discrete law, the following equivalence

$$\left\{a_{ij}^{k,(1)}\right\} \leq_{st} \left\{a_{ij}^{k,(2)}\right\} \Leftrightarrow \overline{a_{ij}^{k}}^{(1)} = \sum_{m=i}^{\infty}\sum_{n=j}^{\infty} a_{m,n}^{k,(1)} \leq \sum_{m=i}^{\infty}\sum_{n=j}^{\infty} a_{m,n}^{k,(2)} = \overline{a_{ij}^{k}}^{(2)}.$$

Equivalently, for $k = 1, 2$.

$$\begin{aligned}
\overline{a_{ij}^{k}}^{(1)} &= \sum_{m=i}^{\infty}\sum_{n=j}^{\infty} \int_0^{\infty} \frac{\left(\lambda_1^{(1)}x\right)^m}{m!} \exp(-\lambda_1^{(1)}x) \frac{\left(\lambda_2^{(1)}x\right)^n}{n!} \exp(-\lambda_2^{(1)}x) dB_k^{(1)}(x) \\
&= \int_0^{\infty} \sum_{m=i}^{\infty}\sum_{n=j}^{\infty} \frac{\left(\lambda_1^{(1)}x\right)^m}{m!} \exp(-\lambda_1^{(1)}x) \frac{\left(\lambda_2^{(1)}x\right)^n}{n!} \exp(-\lambda_2^{(1)}x) dB_k^{(1)}(x) \\
&= \int_0^{\infty} \left[\sum_{m=i}^{\infty} \frac{\left(\lambda_1^{(1)}x\right)^m}{m!} \exp(-\lambda_1^{(1)}x)\right] \left[\sum_{n=j}^{\infty} \frac{\left(\lambda_2^{(1)}x\right)^n}{n!} \exp(-\lambda_2^{(1)}x)\right] dB_k^{(1)}(x) \\
&\leq \int_0^{\infty} \left[\sum_{m=i}^{\infty} \frac{\left(\lambda_1^{(2)}x\right)^m}{m!} \exp(-\lambda_1^{(2)}x)\right] \left[\sum_{n=j}^{\infty} \frac{\left(\lambda_2^{(2)}x\right)^n}{n!} \exp(-\lambda_2^{(2)}x)\right] dB_k^{(2)}(x) \\
&= \overline{a_{ij}^{k}}^{(2)}.
\end{aligned} \tag{9.1}$$

Next, to prove inequality (9.1), we consider the following two functions

$$h_i(x, \lambda_1) = \sum_{m=i}^{\infty} \frac{(\lambda_1 x)^m}{m!} \exp(-\lambda_1 x), \quad \text{and} \quad g_j(x, \lambda_2) = \sum_{n=j}^{\infty} \frac{(\lambda_2 x)^n}{n!} \exp(-\lambda_2 x).$$

By taking the derivatives of the function $h_i(x,\lambda_1)$ with respect to x and λ_1, we get

$$\begin{aligned}\frac{\partial h_i(x,\lambda_1)}{\partial x} &= \sum_{m=i}^{\infty}\lambda_1 m\frac{(\lambda_1 x)^{m-1}}{m\,(m-1)!}\exp(-\lambda_1 x)-\sum_{m=i}^{\infty}\lambda_1\frac{(\lambda_1 x)^{m}}{m!}\exp(-\lambda_1 x)\\ &= \sum_{m=i}^{\infty}\lambda_1\frac{(\lambda_1 x)^{m-1}}{(m-1)!}\exp(-\lambda_1 x)-\sum_{m=i}^{\infty}\lambda_1\frac{(\lambda_1 x)^{m}}{m!}\exp(-\lambda_1 x)\\ &= \lambda_1\frac{(\lambda_1 x)^{i-1}}{(i-1)!}e^{-\lambda_1 x}+\sum_{m=i+1}^{\infty}\lambda_1\frac{(\lambda_1 x)^{m-1}}{(m-1)!}e^{-\lambda_1 x}-\sum_{m=i}^{\infty}\lambda_1\frac{(\lambda_1 x)^{m}}{m!}e^{-\lambda_1 x}\\ &= \lambda_1\frac{(\lambda_1 x)^{i-1}}{(i-1)!}\exp(-\lambda_1 x)>0,\ \ \forall x>0.\end{aligned}$$

In the same way, we have

$$\frac{\partial h_i(x,\lambda_1)}{\partial \lambda_1}=x\frac{(\lambda_1 x)^{i-1}}{(i-1)!}\exp(-\lambda_1 x)>0.$$

Clearly, the function $h_i(x,\lambda_1)$ is an increasing function in x and λ_1.

Following the same procedure as above, we compute the derivatives of $g_j(x,\lambda_2)$ with respect to x and λ_2. So,

$$\begin{aligned}\frac{\partial g_j(x,\lambda_2)}{\partial x} &= \lambda_2\frac{(\lambda_2 x)^{j-1}}{(j-1)!}\exp(-\lambda_2 x)>0,\ \ \forall x>0,\\ \frac{\partial g_j(x,\lambda_2)}{\partial \lambda_2} &= x\frac{(\lambda_2 x)^{j-1}}{(j-1)!}\exp(-\lambda_2 x)>0,\ \ \forall x>0.\end{aligned}$$

Remark that the derivatives of the functions $h_i(x,\lambda_1)$ and $g_j(x,\lambda_2)$ are positive for all the positive values of λ_1 and λ_2, thus the functions $h_i(x,\lambda_1)$ and $g_j(x,\lambda_2)$ are increasing.

