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Comparison Between Some Nonlinear Controllers for the Position Control of Lagrangian-type Robotic Systems

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ABSTRACT This work addresses the set-point control problem of the position state of fully-actuated Lagrangian-type robotic systems by means of some nonlinear control laws. We adopt four different nonlinear control laws: the PD plus gravity compensation controller, the PD plus desired gravity compensation controller, the computed-torque controller and the augmented PD plus gravity compensation controller. An in-depth comparison between these control laws and their application is achieved. Indeed, using some properties, we design some conditions on the matrix feedback gains of the nonlinear controllers ensuring the stability in the closed loop of the zero-equilibrium point and its uniqueness. At the end of this paper, we adopt a planar two-degree-of-freedom robotic manipulator to illustrate via simulation the difference between and the efficiency of the adopted nonlinear controllers.

KEYWORDS
Lagrangian-type robotic systems
Nonlinear dynamics
Approximate linear model
Position feedback control
Nonlinear controllers
Stability
Stabilization
Solution uniqueness

INTRODUCTION

Robotics is a field of activity covering the study, design and manufacture of robots or automated machines (Koditschek 2021). Nowadays, robots are omnipresent in several sectors, and each robot is created and modified in such a way that it can perform certain desired tasks (Biswal and Mohanty 2021; Chai *et al.* 2021; da Costa Barros and Nascimento 2021; Gonzalez-Aguirre *et al.* 2021; González *et al.* 2021; Gualtieri *et al.* 2021; Tipary and Erdos 2021). Robotic systems have been introduced and employed in different fields such as the medical field for the rehabilitation of upper and lower limbs by building robotic-based orthosis, prosthetic leg robot and exoskeleton devices (Ahmed *et al.* 2021; Islam *et al.* 2020; Jafari *et al.* 2023; Kalita *et al.* 2021; Narayan and Dwivedy 2021; Tarnita *et al.* 2022; Wang *et al.* 2021). An historical overview of control theory applied to robotic manipulators and fundamental theoretical foundations of robot control were reported in (Spong 2022).

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The different types of robotic systems can be subdivided into three main classes, depending on the degree number of their actuation (Choukchou-Braham *et al.* 2014; Gritli and Belghith 2021; Krafes *et al.* 2018; Liu and Yu 2013). The first class being the underactuated robotic systems Choukchou-Braham *et al.* (2014); Liu and Yu (2013); Zilong Zhang (2022). This type has less of actuators than the degrees of freedom (DoF), such as the acrobot, the pendubot and the inverted pendulum on a cart, and the inertia wheel inverted pendulum, just to mention a few (Choukchou-Braham *et al.* 2014; Gritli and Belghith 2018, 2021; Krafes *et al.* 2018; Liu and Yu 2013; Parulski *et al.* 2021, 11; Zilong Zhang 2022). The second class is called the fully-actuated robotic systems. In this class, the number of actuators in the robotic system is equal to the degrees of freedom (Li *et al.* 2020; Zhang and Wu 2021). Finally, the third class is the overactuated robotic systems, for which the number of control actuators is more than the number of degrees of freedom such as the over-actuated hexapod robot (Bjelonic *et al.* 2016).

In order to control these different types of mechanical systems, a suitable controller must be designed and hence applied. In the literature, as in (Abbas *et al.* 2021; Abdul-Adheem *et al.* 2021; Choukchou-Braham *et al.* 2014; Gritli 2020; Gu 2013; Kelly *et al.* 2005; Krafes *et al.* 2018; Kurdila and Ben-Tzvi 2019; Liu *et al.* 2020; Liu and Yu 2013; Mobayen *et al.* 2017; Perrusquia *et al.* 2020; Spong *et al.* 2020), we find different control techniques for robotic systems. These controllers can be subdivided into two main families: (1) the linear controllers such as the Proportional-Derivative (PD) control law, the Proportional-Integral-Derivative (PID) control law, the Linear Quadratic Regulator (LQR) control law, and the state-feedback control law (Chawla and Singla 2021; Kelly *et al.* 2005; Narayan and Dwivedy 2021; Singla and Singh 2017), and (2) the nonlinear controllers like the PD plus gravity compensation control law, the computed-torque control (CTC) law, the sliding mode control law, the PD plus desired gravity compensation control law, among others (Hasan and Dhingra 2021; Jiang *et al.* 2020; Kelly *et al.* 2005; Mobayen *et al.* 2017; Nho and Meckl 2003; Perrusquia *et al.* 2020). These several types of control laws have been proven to be applicable for settlement at desired configuration states (position control problem), i.e. to control and then stabilize the robotic system at some desired position/state, or for regulation to reference trajectories (tracking problem) i.e. to control the trajectory of the robot.

Recently, in (Gritli *et al.* 2022; Jenhani *et al.* 2022b,c), and using the developed approximate linear dynamic model, we proposed an affine PD-based control law in order to solve the position control problem of robotic systems. Indeed, in (Gritli *et al.* 2022; Jenhani *et al.* 2022b), the approximate linear dynamic was used to develop the conditions ensuring the stability of the closed-loop robotic system. Moreover, in (Jenhani *et al.* 2022b) we presented a comparison between the affine PD-based control law and the computed-torque control law. The numerical simulations demonstrated the efficiency and validity of the proposed affine PD-based control law. Some LMI stability conditions and improved ones for the stabilization of the controlled robotic system using the affine PD-based controller have been developed in (Jenhani *et al.* 2022f,g). Moreover, we performed in (Jenhani *et al.* 2022a) a comparison between this affine PD-based controller and the PD plus gravity compensation controller and also the PD plus desired gravity compensation controller. In addition, the control problem of underactuated robotic systems via the affine PD-based controller has been considered in (Jenhani *et al.* 2022d). Furthermore, in (Jenhani *et al.* 2022e), we applied an affine PID-based controller to control the Lagrangian robotic systems using the dynamical model which described the difference into the nonlinear dynamics and its approximated linear model.

In this research work, we will be interested in solving the position feedback control problem for fully-actuated Lagrangian-type robotic systems (Choukchou-Braham *et al.* 2014; Kelly *et al.* 2005; Spong *et al.* 2020). Thus, to control the robotic systems and then to ensure stability at the desired position, we will adopt four nonlinear control laws, namely the PD plus gravity compensation controller, the PD plus desired gravity compensation controller, the computed-torque controller, and the augmented PD plus gravity compensation controller. Additionally, we will construct conditions on the matrix feedback gains of these proposed control laws to ensure stabilization at the desired state. Moreover, for each case of control problem, we will develop the conditions on these matrix gains guaranteeing the uniqueness of the solution, that is the zero-equilibrium point of the closed-loop robotic system. Furthermore, a comparison between these controllers will be achieved. Finally, we will adopt a 2-DoF planar robotic manipulator as an illustrative robotic system, to check the validity of the developed conditions of stability and uniqueness of the zero-equilibrium state of the controlled Lagrangian-type robotic system, and hence the efficiency of the adopted nonlinear controllers.

The remaining and following sections of this article are organized like so. The second section presents the dynamic model of Lagrangian robotic systems and the problem considered in this research article. Some useful properties of the nonlinear dynamics of Lagrangian-type mechanical robot systems are given. The third section describes the design procedure of the matrix feedback gains of the PD plus gravity compensation controller. The fourth section is devoted to developing the stabilizing conditions on the feedback gain matrices of the PD plus desired gravity compensation control law. The fifth section presents the computed-torque controller and the condition on its matrix feedback gains. The design of certain conditions on the matrix gains of the augmented PD plus gravity compensation control law will be performed in the sixth section. The seventh section will be dedicated for the simulation results by introducing the 2-DoF manipulator robot. In the eighth section, a discussion about the obtained results and the efficiency of the proposed control laws is presented. Finally, at the end of this paper and in the last section, a conclusion and a future direction of this article will be drawn.

DYNAMIC MODEL OF LAGRANGIAN ROBOTIC SYSTEMS AND PROBLEM FORMULATION

Nonlinear Dynamics of Lagrangian-type Robotic Systems

In this present work, and for the reason of simplicity, we will ignore the presence of frictional and elastic forces, unmodeled dynamics, external disturbances, and structured and unstructured uncertainties in the nonlinear dynamics of Lagrangian-type mechanical robotic (or mechatronic) systems. Therefore, the dynamics of these robotic systems under such assumptions is given by the following nonlinear (and complex) expression:

$$\mathcal{M}(q)\ddot{q} + \mathcal{H}(q,\dot{q})\dot{q} + \mathcal{G}(q) = \mathcal{D}(q)\mathcal{U} \tag{1}$$

where in this previous model (1), we have the following notations:

- $q \in \mathbb{R}^{n\times 1}$ is the positions' vector of the different joints/articulations of the Lagrangian robotic system,
- $\dot{q} \in \mathbb{R}^{n\times 1}$ refers to the vector of corresponding velocities,
- $\ddot{q} \in \mathbb{R}^{n\times 1}$ is the vector of corresponding accelerations,
- $\mathcal{U} \in \mathbb{R}^{n\times 1}$ represents the input vector of available actuators applied to the Lagrangian robotic system,
- $\mathcal{M}(q) \in \mathbb{R}^{n\times n}$ represents the inertia matrix of the robot,
- $\mathcal{H}(q,\dot{q})\dot{q} \in \mathbb{R}^{n\times 1}$, where $\mathcal{H}(q) \in \mathbb{R}^{n\times n}$, is a vector containing two types of terms containing $\dot{q}_i\dot{q}_j$, which are called the centrifugal terms (for the cases $i = j$) and the Coriolis terms (for the cases $i \neq j$), for all $i, j = 1, \ldots, n$,
- $\mathcal{G}(q) \in \mathbb{R}^{n\times 1}$ stands for the gravity matrix of the robotic system, and
- $\mathcal{D}(q) \in \mathbb{R}^{n\times n}$ denotes the input matrix or the distribution matrix of all the actuators applied to the Lagrangian-type robotic system.

Remark 1 *It is worth to indicate that the nonlinear dynamic model (1) has been considered in some previous works, like (Krafes* et al. *2018; Liu* et al. *2020; Liu and Yu 2013). This general for the nonlinear dynamics (1) can model several types of robotic systems manipulator robots and wheeled mobile manipulator robots. Furthermore, in several works of the literature, the vector of control inputs, saying τ, in the dynamic model (1) was taken to be without the matrix $\mathcal{D}(q)$, and then $\tau = \mathcal{D}(q)\mathcal{U}$.*

Problem Formulation

Among the most important axes in the field of robotics research is the control of the robotic system to solve the stabilization problem. Such a stabilization problem can be classified into two main classes. The first is the problem of controlling the position of the robotic system and then bringing it from its current configuration state to some desired one. The second stabilization problem is the trajectory/motion control, where the goal is to move the robot through control to a desired path/trajectory. Thus, to achieve these stabilization problems, we need to design a controller adapted and simple to apply in practice to the robot.

Our main objective in this article is to solve the position control problem of fully-actuated Lagrangian-type robotic systems. Then, our goal is to design an appropriate and simple controller $\mathcal{U}$ for the mechanical system where its (joint) motion is represented by its nonlinear dynamic model (1). Hence, the objective is to find and develop an expression of the nonlinear control law $\mathcal{U}$ that allows the robot to change its current configuration state q to a desired position state q_d. In this paper, we will adopt four nonlinear control laws: (i) the PD plus gravity compensation controller, (ii) the PD plus desired gravity compensation controller, (iii) the computed-torque control law, and (iv) the augmented PD plus gravity compensation controller.

Moreover, we will focus in this work on developing some feasible conditions to help in the right selection of the matrix feedback gains of the controllers ensuring the stabilization of the zero state, as the unique equilibrium, of the controlled Lagrangian robotic system. In addition, we will develop other conditions to guarantee the unicity of the zero origin as the unique possible equilibrium point of the controlled nonlinear dynamical system (1) of the Lagrangian robot systems.

In the sequel, let us consider $\phi = q - q_d$. Then, for the position control of the Lagrangian robot systems, and since q_d is constant, we have $\dot{\phi} = \dot{q}$ and hence $\ddot{\phi} = \ddot{q}$. Thus, the nonlinear dynamics' model (1) can be reformulated under the following equivalent nonlinear dynamic model:

$$\mathcal{M}(q)\ddot{\phi} + \mathcal{H}(q,\dot{\phi})\dot{\phi} + \mathcal{G}(q) = \mathcal{D}(q)\mathcal{U} \tag{2}$$

Such nonlinear system (2) represents the nonlinear dynamics of the position error of the Lagrangian-type robotic system. Thus, hereafter, we will use the nonlinear dynamic model (2) for the development of the proposed nonlinear control laws $\mathcal{U}$ too solve the set-point control problem of the position state of fully-actuated Lagrangian-type robot systems.

Useful Properties and Theorem

In this part, in order to develop some feasible conditions on the matrix gains of the nonlinear control laws to develop in the sequel, we present here some properties on the various matrices in the nonlinear dynamics (1) or the position error dynamics (2) of the robotic systems.