Since the functions $h_i(x,\lambda_1)$ and $g_j(x,\lambda_2)$ are increasing, then their product is an increasing function. In addition, $B_k^{(1)}\le_{st}B_k^{(2)}$. Consequently

$$\int_0^{\infty}h_i(x,\lambda_1^{(1)})g_j(x,\lambda_2^{(1)})dB_k^{(1)}(x)\le\int_0^{\infty}h_i(x,\lambda_1^{(1)})g_j(x,\lambda_2^{(1)})dB_k^{(2)}(x),\ \ k=1,2. \tag{9.2}$$

On the other hand, because of the monotonicity of the product $h_i(x,\lambda_1)g_j(x,\lambda_2)$ is with respect to λ_1 and λ_2, and the fact that $\lambda_1^{(1)}\le\lambda_1^{(2)}$ and $\lambda_2^{(1)}\le\lambda_2^{(2)}$, we obtain

$$\int_0^{\infty}h_i(x,\lambda_1^{(1)})g_j(x,\lambda_2^{(1)})dB_k^{(2)}(x)\le\int_0^{\infty}h_i(x,\lambda_1^{(2)})g_j(x,\lambda_2^{(2)})dB_k^{(2)}(x) \tag{9.3}$$

Finally, inequality (9.1) results from (9.2) and (9.3). □

Proof of Lemma 4.2. By definition of the convex order ($\leqslant_v$), one can write

$$\left\{a_{ij}^{k,(1)}\right\}\leqslant_v\left\{a_{ij}^{k,(2)}\right\}\Longleftrightarrow\overline{\overline{a_{ij}^{k}}}^{(1)}=\sum_{s=i}^{\infty}\sum_{r=j}^{\infty}\overline{a_{sr}^{k}}^{(1)}\leqslant\sum_{s=i}^{\infty}\sum_{r=j}^{\infty}\overline{a_{sr}^{k}}^{(2)}=\overline{\overline{a_{ij}^{k}}}^{(2)}.$$

Equivalently, we have for $k=1,2$.

$$\begin{aligned}
\overline{\overline{a_{ij}^{k}}}^{(1)} &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty}\sum_{m=s}^{\infty}\sum_{n=r}^{\infty}\int_0^{\infty}\frac{\left(\lambda_1^{(1)}x\right)^m}{m!}\exp(-\lambda_1^{(1)}x)\frac{\left(\lambda_2^{(1)}x\right)^n}{n!}\exp(-\lambda_2^{(1)}x)dB_k^{(1)}(x)\\
&= \int_0^{\infty}\sum_{s=i}^{\infty}\sum_{r=j}^{\infty}\sum_{m=s}^{\infty}\sum_{n=r}^{\infty}\frac{\left(\lambda_1^{(1)}x\right)^m}{m!}\exp(-\lambda_1^{(1)}x)\frac{\left(\lambda_2^{(1)}x\right)^n}{n!}\exp(-\lambda_2^{(1)}x)dB_k^{(1)}(x)\\
&= \int_0^{\infty}\left[\sum_{s=i}^{\infty}\sum_{m=s}^{\infty}\frac{\left(\lambda_1^{(1)}x\right)^m}{m!}\exp(-\lambda_1^{(1)}x)\right]\left[\sum_{r=j}^{\infty}\sum_{n=r}^{\infty}\frac{\left(\lambda_2^{(1)}x\right)^n}{n!}\exp(-\lambda_2^{(2)}x)\right]dB_k^{(1)}(x)\\
&= \int_0^{\infty}\left[\sum_{s=i}^{\infty}h_s(x,\lambda_1^{(1)})\right]\left[\sum_{r=j}^{\infty}g_r(x,\lambda_2^{(1)})\right]dB_k^{(1)}(x)\\
&\leqslant \int_0^{\infty}\left[\sum_{s=i}^{\infty}h_s(x,\lambda_1^{(2)})\right]\left[\sum_{r=j}^{\infty}g_r(x,\lambda_2^{(2)})\right]dB_k^{(2)}(x) = \overline{\overline{a_{ij}^{k}}}^{(2)}.
\end{aligned} \tag{9.4}$$

Next, let

$$h_s(x,\lambda_1) = \sum_{m=s}^{\infty}\frac{(\lambda_1 x)^m}{m!}\exp(-\lambda_1 x), \text{ and } g_r(x,\lambda_2) = \sum_{n=r}^{\infty}\frac{(\lambda_2 x)^n}{n!}\exp(-\lambda_2 x).$$

The functions $h_s(x,\lambda_1)$ and $g_r(x,\lambda_2)$ are increasing with respect to λ_1 and λ_2, respectively. Then, the functions defined by

$$\overline{h}_i(x,\lambda_1) = \sum_{s=i}^{\infty}h_s(x,\lambda_1), \text{ and } \overline{g}_j(x,\lambda_2) = \sum_{r=j}^{\infty}g_r(x,\lambda_2),$$

have the same behavior.