Useful Properties

We consider first the following useful properties (Jenhani *et al.* 2022a,c; Kelly *et al.* 2005).

Property 1 *In the nonlinear dynamic model (2), $\mathcal{M}(q)$ is such that:*

$$\mathcal{M}(q) = \mathcal{M}(q)^{\mathrm{T}} > 0, \qquad \forall\, q \in \mathbb{R}^n \tag{3}$$

Property 2 *For all vector $q \in \mathbb{R}^n$ and all vector $\dot{q} \in \mathbb{R}^n$, the inertia matrix $\mathcal{M}(q)$ and the matrix $\mathcal{H}(q,\dot{q})$ in (1) satisfy the following equality constraint:*

$$\xi^{\mathrm{T}}\left[\dot{\mathcal{M}}(q) - 2\mathcal{H}(q,\dot{q})\right]\xi = 0 \tag{4}$$

for all vector $\xi \in \mathbb{R}^n$.

Property 3 *For all vector $q \in \mathbb{R}^n$, the gravity matrix $\mathcal{G}(q)$ satisfies the following Lipschitz condition:*

$$\|\mathcal{G}(x) - \mathcal{G}(y)\| \le k_g \|x - y\| \tag{5}$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, and where k_g satisfies:

$$k_g \ge n \max_{i,j,q} \left| \frac{\partial \mathcal{G}_i(q)}{\partial q_j} \right| \tag{6}$$

or the following condition:

$$k_g \ge \lambda_{max} \left\{ \frac{\partial \mathcal{G}(q)}{\partial q} \right\} \tag{7}$$

where λ_{max} stands for the largest eigenvalue.

Contraction Mapping Theorem

In the sequel and in order to check the uniqueness of the zero solution of an equation constraint, we need to use the contraction map (Jenhani *et al.* 2022c; Kelly *et al.* 2005), which is introduced via the following theorem.

Theorem 1 (Contraction Map (Jenhani *et al.* 2022c)) *Consider $\Psi \subset \mathbb{R}^m$, a parameters' vector $\Theta \in \Psi$ and the continuous nonlinear function $\digamma : \mathbb{R}^n \times \Psi \to \mathbb{R}^n$.*

Suppose that there is a constant scalar $\gamma > 0$ such that for all vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, and all vector $\Theta \in \Psi$, we obtain:

$$\|\digamma(x,\Theta) - \digamma(y,\Theta)\| \le \gamma \|x - y\| \tag{8}$$

If the constant scalar γ is such that the following condition is well verified:

$$\gamma < 1 \tag{9}$$

then, for all vector $\Theta^\star \in \Psi$, the nonlinear function $\digamma(z,\Theta^\star)$ has only one and unique fixed point $z^\star \in \mathbb{R}^n$, satisfying this expression:

$$\digamma(z^\star,\Theta^\star) = z^\star \tag{10}$$

DESIGN OF THE PD PLUS GRAVITY COMPENSATION CONTROL LAW

In this present section, we will consider the first nonlinear control law, namely the PD plus gravity compensation controller, to control the Lagrangian-type robotic system by its own nonlinear dynamics (1) or (2), in order to control it to and hence stabilize it at the desired position vector q_d.

The Nonlinear PD Plus Gravity Compensation Controller

Let us consider the following nonlinear PD plus gravity compensation control law:

$$\mathcal{U} = \mathcal{D}^{-1}(q)(\mathcal{G}(q) + \mathcal{K}_p\phi + \mathcal{K}_v\dot{\phi}) \tag{11}$$

where $\mathcal{K}_p$ and $\mathcal{K}_v$ stand for the two matrix feedback gains of the nonlinear controller (11), that need to be produced in this part. Moreover, notice that $\mathcal{K}_p \in \mathbb{R}^{n\times n}$ and $\mathcal{K}_v \in \mathbb{R}^{n\times n}$.

Therefore, our main objective is to develop some feasible constraints on the two feedback gain matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ of the adopted nonlinear control law (11) ensuring the stability of the zero state of the nonlinear dynamics (2) of the position state error of the Lagrangian robotic systems.

Conditions on the Feedback Gains

The nonlinear dynamic model (2) of the Lagrangian robot systems under the proposed nonlinear controller (11), and hence in the closed loop, is reformulated like so:

$$\mathcal{M}(q)\ddot{\phi} + \mathcal{H}(q,\dot{\phi})\dot{\phi} - \mathcal{K}_p\phi - \mathcal{K}_v\dot{\phi} = 0 \tag{12}$$

Then, in order to design and establish some possible conditions on the two matrix gains that ensure the control to and hence the stabilization of the closed-loop robotic system at the desired configuration state q_d, let us propose the following nonlinear Lyapunov function:

$$\mathcal{V}(\phi,\dot{\phi}) = \frac{1}{2}\dot{\phi}^{\mathrm{T}}\mathcal{M}(q)\dot{\phi} - \frac{1}{2}\phi^{\mathrm{T}}\mathcal{K}_p\phi \tag{13}$$

Notice that since the matrix $\mathcal{M}(q)$ satisfies the Property 1, and by considering the following condition on the feedback gain $\mathcal{K}_p$:

$$\mathcal{K}_p = \mathcal{K}_p^T < 0 \tag{14}$$

it stands as a result that $\mathcal{V}(\phi,\dot{\phi}) > 0$.

The derivative the adopted candidate Lyapunov function (13) is described as follows:

$$\begin{aligned}\dot{\mathcal{V}}(\phi,\dot{\phi}) =& \frac{1}{2}\ddot{\phi}^{\mathrm{T}}\mathcal{M}(q)\dot{\phi} + \frac{1}{2}\dot{\phi}^{\mathrm{T}}\dot{\mathcal{M}}(q)\dot{\phi} \\ &+ \frac{1}{2}\dot{\phi}^{\mathrm{T}}\mathcal{M}(q)\ddot{\phi} - \frac{1}{2}\dot{\phi}^{\mathrm{T}}\mathcal{K}_p\phi - \frac{1}{2}\phi^{\mathrm{T}}\mathcal{K}_p\dot{\phi}\end{aligned} \tag{15}$$

As $\mathcal{M}(q) = \mathcal{M}^{\mathrm{T}}(q)$ (according to Property 1) and $\mathcal{K}_p = \mathcal{K}_p^T$, then expression (15) is simplified as follows:

$$\dot{\mathcal{V}}(\phi,\dot{\phi}) = \frac{1}{2}\dot{\phi}^{\mathrm{T}}\dot{\mathcal{M}}(q)\dot{\phi} + \dot{\phi}^{\mathrm{T}}\mathcal{M}(q)\ddot{\phi} - \dot{\phi}^{\mathrm{T}}\mathcal{K}_p\phi \tag{16}$$

Additionally, by solving for $\mathcal{M}\ddot{q}$ in the closed-loop dynamic model (12) and substituting it in (16), it follows that:

$$\dot{\mathcal{V}}(\phi,\dot{\phi}) = \frac{1}{2}\dot{\phi}^{\mathrm{T}}\dot{\mathcal{M}}(q)\dot{\phi} - \dot{\phi}^{\mathrm{T}}\mathcal{H}(q,\dot{\phi})\dot{\phi} + \dot{\phi}^{\mathrm{T}}\mathcal{K}_v\dot{\phi} \tag{17}$$

Moreover, by taking into consideration Property 2, and then relation (4), it follows from (17) that $\frac{1}{2}\dot{\phi}^{\mathrm{T}}\dot{\mathcal{M}}(q)\dot{\phi} - \dot{\phi}^{\mathrm{T}}\mathcal{H}(q,\dot{\phi})\dot{\phi} = 0$.

Thus, the expression (17) of the derivative of the Lyapunov function can be simplified as follows:

$$\dot{\mathcal{V}}(\phi,\dot{\phi}) = \dot{\phi}^{\mathrm{T}}\mathcal{K}_v\dot{\phi} \tag{18}$$

Therefore, to guarantee the stabilization of the zero point of the position state error dynamics model (12), we should ensure that $\dot{\mathcal{V}}(\phi,\dot{\phi}) < 0$. Hence, the matrix gain $\mathcal{K}_v$ must satisfy this condition:

$$\mathcal{K}_v = \mathcal{K}_v^T < 0 \tag{19}$$

Condition Ensuring Equilibrium Unicity

The equilibrium state must satisfy the following conditions $\ddot{\phi} = 0$ and $\dot{\phi} = 0$. Then, the expression (12) becomes:

$$\mathcal{K}_p\phi = 0 \tag{20}$$

According to condition (14), it follows that $\mathrm{rank}(\mathcal{K}_p) = n$. Then, the only solution of the equality (20) is $\phi = 0$, and then it follows that the position vector $q = q_d$ is the unique admissible solution of the condition (20). Thus, the designed conditions (14) and (19) on the two matrix gains $\mathcal{K}_p$ and $\mathcal{K}_v$, respectively, ensure the stabilization (and hence stability) of the robotic system under the proposed PD plus gravity compensation controller (11) at the desired position vector q_d.

Final Stabilization Conditions

Based on inequalities (14) and (19), and in order to ensure the control and therefore the stabilization of the Lagrangian robotic system, we developed the following conditions on the two gain matrices of the adopted nonlinear PD plus gravity compensation control law (20):

$$\mathcal{K}_p = \mathcal{K}_p^T < 0 \tag{21a}$$

$$\mathcal{K}_v = \mathcal{K}_v^T < 0 \tag{21b}$$

DESIGN OF THE PD PLUS DESIRED GRAVITY COMPENSATION CONTROL LAW

This part is dedicated to present a different and simple controller, namely the PD plus desired gravity compensation controller. As before, our main objective is to develop conditions on the two matrix feedback gains of the proposed control law achieving the control of the Lagrangian robot system and therefore its stabilization at the desired state q_d.

The PD Plus Desired Gravity Compensation Controller

In order to stabilize the dynamical system (2), we will consider the following nonlinear PD plus desired gravity compensation control law:

$$\mathcal{U} = \mathcal{D}(q)^{-1}(\mathcal{G}(q_d) + \mathcal{K}_p\phi + \mathcal{K}_v\dot{\phi}) \tag{22}$$

where $\mathcal{K}_p$ and $\mathcal{K}_v$ are the two matrix feedback gains to design in the sequel.

It is obvious that compared to the control law (11), the gravity matrix $\mathcal{G}(q)$ is here constant in the adopted controller (22) and is evaluated at the desired position q_d and therefore equal to the quantity $\mathcal{G}(q_d)$. Notice that expression of the controller (22) is simpler than expression (11). Indeed, the controller (11) is more complex than (22) in realization in practice since it needs much time to be computed because of the gravity matrix $\mathcal{G}(q)$ that takes more time to be calculated at each iteration, or at each computation and application of the controller $\mathcal{U}$ via expression (11). However, in (22), we only need the computation of the matrix $\mathcal{D}(q)$.

We will focus now on designing the conditions on the matrix feedback gains $\mathcal{K}_p$ and $\mathcal{K}_v$ of the nonlinear control law (22) to ensure the control to and consequently the stabilization of the zero-equilibrium point of the position error dynamics (2) under such control law (22), and then guarantee the control/stabilization of the Lagrangian robot system to/at the desired position state q_d.