On the other side, we have

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\overline{h}_i(x,\lambda_1) &= \lambda_1^2\frac{(\lambda_1 x)^{i-2}}{(i-2)!}\exp(-\lambda_1 x) > 0,\\
\frac{\partial^2}{\partial x^2}\overline{g}_j(x,\lambda_2) &= \lambda_2^2\frac{(\lambda_2 x)^{j-2}}{(j-2)!}\exp(-\lambda_2 x) > 0.
\end{aligned}$$

As a consequence, $\overline{h}_i(x,\lambda_1)$ and $\overline{g}_j(x,\lambda_2)$ are increasing and convex with respect to the variable x. Hence, the product of these two functions is also increasing and convex with respect to the variable x. This leads implies

$$\int_0^{\infty}\overline{h}_i(x,\lambda_1^{(1)})\overline{g}_j(x,\lambda_2^{(1)})dB_k^{(1)}(x) \leq \int_0^{\infty}\overline{h}_i(x,\lambda_1^{(1)})\overline{g}_j(x,\lambda_2^{(1)})dB_k^{(2)}(x). \tag{9.5}$$

Also, through the monotonicity of the product of these two functions $\overline{h}_i(x,\lambda_1)\overline{g}_j(x,\lambda_2)$ with respect to λ_1 and λ_2, we find

$$\int_0^{\infty}\overline{h}_i(x,\lambda_1^{(1)})\overline{g}_j(x,\lambda_2^{(1)})dB_k^{(2)}(x) \leq \int_0^{\infty}\overline{h}_i(x,\lambda_1^{(2)})\overline{g}_j(x,\lambda_2^{(2)})dB_k^{(2)}(x). \tag{9.6}$$

Finally, inequality (9.4) is obtained from (9.5) and (9.6). □

Proof of Lemma 4.3. By definition, we have

$$
\begin{aligned}
a(z) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}^{k} z^i z^j \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \int_0^{\infty} \frac{(\lambda_1 x)^i}{i!} \exp(-\lambda_1 x) \frac{(\lambda_2 x)^j}{j!} \exp(-\lambda_2 x) z^i z^j dB_k(x) \\
&= \int_0^{\infty} \left[\sum_{i=0}^{\infty} \frac{(\lambda_1 x z)^i}{i!} \exp(-\lambda_1 x)\right] \left[\sum_{j=0}^{\infty} \frac{(\lambda_2 x z)^j}{j!} \exp(-\lambda_2 x)\right] dB_k(x) \\
&= \int_0^{\infty} [\exp(-\lambda_1 x(1-z))]\,[\exp(-\lambda_2 x(1-z))]\, dB_k(x) \\
&= \int_0^{\infty} [\exp(-x(\lambda_1+\lambda_2)(1-z)]\, dB_k(x) \\
&= \tilde{B}_k((\lambda_1+\lambda_2)(1-z)).
\end{aligned}
$$

In order to prove that $\left\{a_{ij}^{k,(1)}\right\} \leqslant_L \left\{a_{ij}^{k,(2)}\right\}$, it suffices to establish the following result for the corresponding generating functions

$$a^{k,(1)}(z) \geqslant a^{k,(2)}(z).$$

Equivalently, we have to prove that

$$\tilde{B}_k^{(1)}((\lambda_1^{(1)}+\lambda_2^{(1)})(1-z)) \geq \tilde{B}_k^{(2)}((\lambda_1^{(2)}+\lambda_2^{(2)})(1-z)).$$

That is,

$$\left\{a_{ij}^{k,(1)}\right\} \leq_L \left\{a_{ij}^{k,(2)}\right\} \Longleftrightarrow \tilde{B}_k^{(1)}((\lambda_1^{(1)}+\lambda_2^{(1)})(1-z)) \geq \tilde{B}_k^{(2)}((\lambda_1^{(2)}+\lambda_2^{(2)})(1-z)), \tag{9.7}$$

But $B_k^{(1)} \leq_L B_k^{(2)}$ means that $\tilde{B}_k^{(1)}(s) \geq \tilde{B}_k^{(2)}(s), \forall s > 0$.