Conditions on the Feedback Gains

The nonlinear dynamics (2) under the PD plus desired gravity compensation control law $\mathcal{U}$ expressed by (22) is given as follows:

$$\mathcal{M}(q)\ddot{\phi} + \mathcal{H}(q,\dot{\phi})\dot{\phi} + \mathcal{G}(q) - \mathcal{G}(q_d) - \mathcal{K}_p\phi - \mathcal{K}_v\dot{\phi} = 0 \tag{23}$$

To determinate the stability conditions of the controlled system (22), we consider the following candidate Lyapunov function:

$$\begin{aligned}\mathcal{V}(\phi,\dot{\phi}) =& \frac{1}{2}\dot{\phi}^{\mathrm{T}}\mathcal{M}(q)\dot{\phi} + \mathcal{P}_e(q) - k_u - \frac{1}{2}\phi^{\mathrm{T}}\mathcal{K}_p\phi \\ &- \phi^{\mathrm{T}}\mathcal{G}(q_d) - \frac{1}{2}\mathcal{G}(q_d)^{\mathrm{T}}\mathcal{K}_p^{-1}\mathcal{G}(q_d)\end{aligned} \tag{24}$$

where $\mathcal{P}_e(q)$ defines the potential energy. Such quantity $\mathcal{P}_e(q)$ is linked to the gravity vector $\mathcal{G}(q)$ like so:

$$\mathcal{G}(q) = \frac{\partial\mathcal{P}_e(q)}{\partial q} \tag{25}$$

Moreover, the parameter k_u in (24) is defined like so:

$$k_u = \min_q \{\mathcal{P}_e(q)\} \tag{26}$$

Accordingly, from (24) it follows that:

$$\mathcal{P}_e(q) - k_u \geq 0 \tag{27}$$

Moreover, it is straightforward to demonstrate that:

$$\frac{1}{2}\phi^{\mathrm{T}}\mathcal{K}_p\phi + \phi^{\mathrm{T}}\mathcal{G}(q_d) + \frac{1}{2}\mathcal{G}(q_d)^{\mathrm{T}}\mathcal{K}_p^{-1}\mathcal{G}(q_d) = \frac{1}{2}\begin{bmatrix} \phi \\ \mathcal{G}(q_d) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{K}_p & \mathcal{I} \\ \mathcal{I} & \mathcal{K}_p^{-1} \end{bmatrix} \begin{bmatrix} \phi \\ \mathcal{G}(q_d) \end{bmatrix} \tag{28}$$

By taking the following condition on the feedback gain matrix $\mathcal{K}_p$:

$$\mathcal{K}_p = \mathcal{K}_p^T < 0 \tag{29}$$

it is easy to demonstrate, based on the Schur complement lemma (Gritli and Belghith 2018; Turki *et al.* 2020), that

$$\begin{bmatrix} \mathcal{K}_p & \mathcal{I} \\ \mathcal{I} & \mathcal{K}_p^{-1} \end{bmatrix} \leq 0 \tag{30}$$

Therefore, based on inequality (28), it follows that:

$$\frac{1}{2}\phi^{\mathrm{T}}\mathcal{K}_p\phi + \phi^{\mathrm{T}}\mathcal{G}(q_d) + \frac{1}{2}\mathcal{G}(q_d)^{\mathrm{T}}\mathcal{K}_p^{-1}\mathcal{G}(q_d) \leq 0 \tag{31}$$

Moreover, since $\mathcal{M}(q) = \mathcal{M}(q)^{\mathrm{T}} > 0$ (according to Property 1), and relying on conditions (27) and (31), we demonstrate that the adopted candidate Lyapunov function (24) is positive, and hence we ensure that: $\mathcal{V}(\phi, \dot{\phi}) > 0$.

Based on relation (25), the derivative of the candidate Lyapunov function (24) can be formulated as follows:

$$\begin{aligned}\dot{\mathcal{V}}(\phi, \dot{\phi}) =& \frac{1}{2}\dot{\phi}^{\mathrm{T}}\dot{\mathcal{M}}(q)\dot{\phi} + \dot{\phi}^{\mathrm{T}}\mathcal{M}(q)\ddot{\phi} \\ &+ \dot{\phi}^{\mathrm{T}}\mathcal{G}(q) - \phi^{\mathrm{T}}\mathcal{K}_p\dot{\phi} - \dot{\phi}^{\mathrm{T}}\mathcal{G}(q_d)\end{aligned} \tag{32}$$

From the dynamics (23), we can write the following relation:

$$\mathcal{M}(q)\ddot{\phi} = -\mathcal{H}(q, \dot{\phi})\dot{\phi} - \mathcal{G}(q) + \mathcal{G}(q_d) + \mathcal{K}_p\phi + \mathcal{K}_v\dot{\phi} \tag{33}$$

By replacing this last quantity $\mathcal{M}(q)\ddot{\phi}$ in expression (32), we obtain:

$$\begin{aligned}\dot{\mathcal{V}}(\phi, \dot{\phi}) =& \frac{1}{2}\dot{\phi}^{\mathrm{T}}\dot{\mathcal{M}}(q)\dot{\phi} - \dot{\phi}^{\mathrm{T}}\mathcal{H}(q, \dot{\phi})\dot{\phi} + \dot{\phi}^{\mathrm{T}}\left(\mathcal{G}(q_d) - \mathcal{G}(q)\right) \\ &+ \dot{\phi}^{\mathrm{T}}\mathcal{K}_p\phi + \dot{\phi}^{\mathrm{T}}\mathcal{K}_v\dot{\phi} + \dot{\phi}^{\mathrm{T}}\mathcal{G}(q) \\ &- \phi^{\mathrm{T}}\mathcal{K}_p\dot{\phi} - \dot{\phi}^{\mathrm{T}}\mathcal{G}(q_d)\end{aligned} \tag{34}$$

This previous expression can be simplified as follows:

$$\dot{\mathcal{V}}(\phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^{\mathrm{T}}\dot{\mathcal{M}}(q)\dot{\phi} - \dot{\phi}^{\mathrm{T}}\mathcal{H}(q, \dot{\phi})\dot{\phi} + \dot{\phi}^{\mathrm{T}}\mathcal{K}_v\dot{\phi} \tag{35}$$

By virtue of Property 2, the previous function is simplified and equal to:

$$\dot{\mathcal{V}}(\phi, \dot{\phi}) = \dot{\phi}^{\mathrm{T}}\mathcal{K}_v\dot{\phi} \tag{36}$$

Therefore, to guarantee the stabilization of the zero-equilibrium point of the position error dynamic model (23), the following condition must be satisfied:

$$\mathcal{K}_v = \mathcal{K}_v^T < 0 \tag{37}$$

Condition Ensuring Equilibrium Unicity

Note that at the equilibrium state, we have $\dot{\phi} = 0$ and $\ddot{\phi} = 0$. Then, relation (23) can be rewritten and therefore simplified as follows:

$$\mathcal{G}(q) - \mathcal{G}(q_d) - \mathcal{K}_p\phi = 0 \tag{38}$$

Since the gain matrix $\mathcal{K}_p$ satisfies the condition (29), it is then a non-singular matrix. Hence, we can write from (38) the following relation/condition:

$$\phi = \mathcal{K}_p^{-1}(\mathcal{G}(q) - \mathcal{G}(q_d)) \tag{39}$$

It is straightforward to demonstrate that $\phi = q - q_d = 0$ is a solution of the condition (39). Nevertheless, such solution is not the only one of the constraint (39). This equation (39) can generate other solutions. Then, as the condition (39) depends on the feedback gain $\mathcal{K}_p$, our main goal is to develop a condition on such matrix feedback gain $\mathcal{K}_p$ guaranteeing the equilibrium unicity, that is $\phi = 0$. Then, to achieve this objective, we should refer to the contracting mapping presented in Theorem 1.

Consider the following nonlinear function:

$$F(\phi, q_d) = \mathcal{K}_p^{-1}(\mathcal{G}(q) - \mathcal{G}(q_d)) \tag{40}$$

Then, relying on (8), our main objective is to search for some condition on the parameter γ, with $\gamma < 1$, satisfying the following constraint:

$$\|F(\phi_2, q_d) - F(\phi_1, q_d)\| \leq \gamma \|\phi_2 - \phi_1\| \tag{41}$$

Using equation (40), we can write the following expression:

$$F(\phi_2, q_d) - F(\phi_1, q_d) = \mathcal{K}_p^{-1}(\mathcal{G}(\phi_2) - \mathcal{G}(\phi_1)) \tag{42}$$

and therefore, we obtain:

$$\|F(\phi_2, q_d) - F(\phi_1, q_d)\| = \left\|\mathcal{K}_p^{-1}(\mathcal{G}(\phi_2) - \mathcal{G}(\phi_1))\right\| \tag{43}$$

Moreover, it is evident that:

$$\left\|\mathcal{K}_p^{-1}(\mathcal{G}(\phi_2) - \mathcal{G}(\phi_1))\right\| \leq \left\|\mathcal{K}_p^{-1}\right\| \times \|\mathcal{G}(\phi_2) - \mathcal{G}(\phi_1)\| \tag{44}$$

By taking into account the condition (29) on $\mathcal{K}_p$, inequality (44) is recast as follows:

$$\left\|\mathcal{K}_p^{-1}(\mathcal{G}(\phi_2) - \mathcal{G}(\phi_1))\right\| \leq \lambda_{\max}(-\mathcal{K}_p^{-1}) \|\mathcal{G}(\phi_2) - \mathcal{G}(\phi_1)\| \tag{45}$$

Since

$$\lambda_{\max}(-\mathcal{K}_p^{-1}) = \frac{1}{\lambda_{\min}(-\mathcal{K}_p)} \tag{46}$$

it follows that the inequality condition (45) is reformulated as follows:

$$\left\|\mathcal{K}_p^{-1}(\mathcal{G}(\phi_2) - \mathcal{G}(\phi_1))\right\| \leq \frac{1}{\lambda_{\min}(-\mathcal{K}_p)} \|\mathcal{G}(\phi_2) - \mathcal{G}(\phi_1)\| \tag{47}$$

By considering Property 3 on the gravity matrix $\mathcal{G}(q)$, we get the following expression:

$$\left\|\mathcal{K}_p^{-1}(\mathcal{G}(q) - \mathcal{G}(q_d))\right\| \leq \frac{k_g}{\lambda_{\min}(-\mathcal{K}_p)} \|\phi_2 - \phi_1\| \tag{48}$$

where k_g is determined by means of the condition (6).

Hence, relying on expression (43), it follows that:

$$\|F(\phi_2, q_d) - F(\phi_1, q_d)\| \leq \frac{k_g}{\lambda_{\min}(-\mathcal{K}_p)} \|\phi_2 - \phi_1\| \tag{49}$$

By comparing to condition (41), we can deduce that

$$\gamma = \frac{k_g}{\lambda_{\min}(-\mathcal{K}_p)} \tag{50}$$

Accordingly, by taking into account expressions (39) and (40), and in order to guarantee that the zero state $\phi = 0$ is the unique solution of the following relation:

$$\phi = F(\phi, q_d) \tag{51}$$

the following condition (since $\gamma < 1$) must be satisfied:

$$\frac{k_g}{\lambda_{\min}(-\mathcal{K}_p)} < 1 \tag{52}$$

This previous condition (52) can be rewritten like so:

$$\lambda_{\max}(\mathcal{K}_p) < -k_g \tag{53}$$

Final Stabilization Conditions

Finally, relying on constraints (29), (37) and (53), the conditions on the two matrix feedback gains of the adopted control law (20), that is the PD plus desired gravity compensation controller, guaranteeing the control to and hence the stabilization of the Lagrangian-type robot at the point q_d, are reformulated like so:

$$\mathcal{K}_p = \mathcal{K}_p^T < -k_g\,\mathcal{I}_n \tag{54a}$$

$$\mathcal{K}_v = \mathcal{K}_v^T < 0 \tag{54b}$$

where here in (54a) and in the sequel, $\mathcal{I}_n$ stands for the identity matrix with dimension $(n \times n)$.

Moreover, recall that the constant k_g in (54a) should be computed according to expression (6) or expression (7).

DESIGN OF THE COMPUTED-TORQUE CONTROL LAW

In this part, to control the Lagrangian robot system via its nonlinear dynamics (1) to the desired point q_d, we will adopt a computed-torque controller. Thus, our goal in this present section is to design some possible and realizable conditions on the two gain matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ of such controller guaranteeing the stabilization of the zero-equilibrium state $\phi = 0$ of the position error dynamics (2).

The Computed-Torque Controller

The expression of the proposed computed-torque control law has the following form:

$$\mathcal{U} = \mathcal{D}^{-1}(q)(\mathcal{M}(q)v + \mathcal{H}(q,\dot{\phi})\dot{\phi} + \mathcal{G}(q)) \tag{55}$$

where the input v is given by:

$$v = \mathcal{K}_p\phi + \mathcal{K}_v\dot{\phi} \tag{56}$$

Next, we will determinate some feasible conditions on the two gain matrices of the proposed control law $\mathcal{U}$ defined by (55) and (56) in order to achieve the stabilization of the Lagrangian-type robot system under control.

Conditions on the Feedback Gains

Under the adopted computed-torque controller $\mathcal{U}$ defined in (55)-(56), the nonlinear dynamic model (2) of the Lagrangian robot system becomes like so:

$$\mathcal{M}(q)\ddot{\phi} = \mathcal{M}(q)(\mathcal{K}_p\phi + \mathcal{K}_v\dot{\phi}) \tag{57}$$

Based on Property 1, expression (57) becomes:

$$\ddot{\phi} - \mathcal{K}_v\dot{\phi} - \mathcal{K}_p\phi = 0 \tag{58}$$

First design approach of stability conditions

To look for stability conditions of the zero state (defined by $\phi = 0$ and $\dot{\phi} = 0$) of this previous linear system (58), we take a such candidate Lyapunov function defined as follows:

$$\mathcal{V}(\phi,\dot{\phi}) = \frac{1}{2}\dot{\phi}^{\mathrm{T}}\dot{\phi} - \frac{1}{2}\phi^{\mathrm{T}}\mathcal{K}_p\phi \tag{59}$$

It is clear that if we take the following condition:

$$\mathcal{K}_p = \mathcal{K}_p^T < 0 \tag{60}$$

then, the Lyapunov function (59) is positive.

The derivative with respect to time of the Lyapunov function (59) is formulated like so:

$$\dot{\mathcal{V}}(\phi,\dot{\phi}) = \dot{\phi}^{\mathrm{T}}\ddot{\phi} - \phi^{\mathrm{T}}\mathcal{K}_p\dot{\phi} \tag{61}$$

Using the linear dynamics (58) in expression (61), we obtain then:

$$\dot{\mathcal{V}}(\phi,\dot{\phi}) = \dot{\phi}^{\mathrm{T}}\mathcal{K}_v\dot{\phi} \tag{62}$$

Thus, by taking the following condition on the matrix gain $\mathcal{K}_v$:

$$\mathcal{K}_v = \mathcal{K}_v^T < 0 \tag{63}$$

we obtain then $\dot{\mathcal{V}}(\phi,\dot{\phi}) < 0$.

Therefore, the two conditions guaranteeing the stabilization of the zero state of the linear dynamic model (58) are defined like so:

$$\mathcal{K}_p = \mathcal{K}_p^T < 0 \tag{64a}$$

$$\mathcal{K}_v = \mathcal{K}_v^T < 0 \tag{64b}$$

Second design approach of the stability conditions

In the sequel, we look for designing other conditions on the two feedback gain matrices of the computed-torque controller (55)-(56). Thus, let us consider a decoupled controlled linear dynamics (58) and then the gain matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ are diagonal. Moreover, we impose the desired poles of the controlled system (58). Then, our main objective here is to find conditions on theses two gain matrices ensuring the stabilization of the controlled Lagrangian robot system by imposing some desired motion (defined with respect to the desired poles) of a decoupled stable linear system. Then, suppose that all the position variables, q_i with $i = 1, \cdots, n$, of the Lagrangian-type robot system under the proposed computed-torque control law and then of the linear dynamics (58) are completely decoupled and therefore the desired closed-loop linear dynamics of the position q_i is like so:

$$\ddot{\phi}_i - (p_{1i} + p_{2i})\,\dot{\phi}_i + (p_{1i} \times p_{2i})\,\phi_i = 0 \tag{65}$$

where in the previous model, $\phi_i = q_i - (q_d)_i$, and the two parameters p_{1i} and p_{2i} denote together the desired poles of the controlled robotic system.

It is worth to note that the two poles p_{1i} and p_{2i} should be with negative real parts in order to have a stable linear dynamical system (65). In the case they are complex, they should be imperatively complex conjugate.

Relying on (65), the desired decoupled dynamics of the controlled robotic system defined by such linear reference model given as follows:

$$\ddot{\phi} + \Omega_v\dot{\phi} + \Omega_p\phi = 0 \tag{66}$$

where the matrices Ω_v and Ω_p are diagonal and positive definite ($\Omega_v > 0$ and $\Omega_p > 0$). They are defined like so:

$$\Omega_p = \text{diag}\left(p_{11} \times p_{21}, p_{12} \times p_{22}, \cdots, p_{1n} \times p_{2n}\right) \quad (67a)$$
$$\Omega_v = -\text{diag}\left(p_{11} + p_{21}, p_{12} + p_{22}, \cdots, p_{1n} + p_{2n}\right) \quad (67b)$$

Then, in order to obtain an appropriate choice of Ω_v and Ω_p, the two poles p_{1i} and p_{2i} are adopted to be entirely real. Moreover, we impose that $p_{1i} = p_{2i} = -w_i$, where here the parameter w_i is such that $w_i > 0$, and it denotes the desired natural frequency of the subsystem.

Comparing the closed-loop system (58) with that defined by (66), the two feedback gains are then defined as follows:

$$\mathcal{K}_p = -\Omega_p \quad (68a)$$
$$\mathcal{K}_v = -\Omega_v \quad (68b)$$

Recall here that the two matrix gains $\mathcal{K}_p$ and $\mathcal{K}_v$ are diagonal matrices.

Condition Ensuring Equilibrium Unicity

As previously, at the equilibrium state, we have $\dot{\phi} = 0$ and $\ddot{\phi} = 0$. Then, the controlled dynamics defined by (58) can be simplified as follows:

$$\mathcal{K}_p\phi = 0 \quad (69)$$

Since $\Omega_p > 0$, then according to (68a), it follows that $\mathcal{K}_p < 0$ and then $\text{rank}(\mathcal{K}_p) = n$. Notice that the condition $\mathcal{K}_p < 0$ was already determined in (64a), where $\mathcal{K}_p$ is not diagonal. However, by adopting the desired closed-loop decouple dynamics, the gain matrix $\mathcal{K}_p$ is diagonal. Hence, since $\text{rank}(\mathcal{K}_p) = n$, it follows that the state $\phi = 0$ is the only possible solution of the constraint (69).

Final Stabilization Conditions

In the previous development of the conditions on the two matrix feedback gains $\mathcal{K}_p$ and $\mathcal{K}_v$ of the computed-torque controller (55)-(56), we adopted two approaches. In the first approach using the Lyapunov method, we developed the two conditions (64a) and (64b) on $\mathcal{K}_p$ and $\mathcal{K}_v$, which are not diagonal matrices. Therefore, by taking into account the results achieved to check the unicity of the zero equilibrium, these two conditions (64a) and (64b) on the two matrix feedback gains $\mathcal{K}_p$ and $\mathcal{K}_v$ induce the stabilization of the Lagrangian robot system under the computed-torque controller (55)-(56), at the desired position q_d.

However, by adopting a different design approach by imposing a desired decoupled linear dynamics in the closed loop, we developed the two conditions/expressions (68a) and (68b) on the feedback gains $\mathcal{K}_p$ and $\mathcal{K}_v$. Thus, relying on expressions in (67), and by taking $p_{1i} = p_{2i} = -w_i$, with $w_i > 0$ for all $i = 1, 2, \ldots, n$, the two expressions of $\mathcal{K}_p$ and $\mathcal{K}_v$ are formulated as follows:

$$\mathcal{K}_v = -2\,\text{diag}\left(w_1, w_2, \cdots, w_n\right) \quad (70a)$$
$$\mathcal{K}_p = -\text{diag}\left(w_1^2, w_2^2, \cdots, w_n^2\right) \quad (70b)$$

Accordingly, by selecting positive values of the parameters w_1, w_2, $\cdots$, w_n, the two conditions on the matrix feedback gains $\mathcal{K}_p$ and $\mathcal{K}_v$ defined by (70a) and (70b) ensure the stabilization of the Lagrangian robot system under the computed-torque controller (55) at the desired state q_d.

DESIGN OF THE AUGMENTED PD PLUS GRAVITY COMPENSATION CONTROL LAW

In this present section, we will propose a different controller, namely the PD$^+$ controller or the augmented PD plus gravity compensation control law, to achieve the stabilization of Lagrangian-type robotic systems at the desired position vector q_d. Such augmented control law depends chiefly on the PD plus gravity compensation control law (11). Thus, as in the previous sections, our goal is to build some practicable conditions on the two matrix gains of the augmented control law to ensure the control/stabilization of the Lagrangian robotic system, modeled by its nonlinear dynamic model (1), to/at the desired point q_d.

The Augmented PD Plus Gravity Compensation Controller

The augmented PD plus gravity compensation control law (or simply the PD$^+$ controller) is defined:

$$\begin{aligned}\mathcal{U} =& \mathcal{D}^{-1}(q)\left(\mathcal{G}(q) + \mathcal{K}_p\phi + \mathcal{K}_v\dot{\phi}\right)\\ &- \mathcal{D}^{-1}(q)\left(\mathcal{M}(q)\Omega\dot{\phi} + \mathcal{H}(q,\dot{\phi})\Omega\phi\right)\end{aligned} \quad (71)$$

where the matrix Ω is given as follows:

$$\Omega = \mathcal{K}_v^{-1}\mathcal{K}_p \quad (72)$$

with $\mathcal{K}_p$ and $\mathcal{K}_v$ are the two matrix gains to determine in the sequel.

It is obvious that the control law (71) contains the part of the PD plus gravity compensation controller (11). Such part is the first line in the expression (71). The second line in (71) is the augmented part in the controller and aims at improving the stabilization process of the robotic system.

Then, our objective in the following is to design the conditions on the gain matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ guaranteeing the stabilization of the Lagrangian robotic system under the proposed controller (71) at the desired state q_d.

Condition on the Feedback Gains

By considering and applying the adopted augmented PD plus gravity compensation control law (71), the nonlinear dynamics (2) becomes:

$$\mathcal{M}(q)\ddot{\phi} + \mathcal{H}(q,\dot{\phi})\dot{\phi} = \mathcal{K}_p\phi + \mathcal{K}_v\dot{\phi} - \mathcal{M}(q)\Omega\dot{\phi} - \mathcal{H}(q,\dot{\phi})\Omega\phi \quad (73)$$

This expression (73) of the closed-loop nonlinear dynamics can be rearranged and simplified as follows:

$$\mathcal{M}(q)(\ddot{\phi} + \Omega\dot{\phi}) + \mathcal{H}(q,\dot{\phi})(\dot{\phi} + \Omega\phi) - \mathcal{K}_v(\dot{\phi} + \Omega\phi) = 0 \quad (74)$$

We emphasize that the equilibrium point of the closed-loop system (74) is $\phi = 0$, with $\ddot{\phi} = 0$ and $\dot{\phi} = 0$. The proof will be provided in the next section.

Posing in the sequel

$$\psi = \dot{\phi} + \Omega\phi \quad (75)$$

Then, $\psi = 0$ defines the new equilibrium state. Therefore, using this previous variable change, expression (74) can be written like this:

$$\mathcal{M}(q)\dot{\psi} + (\mathcal{H}(q,\dot{\phi}) - \mathcal{K}_v)\psi = 0 \quad (76)$$

Thus, let us adopt a such candidate Lyapunov function defined as follows:

$$\mathcal{V}(\psi) = \frac{1}{2}\psi^{\mathrm{T}}\mathcal{M}(q)\psi \quad (77)$$

The derivative with respect to time of the adopted candidate Lyapunov function (77) is expressed like so:

$$\dot{\mathcal{V}}(\psi) = \psi^{\mathrm{T}}\mathcal{M}(q)\dot{\psi} + \frac{1}{2}\psi^{\mathrm{T}}\dot{\mathcal{M}}(q)\psi \tag{78}$$

Moreover, by solving $\mathcal{M}\dot{\psi}$ in the closed-loop dynamic model (76) and substituting it into (78), we obtain:

$$\dot{\mathcal{V}}(\psi) = \psi^{\mathrm{T}}\mathcal{K}_v\psi + \psi^{\mathrm{T}}\left(\frac{1}{2}\dot{\mathcal{M}}(q) - \mathcal{H}(q,\dot{\phi})\right)\psi \tag{79}$$

Furthermore, based on Property 2, it follows that expression (79) is simplified like so:

$$\dot{\mathcal{V}}(\psi) = \psi^{\mathrm{T}}\mathcal{K}_v\psi \tag{80}$$

Therefore, to ensure the stability of the transformed closed-loop nonlinear system (76) at the zero-equilibrium point $\psi = 0$, we should have $\dot{\mathcal{V}}(\psi) < 0$, and the following condition on the matrix feedback gain $\mathcal{K}_v$ must be verified:

$$\mathcal{K}_v = \mathcal{K}_v^T < 0 \tag{81}$$

Moreover, since $\psi = \dot{\phi} + \Omega\phi$, and as $\psi = 0$ is the stable equilibrium point of the dynamic model (76), it follows then that the state ϕ evolves with respect to the following linear dynamics:

$$\dot{\phi} + \Omega\phi = 0 \tag{82}$$

The clear condition for the stabilization of the zero equilibrium, $\phi = 0$, is that $\Omega > 0$. Therefore, as the matrix Ω is defined by expression (72) and by taking into account the condition (81) on the gain $\mathcal{K}_v$, we emphasize that the position feedback gain $\mathcal{K}_p$ should satisfy the following condition:

$$\mathcal{K}_p = \mathcal{K}_p^T < 0 \tag{83}$$

Condition Ensuring Equilibrium Unicity

Consider the previously closed-loop nonlinear dynamic model (74) of the Lagrangian robotic system. As in the previous parts, the equilibrium of such system must satisfy $\dot{\phi} = 0$ and $\ddot{\phi} = 0$. Then, according to these evaluations and by taking into account expression (72), the nonlinear dynamics (74) at the equilibrium is simplified as follows:

$$\mathcal{K}_p\phi = \mathcal{H}(q,0)\Omega\phi \tag{84}$$

Relying on condition (83), it follows that $\mathrm{rank}(\mathcal{K}_p) = n$. Then, relation (84) is rewritten like so:

$$\phi = \mathcal{K}_p^{-1}\mathcal{H}(\phi,0)\Omega\phi \tag{85}$$

It is easy to show that $\phi = 0$ is a solution of the obtained expression (85), however it is not the only one. It is feasible to obtain other solutions satisfying this equation (85). Thus, our main goal in the sequel is to design some possible but feasible conditions on the two matrix gains $\mathcal{K}_v$ and $\mathcal{K}_p$ guaranteeing the uniqueness of the solution $\phi = 0$. Then, to achieve this objective, we will rely on Theorem 1 of the contracting mapping.