In particular, for $s = (\lambda_1^{(1)}+\lambda_2^{(1)})(1-z)$, we have

$$\tilde{B}_k^{(1)}((\lambda_1^{(1)}+\lambda_2^{(1)})(1-z)) \geqslant \tilde{B}_k^{(2)}((\lambda_1^{(1)}+\lambda_2^{(1)})(1-z)). \tag{9.8}$$

Since a Laplace transform is a decreasing function, then the inequalities $\lambda_1^{(1)} \leqslant \lambda_1^{(2)}$ and $\lambda_2^{(1)} \leqslant \lambda_2^{(2)}$ imply the following inequality

$$\tilde{B}_k^{(2)}((\lambda_1^{(1)}+\lambda_2^{(1)})(1-z)) \geqslant \tilde{B}_k^{(2)}((\lambda_1^{(2)}+\lambda_2^{(2)})(1-z)). \tag{9.9}$$

Finally, inequality (9.7) arises from inequalities (9.8) and (9.9). □

Proof of Theorem 5.1. At first, we show that the transition operator is monotone with respect to the stochastic order. It is enough to check inequality (5.1) through the following cases

Case 1: If $n \geqslant 1$, we have:

$$r_{(k,n,m)(1,i,j)} = a^1_{i-n+1,j-m}.$$

By definition, we get

$$
\begin{aligned}
\overline{r}_{(k,n,m)(1,i,j)} &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} r_{(k,n,m)(1,s,r)} = \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} a^1_{s-n+1,r-m}, \\
\overline{r}_{(k,n-1,m-1)(1,i,j)} &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} a^1_{s-n+2,r-m+1}.
\end{aligned}
$$

To prove inequality (5.1), we have

$$\begin{aligned}\overline{r}_{(k,n,m)(1,i,j)} - \overline{r}_{(k,n-1,m-1)(1,i,j)} &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} a^1_{s-n+1,r-m} - \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} a^1_{s-n+2,r-m+1} \\ &= a^1_{i-n+1,j-m} \geqslant 0.\end{aligned}$$

Hence, inequality (5.1) is satisfied for $n = 1$.

Case 2: If $n = 0$, we have:

$$r_{(k,n,m)(2,i,j)} = \frac{\lambda_2}{\lambda_1+\lambda_2+m\theta} a^2_{i,j-m} + \frac{m\theta}{\lambda_1+\lambda_2+m\theta} a^2_{i,j-m+1},$$

such as

$$\begin{aligned}\overline{r}_{(k,n,m)(2,i,j)} &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} r_{(k,n,m)(2,s,r)} \\ &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty}\left[\frac{\lambda_2}{\lambda_1+\lambda_2+m\theta} a^2_{s,r-m} + \frac{m\theta}{\lambda_1+\lambda_2+m\theta} a^2_{s,r-m+1}\right] \\ &= \frac{\lambda_2}{\lambda_1+\lambda_2+m\theta} \overline{a}^2_{i,j-m} + \frac{m\theta}{\lambda_1+\lambda_2+m\theta} \overline{a}^2_{i,j-m+1} \\ &= \frac{\lambda_2}{\lambda_1+\lambda_2+m\theta} a^2_{i,j-m} + \frac{\lambda_2+m\theta}{\lambda_1+\lambda_2+m\theta} \overline{a}^2_{i,j-m+1} \\ &= \frac{\lambda_2+m\theta}{\lambda_1+\lambda_2+m\theta} \overline{a}^2_{i,j-m} - \frac{m\theta}{\lambda_1+\lambda_2+m\theta} a^2_{i,j-m},\end{aligned}$$

and

$$\begin{aligned}\overline{r}_{(k,n-1,m-1)(2,i,j)} &= \frac{\lambda_2}{\lambda_1+\lambda_2+(m-1)\theta} a^2_{i,j-m+1} + \frac{\lambda_2+(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta} \overline{a}^2_{i,j-m+2} \\ &= \frac{\lambda_2}{\lambda_1+\lambda_2+(m-1)\theta} a^2_{i,j-m+1} + \frac{\lambda_2+(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta} \overline{a}^2_{i,j-m+1} \\ &\quad - \frac{\lambda_2+(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta} a^2_{i,j-m+1} \\ &= \frac{\lambda_2+(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta} \overline{a}^2_{i,j-m+1} - \frac{(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta} a^2_{i,j-m+1}.\end{aligned}$$

Finally, we obtain

$$\overline{r}_{(k,n,m)(2,i,j)} - \overline{r}_{(k,n-1,m-1)(2,i,j)} \geq 0.$$

Thus, inequality (5.1) is satisfied for $n = 0$.

Therefore, the operator Θ is monotone with respect to the order $\leq_{st}$.

Secondly, we prove that the operator Θ is monotone with respect to the convex order. To do this, it is enough to check inequality (5.2).

Case 1: $n \geqslant 1$. Let

$$r_{(k,n,m)(1,i,j)} = a^1_{i-n+1,j-m},$$

with

$$\begin{aligned}
\overline{\overline{r}}_{(k,n,m)(1,i,j)} &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty}\overline{r}_{(k,n,m)(1,s,r)} = \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n+1,r-m},\\
\overline{\overline{r}}_{(k,n-1,m-1)(l,i,j)} &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n+2,r-m+1},\\
\overline{\overline{r}}_{(k,n+1,m+1)(l,i,j)} &= \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n,r-m-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\overline{\overline{r}}_{(k,n-1,m-1)(l,i,j)} + \overline{\overline{r}}_{(k,n+1,m+1)(l,i,j)} - 2\overline{\overline{r}}_{(k,n,m)(l,i,j)} =\\
&\quad = \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n+2,r-m+1} + \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n,r-m-1} - 2\sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n+1,r-m}\\
&\quad = \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n+2,r-m+1} + \overline{a}^{1}_{i-n,j-m-1} + \sum_{s=i+1}^{\infty}\sum_{r=j+1}^{\infty} \overline{a}^{1}_{s-n,r-m-1}\\
&\qquad - 2\sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n+1,r-m}\\
&\quad = \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n+2,r-m+1} + \overline{a}^{1}_{i-n,j-m-1} - \sum_{s=i}^{\infty}\sum_{r=j}^{\infty} \overline{a}^{1}_{s-n+1,r-m}\\
&\quad = \overline{a}^{1}_{i-n,j-m-1} - \overline{a}^{1}_{i-n+1,j-m}\\
&\quad = a^{1}_{i-n,j-m-1} \geqslant 0.
\end{aligned}$$

Thus, inequality (5.2) is verified.