To apply such contracting mapping, let us pose first the following nonlinear function $\varphi(\phi)$:

$$\varphi(\phi) = \mathcal{K}_p^{-1}\mathcal{H}(\phi,0)\Omega\phi \tag{86}$$

and then expression (85) becomes:

$$\phi = \varphi(\phi) \tag{87}$$

Then, to guarantee the uniqueness of the zero solution (that is $\phi = 0$) and from expression (85), it is easy to expand the following inequality:

$$\begin{aligned}\|\varphi(\phi_1) - \varphi(\phi_2)\| \leq &\left\|\mathcal{K}_p^{-1}\right\| \times \|\Omega\| \\ &\times \|\mathcal{H}(\phi_1,0)\phi_1 - \mathcal{H}(\phi_2,0)\phi_2\|\end{aligned} \tag{88}$$

Using expression of the matrix Ω defined by (72), and by taking into consideration the two conditions (83) and (81) respectively on $\mathcal{K}_p$ and $\mathcal{K}_v$, we can develop the following relation:

$$\|\Omega\| \leq \lambda_{\max}(-\mathcal{K}_v^{-1})\lambda_{\max}(-\mathcal{K}_p) \tag{89}$$

Since

$$\lambda_{\max}(-\mathcal{K}_v^{-1}) = \frac{1}{\lambda_{\min}(-\mathcal{K}_v)} \tag{90}$$

it follows then that condition (89) becomes:

$$\|\Omega\| \leq \frac{\lambda_{\max}(-\mathcal{K}_p)}{\lambda_{\min}(-\mathcal{K}_v)} \tag{91}$$

Therefore, relying on condition (91), expression (88) is simplified as follows:

$$\begin{aligned}\|\varphi(\phi_1) - \varphi(\phi_2)\| \leq &\frac{1}{\lambda_{\min}(-\mathcal{K}_p)}\frac{\lambda_{\max}(-\mathcal{K}_p)}{\lambda_{\min}(-\mathcal{K}_v)} \\ &\|\mathcal{H}(\phi_1,0)\phi_1 - \mathcal{H}(\phi_2,0)\phi_2\|\end{aligned} \tag{92}$$

Supposing that the matrix $\mathcal{H}(\phi,\dot{\phi})\phi$ satisfies the following Lipschitz constraint:

$$\|\mathcal{H}(\phi_1,\dot{\phi}_1)\phi_1 - \mathcal{H}(\phi_2,\dot{\phi}_2)\phi_2\| \leq k_c\,\|\phi_1 - \phi_2\| \tag{93}$$

where k_c is some positive constant like so:

$$k_c \geq n^2 \max_{k,i,j,q,\dot{q}} \left|\mathcal{H}_{i,j}(q_k,\dot{q}_k)\right| \tag{94}$$

Using then the constraint (93) in (92), we obtain hence the following condition:

$$\|\varphi(\phi_1) - \varphi(\phi_2)\| \leq \frac{1}{\lambda_{\min}(-\mathcal{K}_p)}\frac{\lambda_{\max}(-\mathcal{K}_p)}{\lambda_{\min}(-\mathcal{K}_v)}k_c\,\|\phi_1 - \phi_2\| \tag{95}$$

Relying on the contraction mapping in Theorem 1, and to guarantee that $\phi = 0$ is a solution and the unique one of the equation (85) or (87), we obtain the following condition:

$$\frac{k_c}{\lambda_{\min}(-\mathcal{K}_p)}\frac{\lambda_{\max}(-\mathcal{K}_p)}{\lambda_{\min}(-\mathcal{K}_v)} < 1 \tag{96}$$

As we can write the following equivalent equalities:

$$\lambda_{\min}(-\mathcal{K}_p) = -\lambda_{\max}(\mathcal{K}_p) \tag{97a}$$
$$\lambda_{\max}(-\mathcal{K}_p) = -\lambda_{\min}(\mathcal{K}_p) \tag{97b}$$
$$\lambda_{\min}(-\mathcal{K}_v) = -\lambda_{\max}(\mathcal{K}_v) \tag{97c}$$

then the previous condition (96) can be recast as follows:

$$\frac{k_c}{\lambda_{\max}(\mathcal{K}_p)}\frac{\lambda_{\min}(\mathcal{K}_p)}{\lambda_{\max}(\mathcal{K}_v)} > -1 \tag{98}$$

From this condition (98), and since $\mathcal{K}_p < 0$ and $\mathcal{K}_v < 0$, therefore we expand such condition on the feedback gain $\mathcal{K}_v$, given as follows:

$$\lambda_{\max}(\mathcal{K}_v) < -k_c\frac{\lambda_{\min}(\mathcal{K}_p)}{\lambda_{\max}(\mathcal{K}_p)} \tag{99}$$

that ensures the uniqueness of the zero solution $\phi = 0$ of the nonlinear equation (85).

Final Stabilization Conditions

By considering the three previously established conditions (81), (83) and (99) on the feedback gains, we obtain the following two simplified conditions:

$$\mathcal{K}_p = \mathcal{K}_p^T < 0 \tag{100a}$$

$$\mathcal{K}_v = \mathcal{K}_v^T < -k_c \frac{\lambda_{\min}(\mathcal{K}_p)}{\lambda_{\max}(\mathcal{K}_p)} \mathcal{I}_n \tag{100b}$$

We can conclude therefore that the two conditions (100a) and (100b) on the two feedback gain matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ of the proposed augmented PD plus gravity compensation controller (71), guarantee the control of the Lagrangian robot system to the desired point q_d and hence its stabilization at q_d.

SIMULATION RESULTS

This present section is devoted to presenting the numerical simulation and graphical results of the different control laws proposed in this research work, and then to illustrate the efficiency of the built conditions of the two feedback gain matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ of these control laws to control and stabilize the position of Lagrangian-type robot systems to the desired point q_d. Then, we will propose the planar 2-DoF robot manipulator shown in Fig. 1 as an illustrative example. Such robot system contains two joints/articulations with absolute angular positions q_1 and q_2. Thus, let the vector $q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$ defines the state vector of absolute angular positions.

Using the nonlinear dynamics (1) and according to the relative position coordinates q_1 and q_2, the various matrices of the planar 2-DoF manipulator robot are described as follows (Gritli *et al.* 2022; Jenhani *et al.* 2022a,b,c):

$$\mathcal{M}(q) = \begin{bmatrix} m_1 a_1^2 + m_2 l_1^2 + I_1 & m_2 l_1 a_2 \cos(q_1 - q_2) \\ m_2 l_1 a_2 \cos(q_1 - q_2) & m_2 a_2^2 + I_2 \end{bmatrix} \tag{101a}$$

$$\mathcal{H}(q, \dot{q}) = \sin(q_1 - q_2) \begin{bmatrix} 0 & m_2 l_1 a_2 \dot{q}_2 \\ -m_2 l_1 a_2 \dot{q}_1 & 0 \end{bmatrix} \tag{101b}$$

$$\mathcal{G}(q) = g \begin{bmatrix} (m_1 a_1 + m_2 l_1)\cos(q_1) \\ m_2 a_2 \cos(q_2) \end{bmatrix} \tag{101c}$$

$$\mathcal{D}(q) = \mathcal{D} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{101d}$$

Moreover, the values and descriptions of the various parameters found in these matrices (101) are defined in Table 1.

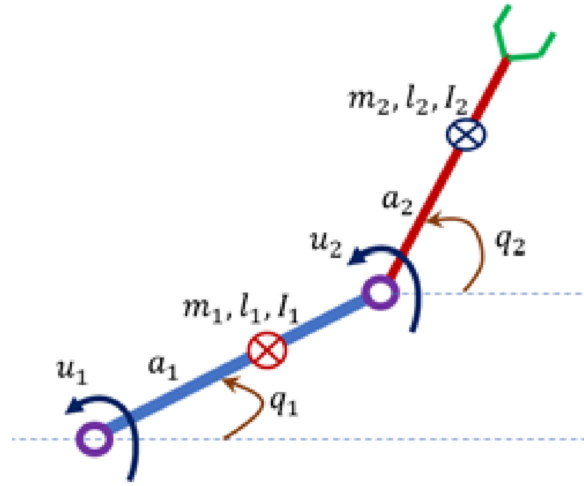


Figure 1 The adopted two-degree-of-freedom planar manipulator robot and its associated geometric and inertial parameters.

It is clear that the manipulator robotic system is composed of two links that are both controlled. Then, the control input vector $\mathcal{U}$ is composed of two control sub-inputs u_1 and u_2. The first joint is controlled via the input u_1, whereas the second joint is controlled via the second input u_2. Thus, we have $\mathcal{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

Let consider in the sequel the desired configuration state q_d to be as follows:

$$q_d = \begin{bmatrix} 90° \\ -45° \end{bmatrix} \tag{102}$$

Furthermore, and in order to make a comparison with the evolution of the controlled manipulator robot, all the simulation simulations with the proposed control laws start at the same initial position point:

$$q_0 = \begin{bmatrix} 0° \\ 0° \end{bmatrix} \tag{103}$$

Table 1 Used parameters and their values for the numerical simulation of the 2-DoF manipulator robotic system in Fig. 1.

Parameter	Value	Description
m_1	2 Kg	Mass of the first pendulum of the manipulator robot
a_1	0.375 m	Distance taken from the first articulation of the robot to the center of mass (CoM) of its first pendulum
m_2	1 Kg	Mass of the second pendulum of the manipulator robot
a_2	0.25 m	Distance taken from the second joint to the CoM of the second pendulum of the robotic manipulator
l_1	0.5 m	Length of the first pendulum of the manipulator robot
I_1	0.02 kg.m^2	Rotational inertia parameter of the first pendulum of the manipulator robotic system
l_2	0.4 m	Length of the second pendulum of the manipulator robot
I_2	0.01 kg.m^2	Rotational inertia parameter of the second pendulum of the manipulator robot
g	9.81 m/s^2	Gravitational constant

Simulation Results Obtained Using the PD Plus Gravity Compensation Controller

In this first part, we will consider and use the PD plus gravity compensation controller (11). Thus, based on the two conditions (21a) and (21b), the feedback gains $\mathcal{K}_v$ and $\mathcal{K}_p$ of the adopted nonlinear control law are computed to be like so:

$$\mathcal{K}_p = \begin{bmatrix} -20.0 & 5.0 \\ 5.0 & -10.0 \end{bmatrix} \tag{104a}$$

$$\mathcal{K}_v = \begin{bmatrix} -10.0 & 5.0 \\ 5.0 & -20.0 \end{bmatrix} \tag{104b}$$

It is straightforward to verify that the eigenvalues of two matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ are real (since they are symmetric) and negative. Therefore, the two inequality constraints (21a) and (21b) on these feedback gains have been well satisfied.

It is worth to mention that the values of the two gain matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ adopted in (104a) and (104b), are selected in order to compare with the augmented PD plus gravity compensation control law (71) by choosing the same gains (113a) and (113b).

Using the two feedback gain matrices (104a) and (104b) in the adopted PD plus gravity compensation controller defined by expression (11), the 2-DoF robot manipulator will be then controlled to and stabilize at the point q_d. Fig. 2(a) demonstrates the temporal evolution of the two angular positions q_1 and q_2 (or the position vector q) of the robotic system in question. It is clear from the curves of q_1 and q_2 that these two states converge to q_d. Moreover, Fig. 2(b) demonstrates that the temporal variation of the angular velocities $\dot{q}_1$ and $\dot{q}_2$ tends to zero. Furthermore, Fig. 2(c) presents the evolution of the PD plus gravity compensation controller $\mathcal{U}$. It is obvious that when the robotic system is controlled and hence stabilized at q_d, the control subinputs u_1 and u_2 (of the control law $\mathcal{U}$) converge towards the constant value 1.7342.