Case 2: $n = 0$. Let

$$r_{(k,n,m)(2,i,j)} = \frac{\lambda_2}{\lambda_1+\lambda_2+m\theta}a^{2}_{i,j-m} + \frac{m\theta}{\lambda_1+\lambda_2+m\theta}a^{2}_{i,j-m+1},$$

such as

$$\begin{aligned}
\overline{\overline{r}}_{(k,n,m)(2,i,j)} &= \frac{\lambda_2+m\theta}{\lambda_1+\lambda_2+m\theta}\overline{\overline{a}}^{2}_{i,j-m+1} + \frac{\lambda_2}{\lambda_1+\lambda_2+m\theta}\overline{a}^{2}_{i,j-m}\\
&= \frac{\lambda_2+m\theta}{\lambda_1+\lambda_2+m\theta}\left[\overline{\overline{a}}^{2}_{i,j-m} - \overline{a}^{2}_{i,j-m}\right] + \frac{\lambda_2}{\lambda_1+\lambda_2+m\theta}\overline{a}^{2}_{i,j-m}\\
&= \frac{\lambda_2+m\theta}{\lambda_1+\lambda_2+m\theta}\overline{\overline{a}}^{2}_{i,j-m} - \frac{m\theta}{\lambda_1+\lambda_2+m\theta}\overline{a}^{2}_{i,j-m},\\
\overline{\overline{r}}_{(k,n-1,m-1)(l,i,j)} &= \frac{\lambda_2+(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta}\overline{\overline{a}}^{2}_{i,j-m+1} - \frac{(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta}\overline{a}^{2}_{i,j-m+1},\\
\overline{\overline{r}}_{(k,n+1,m+1)(l,i,j)} &= \frac{\lambda_2+(m+1)\theta}{\lambda_1+\lambda_2+(m+1)\theta}\overline{\overline{a}}^{2}_{i,j-m} - \frac{(m+1)\theta}{\lambda_1+\lambda_2+(m+1)\theta}\ \overline{a}^{2}_{i,j-m-1}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\overline{\overline{r}}_{(k,n-1,m-1)(l,i,j)} + \overline{\overline{r}}_{(k,n+1,m+1)(l,i,j)} - 2\overline{\overline{r}}_{(k,n,m)(l,i,j)} = \\
&= \frac{\lambda_2 + (m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta}\left[\overline{\overline{a}}^2_{i,j-m} - \overline{a}^2_{i,j-m}\right] - \frac{(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta}\left[\overline{a}^2_{i,j-m} - a^2_{i,j-m}\right] \\
&\quad + \frac{\lambda_2 + (m+1)\theta}{\lambda_1+\lambda_2+(m+1)\theta}\,\overline{\overline{a}}^2_{i,j-m} + \frac{\lambda_2}{\lambda_1+\lambda_2+(m+1)\theta}\left[\overline{a}^2_{i,j-m} - a^2_{i,j-m-1}\right] \\
&\quad - \left[\frac{\lambda_2+m\theta}{\lambda_1+\lambda_2+m\theta}\left(\overline{\overline{a}}^2_{i,j-m} - \overline{a}^2_{i,j-m}\right) + \frac{\lambda_2}{\lambda_1+\lambda_2+m\theta}\overline{a}^2_{i,j-m}\right] \\
&\quad - \left[\frac{\lambda_2+m\theta}{\lambda_1+\lambda_2+m\theta}\overline{\overline{a}}^2_{i,j-m} - \frac{m\theta}{\lambda_1+\lambda_2+m\theta}\overline{a}^2_{i,j-m}\right], \\
&= \overline{\overline{a}}^2_{i,j-m}\left[-\frac{2\theta^2\lambda_1}{(\lambda_1+\lambda_2+m\theta)(\lambda_1+\lambda_2+(m-1)\theta)(\lambda_1+\lambda_2+(m+1)\theta)}\right] \\
&\quad + \overline{a}^2_{i,j-m}\left[\frac{2\theta(m\theta\lambda_1+\lambda_1\lambda_2+\theta\lambda_2+\lambda_1^2+\theta\lambda_1)}{(\lambda_1+\lambda_2+m\theta)(\lambda_1+\lambda_2+(m-1)\theta)(\lambda_1+\lambda_2+(m+1)\theta)}\right] \\
&\quad + a^2_{i,j-m}\,\frac{(m-1)\theta}{\lambda_1+\lambda_2+(m-1)\theta} + a^2_{i,j-m-1}\,\frac{\lambda_2}{\lambda_1+\lambda_2+(m+1)\theta} \geqslant 0.
\end{aligned}
$$

Thus, inequality (5.2) is satisfied for $n=0$ and $n=1$. Then, the operator Θ is monotone with respect to the convex order. $\square$

Proof of Theorem 6.1. To show that the two operators are comparable with respect to the stochastic order, we have to prove the following inequality

$$\overline{r}^{(1)}_{(k,n,m)(l,i,j)} \leq \overline{r}^{(2)}_{(k,n,m)(l,i,j)}, l = 1, 2. \tag{9.10}$$

For the case $n=0$ and $l=2$, inequality (9.10) can be written as follows:

$$
\begin{aligned}
&\frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)}+\lambda_2^{(1)}+m\theta^{(1)}}\overline{a^2}^{(1)}_{i,j-m} - \frac{m\theta^{(1)}}{\lambda_1^{(1)}+\lambda_2^{(1)}+m\theta^{(1)}}a^{2,(1)}_{i,j-m} \\
&\leq \frac{\lambda_2^{(2)} + m\theta^{(2)}}{\lambda_1^{(2)}+\lambda_2^{(2)}+m\theta^{(2)}}\overline{a^2}^{(2)}_{i,j-m} - \frac{m\theta^{(2)}}{\lambda_1^{(2)}+\lambda_2^{(2)}+m\theta^{(2)}}\ a^{2,(2)}_{i,j-m}.
\end{aligned} \tag{9.11}
$$

According to Lemma 4.1, we have

$$\left\{a^{k,(1)}_{i,j}\right\} \leq_{st} \left\{a^{k,(2)}_{i,j}\right\}. \tag{9.12}$$

Moreover, we have by hypothesis of Theorem 6.1 that $\lambda_1^{(1)} \leq \lambda_1^{(2)}$, $\lambda_2^{(1)} \leq \lambda_2^{(2)}$, and $\theta^{(1)} \geq \theta^{(2)}$. This implies that

$$\frac{\lambda_1^{(1)}+\lambda_2^{(1)}}{\theta^{(1)}} \leq \frac{\lambda_1^{(2)}+\lambda_2^{(2)}}{\theta^{(2)}}.$$

Since the function $x \to \frac{m}{x+m}$ is decreasing, we get

$$\frac{m\theta^{(1)}}{\lambda_1^{(1)}+\lambda_2^{(1)}+m\theta^{(1)}} \geq \frac{m\theta^{(2)}}{\lambda_1^{(2)}+\lambda_2^{(2)}+m\theta^{(2)}}. \tag{9.13}$$

In fact $\lambda_2^{(1)} \leq \lambda_2^{(2)}$ and $\theta^{(1)} \geq \theta^{(2)}$ imply that $\frac{\lambda_2^{(1)}}{\theta^{(1)}} \leq \frac{\lambda_2^{(2)}}{\theta^{(2)}}$ and as the function $x \to \frac{x}{x+m}$ is increasing, then we have

$$\frac{\lambda_2^{(1)}}{\lambda_2^{(1)}+m\theta^{(1)}} \leq \frac{\lambda_2^{(2)}}{\lambda_2^{(2)}+m\theta^{(2)}}.$$

This implies $\lambda_2^{(1)} + m\theta^{(1)} \geq \lambda_2^{(2)} + m\theta^{(2)}$.

From inequalities $\lambda_1^{(1)} \le \lambda_1^{(2)}$ and $\lambda_2^{(1)} + m\theta^{(1)} \ge \lambda_2^{(2)} + m\theta^{(2)}$, we have

$$\frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)}} \ge \frac{\lambda_2^{(2)} + m\theta^{(2)}}{\lambda_1^{(2)}}.$$

Then, as $x \to \frac{x}{x+m}$, we get

$$\frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} \ge \frac{\lambda_2^{(2)} + m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}. \tag{9.14}$$

Using inequalities (9.12)-(9.14), we obtain

$$\begin{aligned}
\overline{r}^{(1)}_{(k,n,m)(2,i,j)} - \overline{r}^{(2)}_{(k,n,m)(2,i,j)} &= \frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} \overline{a^2}^{(1)}_{i,j-m} - \frac{m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} a^{2,(1)}_{i,j-m} \\
&\quad - \frac{\lambda_2^{(2)} + m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}} \overline{a^2}^{(2)}_{i,j-m} + \frac{m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}} a^{2,(2)}_{i,j-m} \\
&\le \frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} \overline{a^2}^{(1)}_{i,j-m} - \frac{m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}} a^{2,(2)}_{i,j-m} \\
&\quad - \frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} \overline{a^2}^{(1)}_{i,j-m} + \frac{m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}} a^{2,(2)}_{i,j-m} = 0.
\end{aligned}$$

Thus, inequality (9.11) is satisfied for the cases $n = 0$ and $l = 2$.