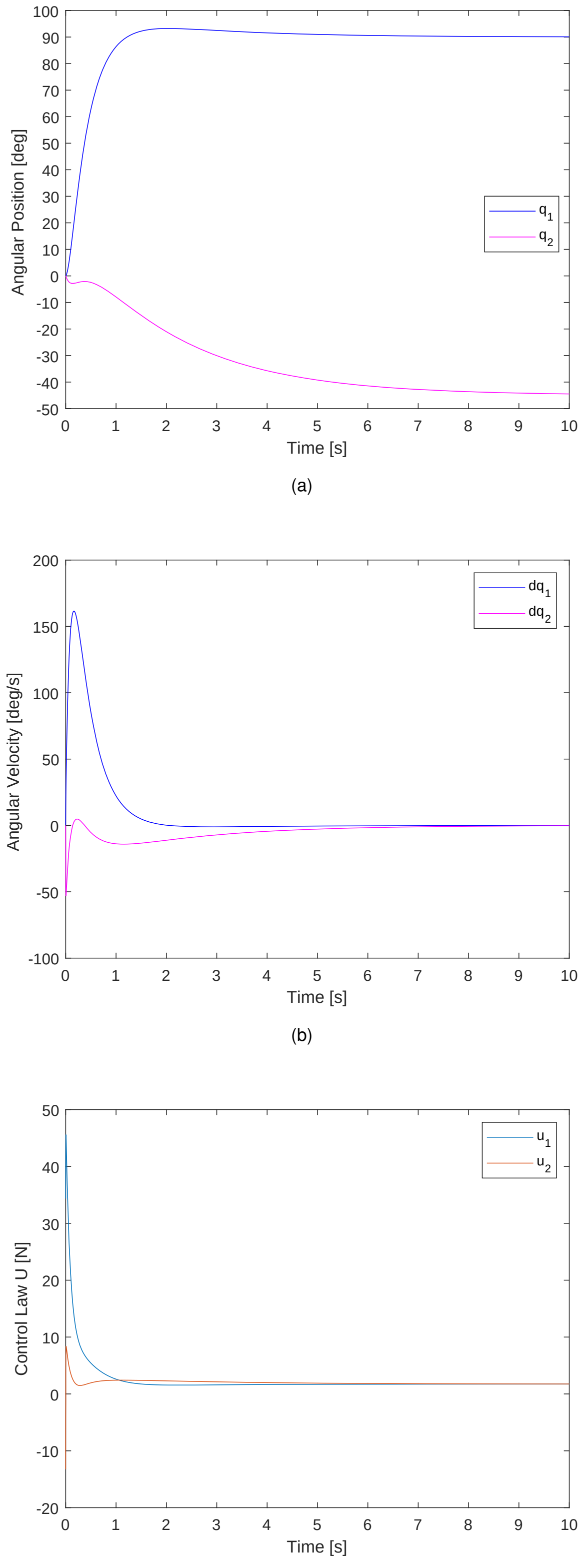
As noted previously, the two gains (104a) and (104b) adopted for the PD with gravity compensation control law are equal to (113a) and (113b), which will be adopted for the augmented PD plus gravity compensation control law. We consider now other values of the matrix feedback gains $\mathcal{K}_v$ and $\mathcal{K}_p$ different to those in (104a) and (104b) as follows:

$$\mathcal{K}_p = \begin{bmatrix} -2.0 & 1.0 \\ 1.0 & -2.0 \end{bmatrix} \tag{105a}$$

$$\mathcal{K}_v = \begin{bmatrix} -2.0 & 1.0 \\ 1.0 & -2.0 \end{bmatrix} \tag{105b}$$

Notice that the eigenvalues of these matrix gains (105a) and (105b) are -1 and -3. Then, the two conditions (21a) and (21b) are well satisfied. We emphasize that these two gains (105a) and (105b) will not satisfy the conditions (100a) and (100b) for the augmented PD plus gravity compensation control law.

By introducing the two feedback gain matrices (105a) and (105b) in the PD plus gravity compensation controller defined by expression (11), we obtain the results in Fig. 3 revealing the control and hence stabilization of the 2-DoF robotic manipulator at q_d. Moreover, the Fig. 3(a) demonstrates the temporal evolution of the positions q_1 and q_2. It is clear from the curves of q_1 and q_2 that these two states converge to q_d. In addition, the Fig. 3(b) depicts the variation of the angular velocities $\dot{q}_1$ and $\dot{q}_2$. Compared to the results in Fig. 2(a) and Fig. 2(b), the stabilization/convergence time at/to the desired point q_d is almost the same, about 7 $[s]$. Nevertheless, the obvious difference between the two results lies in the response of the angular velocities of the two links. Indeed, in Fig. 2(a), the angular velocity $\dot{q}_1$ reaches a high value about 150 $[deg/s]$ before its rapid and asymptotic decrease to zero. However, in Fig. 3(a), the angular velocity $\dot{q}_1$ reaches a relatively small value around 60 $[deg/s]$ before its convergence to zero. In contrast, the angular velocity $\dot{q}_2$ in Fig. 2(a) decreases to the minimal value $\approx -50\,[deg/s]$, whereas in Fig. 3(a) reaches the minimal value $\approx -80\,[deg/s]$.



(c)

Figure 2 Temporal evolution of: (a) the two angular positions q_1 and q_2, (b) the two corresponding angular velocities $\dot{q}_1$ and $\dot{q}_2$, and (c) the PD with gravity compensation control law $\mathcal{U} = [u_1 \;\; u_2]^{\mathrm{T}}$, by adopting the two matrix feedback gains (104a) and (104b).

Furthermore, the curves in Fig. 3(c) illustrate the temporal variation of the controller $\mathcal{U}$ applied to the manipulator robot system. As in Fig. 2(c), when the manipulator robot was control to and hence stabilized at q_d, the two inputs of control u_1 and u_2 converge together towards the constant effort 1.7342. Compared to the results in Fig. 2(c) where u_1 (resp. u_2) reached the maximal value about 45 $[N]$ (resp. $-15\,[N]$), in Fig. 3(c) the controller u_1 (resp. u_2) reaches the maximal value around 16 $[N]$ (resp. 3 $[N]$). Hence, the difference between the two control results is evident.

Numerical Results Obtained with the PD Plus Desired Gravity Compensation Controller

In this subsection, we will focus on the simulation results obtained by applying the the PD plus desired gravity compensation controller. Such control law and the associated conditions on its feedback gains are presented in the forth section. Expression of such controller is defined by (22) and the designed stabilization conditions are defined by the two constraints (54a) and (54b). The computation of the value of the constant k_g in the condition (54a) according to expression (6) gives:

$$k_g = 12.2625 \tag{106}$$

Then, according to these conditions (54a) and (54b), and by adopting the previous value of k_g in (106), we select the following matrices of $\mathcal{K}_p$ and $\mathcal{K}_v$:

$$\mathcal{K}_p = \begin{bmatrix} -15.0 & -1.0 \\ -1.0 & -20.0 \end{bmatrix} \tag{107a}$$

$$\mathcal{K}_v = \begin{bmatrix} -2.0 & -1.0 \\ -1.0 & -6.0 \end{bmatrix} \tag{107b}$$

The eigenvalues of the matrix $\mathcal{K}_p$ are -20.1926 and -14.8074. Since, $\lambda_{\max}(\mathcal{K}_p) = -14.8074 < -k_g = -12.2625$, then the first condition (54a) has been well respected. Moreover, it is obvious that the matrix $\mathcal{K}_v$ is negative definite.

By introducing the two matrix feedback gains (107a) and (107b) into the PD plus the desired gravity compensation controller (22), the manipulator robot system will be then controlled to and therefore stabilized at q_d. Figure 4(a) presents the angular positions q_1 and q_2, where q_1 and q_2 converge towards q_d. Additionally, we reveal from Fig. 4(b) that the temporal behavior of the two corresponding angular velocities $\dot{q}_1$ and $\dot{q}_2$ of the robotic system converge to zero. Moreover, Fig. 4(c) shows the behavior of the proposed PD plus the desired gravity compensation controller $\mathcal{U}$. Obviously, the two sub-controllers u_1 and u_2 converge together to the same constant value 1.7342, as in the previous control case.

Actually, in order to demonstrate the importance of the unicity condition of the zero solution established for the development of the stabilizing gains of the control law $\mathcal{U}$, we slightly modified the matrix $\mathcal{K}_p$ in order that the first condition (54a) will be not respected. Thus, we select the following matrix gain $\mathcal{K}_p$:

$$\mathcal{K}_p = \begin{bmatrix} -10.0 & -1.0 \\ -1.0 & -20.0 \end{bmatrix} \tag{108}$$

Such matrix (108) has the following eigenvalues : -20.0990 and -9.9010. It is evident that $\lambda_{\max}(\mathcal{K}_p) = -9.9010 > -k_g = -12.2625$. Therefore, the first condition (54a) has not been satisfied.

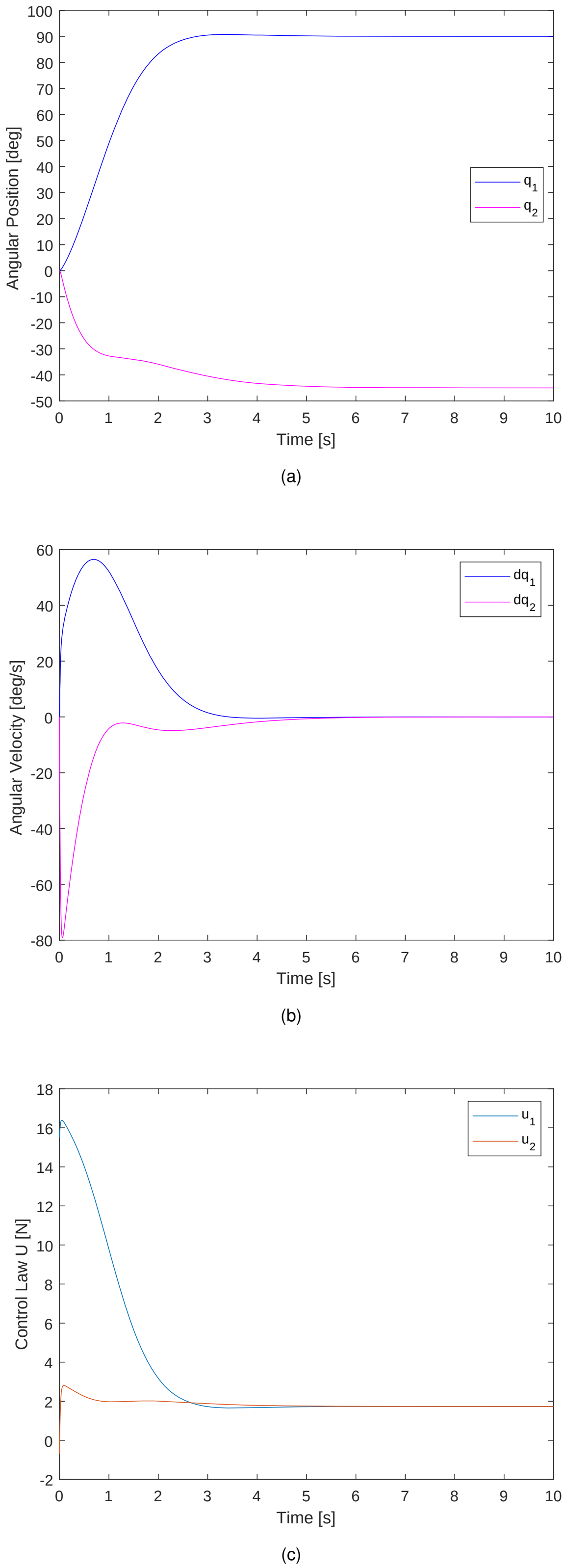


Figure 3 Temporal evolution of: (a) the two angular positions q_1 and q_2, (b) the two corresponding angular velocities $\dot{q}_1$ and $\dot{q}_2$, and (c) the PD plus gravity compensation controller $\mathcal{U} = [u_1 \ \ u_2]^{\mathrm{T}}$ by adopting the two matrix feedback gains (105a) and (105b).

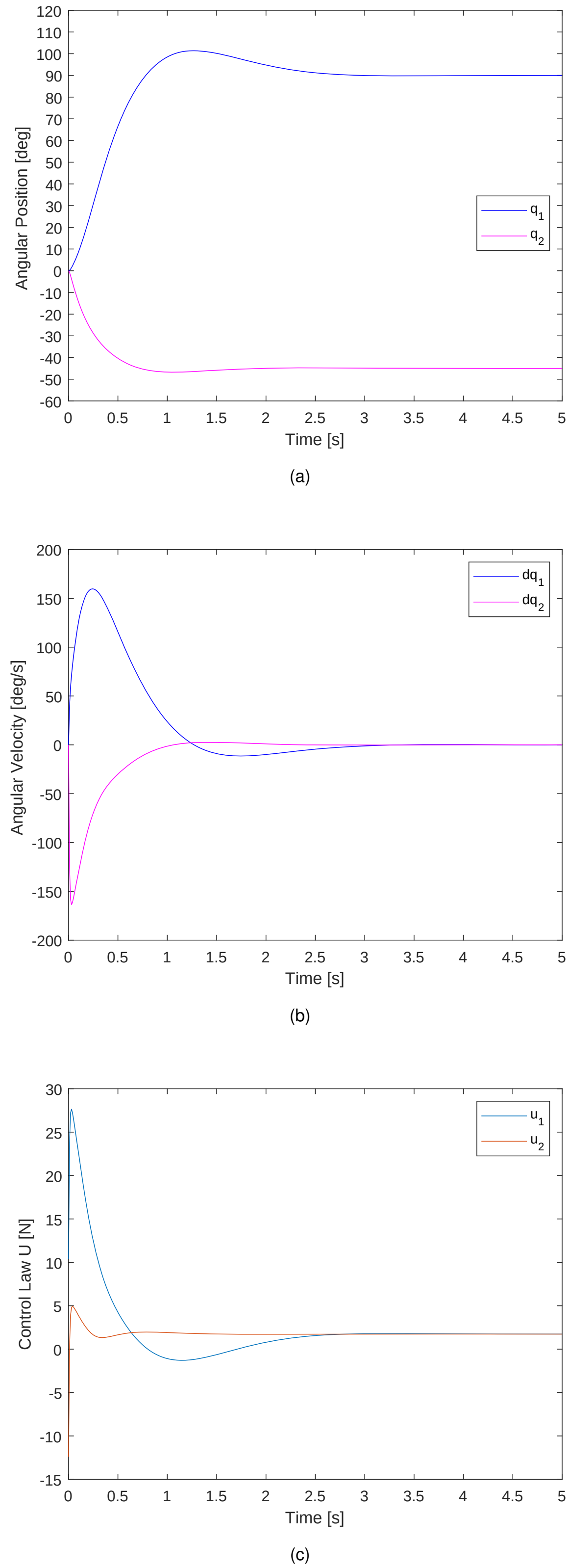


Figure 4 Temporal variation of: (a) the two angular positions q_1 and q_2, (b) the two corresponding angular velocities $\dot{q}_1$ and $\dot{q}_2$, and (c) the PD plus desired gravity compensation control law, by adopting the two matrix feedback gains (107a) and (107b).