Now, for cases $n = 0$ and $l = 1$, inequality (9.10) can be written as follows:

$$\frac{\lambda_1^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} \overline{a^1}^{(1)}_{i,j-m} \le \frac{\lambda_1^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}} \overline{a^1}^{(2)}_{i,j-m}. \tag{9.15}$$

In addition, we have

$$\frac{\lambda_2 + m\theta}{\lambda_1 + \lambda_2 + m\theta} = 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2 + m\theta}.$$

Therefore,

$$\frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} \ge \frac{\lambda_2^{(2)} + m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}},$$

can be rewritten as

$$1 - \frac{\lambda_1^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} \ge 1 - \frac{\lambda_1^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}.$$

This gives

$$\frac{\lambda_1^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}} \le \frac{\lambda_1^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}. \tag{9.16}$$

From inequality (9.12) and (9.16), inequality (9.15) is verified.

For cases $n \ge 1$ and $l = 1$, one can write

$$r_{(k,n,m)(1,i,j)} = a^1_{i-n+1,j-m}.$$

Then, inequality (9.10) can be written directly as:

$$\overline{a^1}^{(1)}_{i-n+1,j-m} \le \overline{a^1}^{(2)}_{i-n+1,j-m}. \tag{9.17}$$

Thus, inequality (9.17) is verified using Lemma 4.1. Therefore, inequality (9.10) is verified $\forall l = 1, 2$.

Next, to show that the two operators are comparable with respect to the convex order, we have to prove the following inequality

$$\overline{\overline{r}}^{(1)}_{(k,n,m)(l,i,j)} \leq \overline{\overline{r}}^{(2)}_{(k,n,m)(l,i,j)}, l = 1, 2. \tag{9.18}$$

In case $n = 0$ and $l = 2$, we have

$$\overline{\overline{r}}_{(k,n,m)(2,i,j)} = \frac{\lambda_2 + m\theta}{\lambda_1 + \lambda_2 + m\theta}\overline{\overline{a}}^2_{i,j-m} - \frac{m\theta}{\lambda_1 + \lambda_2 + m\theta}\overline{a}^1_{i,j-m}.$$

Then, to obtain (9.18), it suffices to check that

$$\begin{aligned}\frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}}\overline{\overline{a^2}}^{(1)}_{i,j-m} - \frac{m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}}\overline{a^2}^{(1)}_{i,j-m} \quad &\leq \\ \frac{\lambda_2^{(2)} + m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}\overline{\overline{a^2}}^{(2)}_{i,j-m} - \frac{m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}\ \overline{a^2}^{(2)}_{i,j-m}&.\end{aligned} \tag{9.19}$$

Indeed, according to Lemma 4.2 and by inequalities (9.13) and (9.14), we obtain

$$\begin{aligned}\overline{\overline{r}}^{(1)}_{(k,n,m)(2,i,j)} - \overline{\overline{r}}^{(2)}_{(k,n,m)(2,i,j)} &= \frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}}\overline{\overline{a^2}}^{(1)}_{i,j-m} - \frac{m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}}\overline{a^2}^{(1)}_{i,j-m} \\ &- \frac{\lambda_2^{(2)} + m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}\overline{\overline{a^2}}^{(2)}_{i,j-m} + \frac{m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}\ \overline{a^2}^{(2)}_{i,j-m} \\ &\leq \frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}}\overline{\overline{a^2}}^{(1)}_{i,j-m} - \frac{m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}\ \overline{a^2}^{(2)}_{i,j-m} \\ &- \frac{\lambda_2^{(1)} + m\theta^{(1)}}{\lambda_1^{(1)} + \lambda_2^{(1)} + m\theta^{(1)}}\overline{\overline{a^2}}^{(1)}_{i,j-m} + \frac{m\theta^{(2)}}{\lambda_1^{(2)} + \lambda_2^{(2)} + m\theta^{(2)}}\ \overline{a^2}^{(2)}_{i,j-m} = 0.\end{aligned}$$

Thus, inequality (9.19) is checked.

For the rest of the proof, concerning the convex order, we have to follow the same steps as in the case of the stochastic order. □

Proof of Theorem 7.1. According to Theorem 6.1, the inequalities $\lambda_2^{(1)} \leq \lambda_2^{(2)}, \lambda_1^{(1)} \leq \lambda_1^{(2)}, \theta^{(1)} \geq \theta^2, B_1^{(1)} \leq_{so} B_1^{(2)}, B_2^{(1)} \leq_{so} B_2^{(2)}$ imply $\Theta^{(1)} \leq_{so} \Theta^{(2)}$, i.e., for any distribution r we have the following inequality

$$\Theta^{(1)}r \leq_{so} \Theta^{(2)}r\ . \tag{9.20}$$

From Theorem 5.1, the operator $\Theta^{(2)}$ associated with the embedded Markov chain, of the second system, is monotone. That is, for any two distributions $r_1^{(2)}$, $r_2^{(2)}$ such as $r_1^{(2)} \leq_{so} r_2^{(2)}$, we have

$$\Theta^{(2)}r_1^{(2)} \leq_{so} \Theta^{(2)}r_2^{(2)},$$

However, from inequality (9.20), one obtains

$$\Theta^{(1)}r^{(1)} \leq_{so} \Theta^{(2)}r^{(1)}. \tag{9.21}$$

Then, there exists a probability $r_1^{(2)}$ such that

$$\Theta^{(2)}r^{(1)} \leq_{so} \Theta^{(2)}r_1^{(2)}. \tag{9.22}$$

By combining inequalities (9.21) and (9.22), we get

$$\Theta^{(1)}r^{(1)} \leq_{so} \Theta^{(2)}r^{(2)}, \tag{9.23}$$

for any two distributions $r^{(1)}$ and $r^{(2)}$.