The objective behind this previous modification is to show the evolution of the robotic system and its stabilization at a position state completely different to the desired one q_d. By using then the PD plus desired gravity compensation controller $\mathcal{U}$ according the position gain $\mathcal{K}_p$ in (108) and the velocity gain $\mathcal{K}_v$ in (107b), using its nonlinear dynamic model (1), we demonstrate the simulation results in Fig. 5. It is clear from the temporal variation of the angular position of the manipulator robot in Fig. 5(a) and that of the angular velocity in Fig. 5(b), the robot is stabilized at the position $q_\infty = \begin{bmatrix} 27.2468^\circ & -42.1061^\circ \end{bmatrix}^T$. Thus, this final state q_∞ is entirely different to the desired one q_d, which was already defined in (102). Moreover, the evolution of the controller in Fig. 5(a) is completely different to that in Fig. 4(c). These results demonstrate accordingly the importance of establishing the condition on the gains of the controller to guarantee the uniqueness of the desired position q_d when the robotic system is stabilized.

Numerical Results Obtained Under the Computed-Torque Control Law

The computed-torque controller is defined by expressions (55) and (56). The designed conditions on the two matrix feedback gains $\mathcal{K}_p$ and $\mathcal{K}_v$ are (70a) and (70b). Using these conditions and by taking $w_1 = w_2 = 2$, we get the following diagonal matrices of $\mathcal{K}_p$ and $\mathcal{K}_v$:

$$\mathcal{K}_p = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \tag{109a}$$

$$\mathcal{K}_v = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \tag{109b}$$

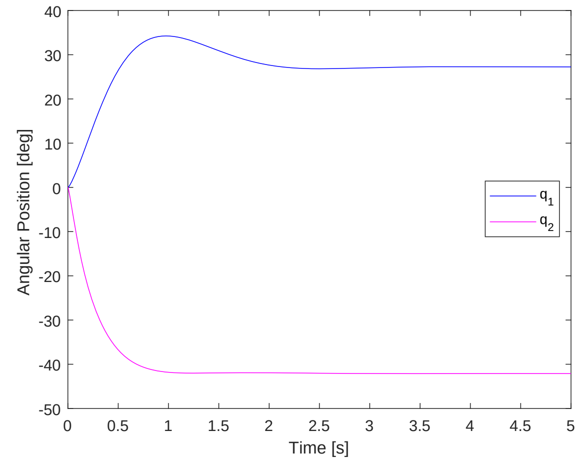
Thus, by introducing these feedback gains in the proposed computed-torque control law (55)-(56), the robotic system, that is the 2-DoF manipulator robot, will be controlled to q_d. Figure 6(a) denotes the dynamic behavior of q_1 and q_2 of the manipulator robot, where q_1 and q_2 converge towards q_d. Thus, the temporal simulation of the corresponding angular velocities is illustrated in Fig. 6(b), where it reveals that $\dot{q}_1$ and $\dot{q}_2$ converge together to zero. Moreover, the Fig. 6(c) shows the temporal behavior of the applied computed-torque controller. In the present case, the control subinputs u_1 and u_2 converge together to 1.7342, as in the two previous cases.

Numerical Results Obtained with the Augmented PD Plus Gravity Compensation Control Law

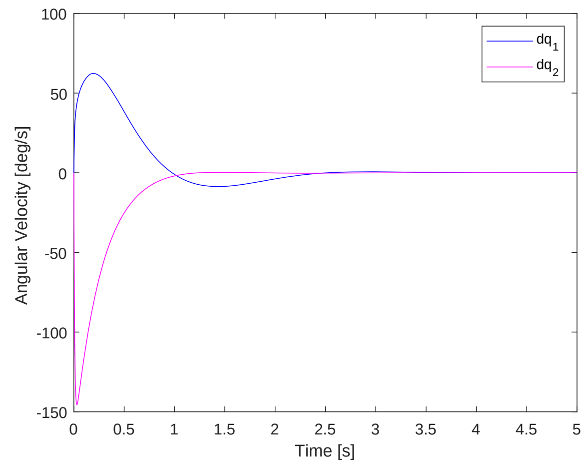
The augmented PD with gravity compensation controller is defined by expression (71). Moreover, the established stabilization conditions are defined by the two inequality constraints (100a) and (100b). According to condition (100b), the feedback gain $\mathcal{K}_v$ depends on the constant k_c and the gain matrix $\mathcal{K}_p$. Recall that the constant k_c should be computed via condition (94). According to expression (101b) of the matrix $\mathcal{H}(q, \dot{q})$ and the values of the parameters of the 2-DoF manipulator robotic system in Table 1, it easy to show that condition (94) leads to the following inequality:

$$k_c \geq 0.5 \tag{110}$$

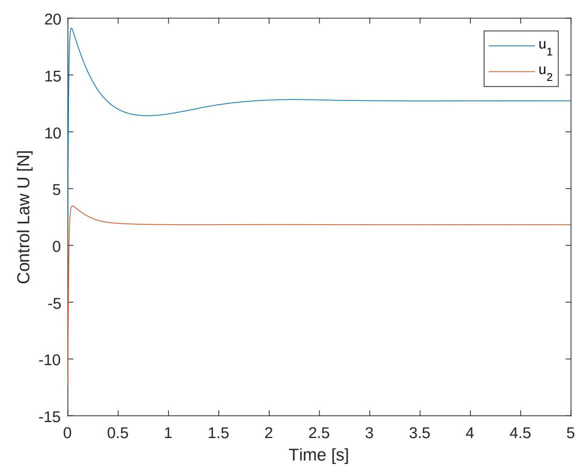
We take here the same gain matrix $\mathcal{K}_p$ adopted for the PD plus gravity compensation controller, that is (104a). The value



(a)

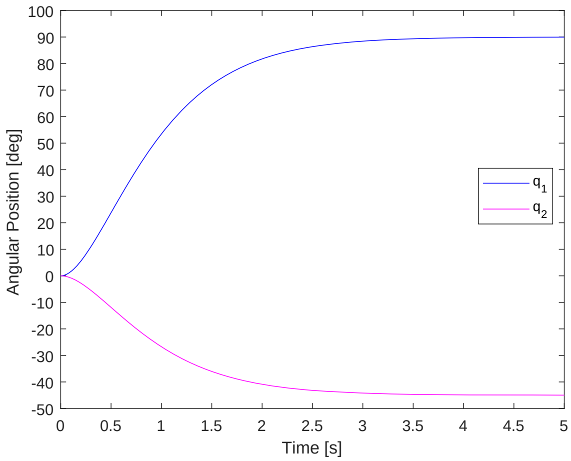


(b)

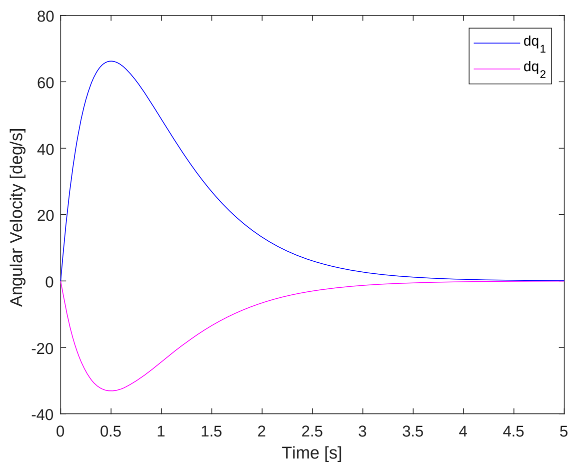


(c)

Figure 5 Simulation results of the controlled robot by selecting a different matrix (108) of $\mathcal{K}_p$: (a) q_1 and q_2, (b) $\dot{q}_1$ and $\dot{q}_2$, (c) the applied PD plus desired gravity compensation controller, by adopting the position gain $\mathcal{K}_p$ in (108) and the velocity gain $\mathcal{K}_v$ in (107b).



(a)



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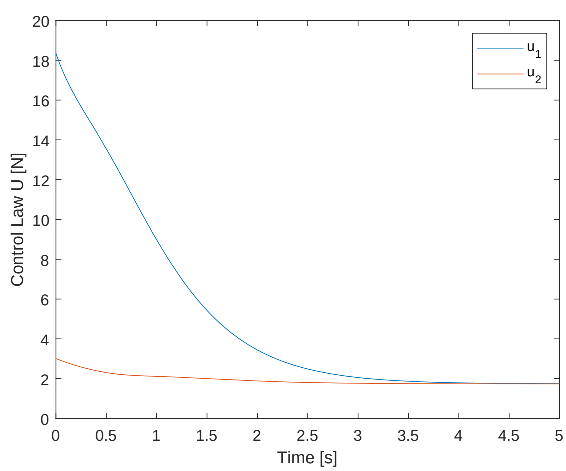


(c)

Figure 6 Temporal variation of: (a) the two angular positions q_1 and q_2 of the manipulator robot, (b) the two corresponding angular velocities $\dot{q}_1$ and $\dot{q}_2$, and (c) the applied computed-torque controller $\mathcal{U}$, by adopting the two matrix feedback gains (109a) and (109b).

of the feedback gain $\mathcal{K}_v$ should be selected via condition (100b). According to the matrix (104a) of $\mathcal{K}_p$, we obtain:

$$\lambda_{\min}(\mathcal{K}_p) = -22.0711 \tag{111a}$$

$$\lambda_{\max}(\mathcal{K}_p) = -7.9289 \tag{111b}$$

Hence, using expressions (111a) and (111b) and inequality (110), it is straightforward to show that according to the condition (100b), the feedback gain $\mathcal{K}_v$ should satisfy and be selected according to the following constraint:

$$\mathcal{K}_v = \mathcal{K}_v^T < -1.3918\, \mathcal{I}_n \tag{112}$$

Then, to satisfy this condition (112), we will adopt the same value (104b) of the feedback gain $\mathcal{K}_v$ adopted for the PD plus gravity compensation control law. Hence, for the controller in question, we will take the following values of $\mathcal{K}_p$ and $\mathcal{K}_v$:

$$\mathcal{K}_p = \begin{bmatrix} -20.0 & 5.0 \\ 5.0 & -10.0 \end{bmatrix} \tag{113a}$$

$$\mathcal{K}_v = \begin{bmatrix} -10.0 & 5.0 \\ 5.0 & -20.0 \end{bmatrix} \tag{113b}$$

It is obvious that these gains (113a) and (113b) are similar to those in (104a) and (104b) of the PD plus gravity compensation controller. The eigenvalues of the two matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ are -22.0711 and -7.9289. It is clear that the condition (112) is well respected.

Using these two gains (113a) and (113b) in the augmented PD controller defined in (71), the planar 2-DoF manipulator is then controlled to q_d as revealed via Fig. 7(a), where convergence of the two angular positions q_1 and q_2 towards q_d is clear. In the Fig. 7(b), we show that the two angular velocities converges to zero, which justifies the stabilization at the desired point q_d. Moreover, Fig. 7(c) illustrates the applied controller $\mathcal{U}$. As in the previous cases, u_1 and u_2 converge progressively to the same value, which is almost equal to 1.734.