Inequality (9.23) can be rewritten as

$$\begin{aligned}\Theta^{(1)}r^{(1)} &= P\left(X_d^{(1)} = (1,i,j)\right) = P\left(X_d^{(1)} = (2,i,j)\right)\\ &\leq_{so} P\left(X_d^{(2)} = (1,i,j)\right) = P\left(X_d^{(2)} = (2,i,j)\right) = \Theta^{(2)}r^{(2)}.\end{aligned}$$

Finally, when $d \longrightarrow \infty$, we have $\left\{\pi_{i,j}^{(1)}\right\} \leq_{so} \left\{\pi_{i,j}^{(2)}\right\}$.

□

Proof of Theorem 7.2. Denote by $\Sigma^{(1)}$ our system (i.e., an $M_2/G_2/1$ retrial queue with a non-preemptive priority customers) with parameters :

- Arrival rate of high-priority customers $\lambda_1^{(1)} = \lambda_1$,
- Arrival rate of low-priority customers $\lambda_2^{(1)} = \lambda_2$,
- Retrial rate $\theta^{(1)} = \theta$,
- First moment of high-priority customers $\beta_1^{1,(1)} = \beta_1^1$,
- First moment of low-priority customers $\beta_1^{2,(1)} = \beta_1^2$.

Next, let $\Sigma^{(2)}$ be an auxiliary $M_2/M_2/1$ retrial queue with a non-preemptive priority customers having the following parameters :

- Arrival rate of high-priority customers $\lambda_1^{(2)} = \lambda_1$,
- Arrival rate of low-priority customers $\lambda_2^{(2)} = \lambda_2$,
- retrial rate $\theta^{(2)} = \theta$,
- first moment of high-priority customers $\beta_1^{1,(2)} = \beta_1^1$,
- first moment of low-priority customers $\beta_1^{2,(2)} = \beta_1^2$.

With $B_1^{(2)} \equiv B_1^{\exp}$ and $B_2^{(2)} \equiv B_2^{\exp}$, where

$$B_1^{\exp}(x) = \begin{cases} 1 - e^{-\frac{x}{\beta_1^1}}, & \text{if } x \geq 0,\\ 0, & \text{if } x < 0,\end{cases}$$

$$B_2^{\exp}(x) = \begin{cases} 1 - e^{-\frac{x}{\beta_1^2}}, & \text{if } x \geq 0,\\ 0, & \text{if } x < 0.\end{cases}$$

If $B_k(x)$ is NBUE, then $B_k(x) \leqslant_v B_k^{\exp}(x)$, $k = 1,2$ and if $B_k(x)$ is NWUE, then $B_k(x) \geqslant_v B_k^{\exp}(x)$, $k = 1,2$ (see Proposition 3.3). Moreover, the following conditions of Theorem 7.1 are satisfied:

$\lambda_1^{(1)} = \lambda_1^{(2)}$, $\lambda_2^{(1)} = \lambda_2^{(2)}$, $\theta^{(1)} = \theta^{(2)}$, $B_1^{(1)}(x) \leq_v B_1^{\exp}(x)$ (respectively, $B_1^{(1)}(x) \geqslant_v B_1^{\exp}(x)$) and $B_2^{(1)}(x) \leq_v B_2^{\exp}(x)$ (respectively, $B_2^{(1)}(x) \geqslant_v B_2^{\exp}(x)$).

Thus, the joint stationary distribution at a departure epoch in the $M_2/G_2/1$ retrial queue with priority customers is less (respectively, greater) than the corresponding joint stationary distribution in the $M_2/M_2/1$ retrial queue with priority customers if $B_k(x)$, $k = 1,2$, is NBUE (resp., if $B_k(x)$, $k = 1,2$, is NWUE). □

10. Conclusion

This paper examines a non-preemptive priority retrial queue with two types of customers, high-priority (class 1) and low-priority (class 2), and different service time distributions. We employ the stochastic comparison method, based on the general theory of stochastic orders, to derive several stochastic comparison properties in the sense of strong stochastic ordering and convex ordering. Specifically, the stochastic inequalities provide simple and insensitive bounds, both lower and upper, for the joint stationary distribution of the number of customers at departure epochs of a non-preemptive priority retrial queue. Furthermore, we illustrate the impact of the arrival rate of high-priority customers on the performance of the retrial queue through numerical examples. These findings have important implications for practitioners and decision-makers in the design and operation of non-preemptive priority retrial queues with different customer classes and service time distributions.

As a direction for future research, similar models with working vacations could be studied. Additionally, these models could be further investigated with the inclusion of breakdowns and repairs.

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