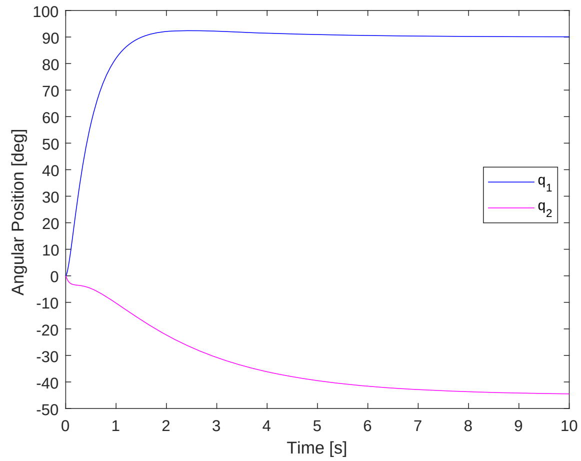
We consider now two other values of the two feedback gain matrices $\mathcal{K}_p$ and $\mathcal{K}_v$, and then we select the same gains (105a) and (105b) that were adopted for the PD plus gravity compensation controller. Since the eigenvalues of these two matrices $\mathcal{K}_p$ and $\mathcal{K}_v$ are -1 and -3, it follows then that the condition (112) was not verified. To satisfy this stabilization condition (112), we make a slight change in the two matrices $\mathcal{K}_p$ and $\mathcal{K}_v$, and we will take the following values:

$$\mathcal{K}_p = \begin{bmatrix} -3.0 & 1.0 \\ 1.0 & -2.0 \end{bmatrix} \tag{114a}$$

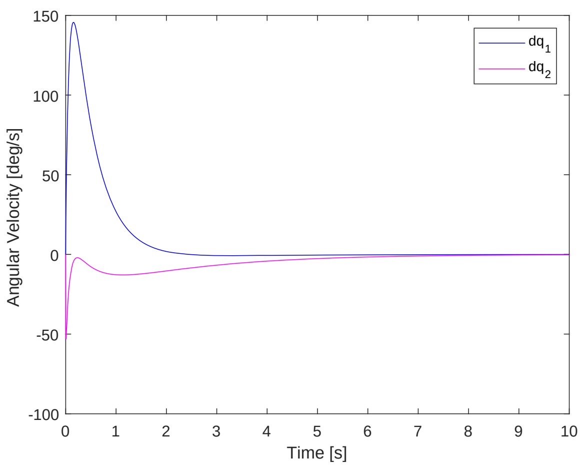
$$\mathcal{K}_v = \begin{bmatrix} -2.0 & 1.0 \\ 1.0 & -3.0 \end{bmatrix} \tag{114b}$$

It is easy to show that the eigenvalues of $\mathcal{K}_p$ and $\mathcal{K}_v$ are -1.3820 and -3.6180. Moreover, we can demonstrate via condition (100b) that the matrix $\mathcal{K}_v$ satisfies the following constraint:

$$\mathcal{K}_v = \mathcal{K}_v^T < -1.3090\, \mathcal{I}_n \tag{115}$$



(a)



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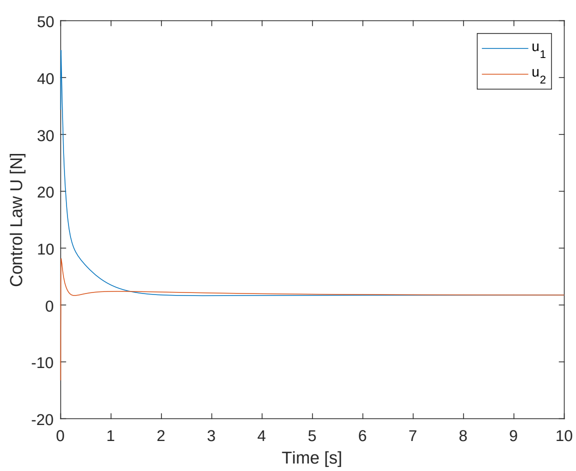

(c)

Figure 7 Temporal variation of: (a) the two angular positions q_1 and q_2 of the manipulator robot, (b) the two corresponding angular velocities $\dot{q}_1$ and $\dot{q}_2$, and (c) the augmented PD plus gravity compensation controller, by adopting the two feedback gains (113a) and (113b).

which is respected by adopting the gain (114b), since $\lambda_{\max}(\mathcal{K}_v) = -1.3820$.

Using then the two matrix feedback gains (114a) and (114b) in the adopted augmented PD plus gravity compensation controller defined by expression (71), the 2-DoF manipulator robot will be controlled to q_d. Fig. 8 shows the obtained outputs of the controlled manipulator robot. The Fig. 8(a) reveals the convergence of q_1 and q_2 towards q_d. Moreover, the Fig. 8(b) shows that the two angular velocities $\dot{q}_1$ and $\dot{q}_2$ converge to zero. In addition, the Fig. 8(c) depicts the controller $\mathcal{U}$ applied to the robotic system. As in the previous cases, the applied control subinputs u_1 and u_2 converge progressively and simultaneously to 1.7342.

Compared to the results obtained in Fig. 3 for the stabilization by means of the PD plus gravity compensation controller, the simulation results illustrated in Fig. 8 are found to be almost similar. The slight difference lies in the maximal values reached by the two subinputs u_1 and u_2 and also in the maximum values reached by the two angular velocities $\dot{q}_1$ and $\dot{q}_2$. This small difference can be explained by the fact that the adopted feedback gains (114a) and (114b) are higher (in terms of eigenvalues) than those in (105a) and (105b).

DISCUSSION

It is important to specify that the four proposed nonlinear controllers can control the position of robotic systems, and more particularly the 2-DoF manipulator in Fig. 1, and stabilize it at some desired state. As we have shown previously in the Fig. 2(a), the Fig. 4(a), the Fig. 6(a) and the Fig. 7(a), the manipulator robot is controlled to the desired point q_d. Furthermore, in all four cases, the angular velocity presented in Fig. 2(b), Fig. 4(b), Fig. 6(b) and Fig. 7(b) converges to zero and therefore the robot is well stabilized. In addition, as reported previously and revealed from Fig. 2(c), Fig. 4(c), Fig. 6(c) and Fig. 7(c), the control subinputs u_1 and u_2 converge together to the constant value $u_\infty \approx 1.734$. Such control effort is the necessary amount needed to keep the robotic manipulator at the desired state q_d. This result reveals that at the stabilization at the desired equilibrium point q_d, a small control effort is required/applied.

Furthermore, we showed that using the PD plus gravity compensation control law (11), the robotic system has been stabilized in almost 10 seconds, although the selected gains $\mathcal{K}_p$ and $\mathcal{K}_v$ are relatively high. But, by using the PD plus desired gravity compensation controller (22), almost 4 seconds were needed for the stability of the robot system at q_d. Furthermore, by applying the computed-torque control law (55)-(56), the manipulator robot was found to be stabilized at the desired point q_d in about 5 seconds. In addition, by applying the augmented PD plus gravity compensation controller (71), the robotic system has been stabilized in almost 6 seconds.

It is worth to note that compared to the PD plus gravity compensation controller (11), the augmented version (71) is more efficient. Indeed, by observing the plots of the angular velocity illustrated in Fig. 2(b) and Fig. 7(b), it is clear that by applying the PD plus gravity compensation controller (11), the angular velocities of the two joints reach higher values than those obtained by applying the augmented PD controller (71). Furthermore, according to Fig. 2(c) and Fig. 7(c), the control efforts u_1 and u_2 applied to the two joints is slightly small for the case of the augmented PD plus gravity compensation control law. This result demonstrates that the augmented/added term in the augmented control law (71), compared to the PD plus gravity compensation controller (11), contributes in (slightly) reducing the controller effort applied to the robot system.

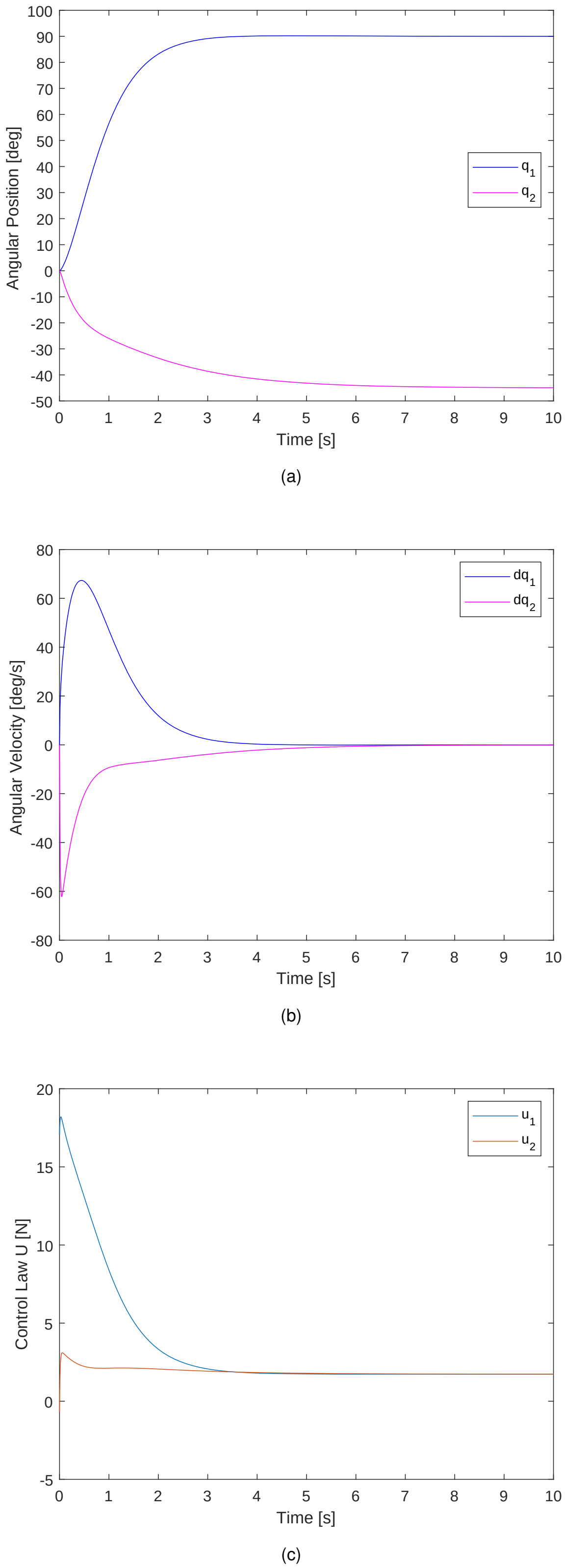

Figure 8 Temporal variation of: (a) the two angular positions q_1 and q_2 of the manipulator robot, (b) the two corresponding angular velocities $\dot{q}_1$ and $\dot{q}_2$, and (c) the augmented PD plus gravity compensation controller, by adopting the two feedback gains (114a) and (114b).

After this previous discussion, we can conclude that the four controllers proved to be globally effective for the control of Lagrangian-type robotic systems, for at least for the planar 2-DoF manipulator in Fig. 1. Nevertheless, more experiences should be achieved to further study the efficiency of these four controllers by considering the effect of external disturbances, unmodeled dynamics and the presence of frictions in the nonlinear dynamic model of the Lagrangian-like robotic systems. In these cases, robust controllers should be designed to deal with these uncertainties and disturbances.

From practical point-of-view, it is important to indicate that the design and application of the four proposed nonlinear controllers require the previous knowledge of all and some parts of the dynamic model of the robotic system to control. In fact, the PD plus desired gravity compensation controller (22) only needs information on a single matrix, namely the distribution input matrix $\mathcal{D}(q)$, which needs to be evaluated on-line at each time. Moreover, the PD plus gravity compensation controller (11) needs the previous knowledge of the matrix $\mathcal{D}(q)$ and also of the gravity matrix $\mathcal{G}(q)$. Thus, this controller is more complicated and needs more time to be executed than the PD plus desired gravity compensation controller. Indeed, the control law (11) requires information on the measurement of the position vector q and the velocity vector $\dot{q}(t)$ at each instant while the robotic system is moving for the computation of the two matrices and $\mathcal{D}(q)$ and $\mathcal{G}(q)$. Furthermore, the computed-torque controller (55)-(56) depends on all the matrices of the nonlinear dynamic model (1) of the Lagrangian robot system. It is usually called as the model-based controller, which explicitly use the full knowledge of the matrices $\mathcal{M}(q)$, $\mathcal{H}(q,\dot{q})$, $\mathcal{G}(q)$ and $\mathcal{D}(q)$ of the dynamic robot model (1). Similarly, the augmented PD plus gravity compensation controller (71) requires the full knowledge of the previous four matrices of the robotic system for its computation. Compared to the computed-torque control law, the augmented controller contains more nonlinear terms, and then its computation requires much time.

CONCLUSION AND FUTURE WORKS

In this research work, we adopted four different nonlinear controllers to solve the control and stabilization problem of the Lagrangian-type robotic systems to some set-point position. Then, the main goal was to control the robotic system through its nonlinear dynamic model to change its current configuration state into a desired position state. In order to achieve this objective, we adopted the PD plus gravity compensation controller, the PD plus desired gravity compensation controller, the computed-torque controller and the augmented PD plus gravity compensation controller. In addition, by applying these control laws in the nonlinear dynamic model of the Lagrangian robotic systems, we developed some feasible conditions on the gain matrices $\mathcal{K}_v$ and $\mathcal{K}_p$ ensuring the stability at the desired state and also guaranteeing the uniqueness of the desired equilibrium. Finally, we proposed the planar manipulator robot with 2-DoF, as an illustrative example, in order to present the simulation results by using the different adopted nonlinear controllers and a comparison between them was therefore achieved.

A a possible future direction of this research work, we aim at analyzing the efficiency of the adopted nonlinear controllers by considering the effect of external perturbations, unmodeled dynamics and parametric uncertainties.

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Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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