Abstract

In this paper, we first introduced the steps that need to be taken to get the set-family that goes with a hoarded graph, as well as an example of how these steps could be used. Then, we explained what a topological hoarded graph is and showed when a set-family induced by a topological hoarded graph is a topology on a set. We also presented some useful facts about topological hoarded graphs.

1. Introduction

A subfamily $S^{(n)}_m$ (or shortly $S^{(n)}$) of $n$-times-iterated power set of a set $X$ is called a $n$-set-family on $X$. In particular, we use the convention that the 0-set-family $S^{(0)}$ is a subset of $X$. We denote $m$-times generalized union of a family $S^{(n)}$ by $\bigcup^m S^{(n)}$, that is,

$$\bigcup^m S^{(n)} = \bigcup_{m \text{ times}} S^{(n)}$$

where $1 \leq m \leq n$. For simplicity, we adopt the convention $\bigcup^0 F^{(n)} = F^{(n)}$. Let $I$ be a partially ordered set with the least element. An indexed family $\{A_i\}_{i \in I}$ whose the least-indexed element is empty, i.e., in which $A_0 = \emptyset$ where $i_0 = \min I$ is said to be first-empty. We denote the set of all integers $\geq k$ and $\leq n$ where $k, n \in \mathbb{Z}$ by $I_{nk}$.

Given a digraph $G = (V, A)$. The sets of heads and tails of all arcs in $G$ is denoted by $V_h(G)$ and $V_t(G)$, respectively. Hence the set $V(G)$ of its all endpoints is union of $V_t(G)$ and $V_h(G)$. Furthermore, we denote the set of all heads of all v-tailed arcs in $G$ by $V_h(G; v)$, or in short $V_h(v)$; and similarly the sets of all tails of all v-headed arcs in $G$ by $V_t(G; v)$, or in short $V_t(v)$. A path in $G$ whose the first and last vertices are in $V'$ and $V''$, respectively, where $V', V'' \subseteq V$, is denoted by $p_{V', V''}$. Especially, we prefer to use the element of that set in the notation if $V'$ or $V''$ is a singleton, and the dot symbol is used instead of unknown sets in the notation $p_{V'_1, \ldots, V''_n}$. The set of last vertices of all directed paths $p_{V_0 \rightarrow W}$ in $G$ where $W \subseteq V$ is denoted by $V_t(v \rightarrow W; G)$, or in short $V_t(v \rightarrow W)$, and similarly the set of first vertices of all directed paths $p_{W \rightarrow v}$ in $G$ by $V_t(W \rightarrow v; G)$, or in short $V_t(W \rightarrow v)$. We prefer to use the notation $V_t(v)$ and $V_f(v)$ instead if $W$ is not particular. The length of a directed path in $G$ is the number of arcs on it. A directed path with length $n$ in $G$ is called a $n$-directed path. Let $G[A']$ denote a subgraph $G'$ of $G$. Also, we denote a vertex-induced subgraph by $V' \subseteq V$ of $G$ by $G[V', V']$, and denote an edge-induced subgraph by $A' \subseteq A$ of $G$ by $G[A', A']$ (for detailed information, see [1-3, 6-11]). The pair $v, w$ of vertices in $G$ is called semiconnected if $G$ contains a directed path from $v$ to $w$ or vice versa; the pair is called non-semiconnected if they are not semiconnected (see [5]).

We introduced the notion of cumulative graph as a subclass of acyclic digraphs [4]. We recall that a $n$-cumulative graph $G = (V, A, B)$ with first-empty indexed families $V = \{V_i\}_{i \in I^I}$, $A = \{A_i\}_{i \in I^I}$ and $B = \{B_i\}_{i \in I^I}$ is an acyclic digraph $G = (\bigcup V, \bigcup (A \cup B))$ satisfying the following : (i) $V_i = V(G[A_i, A_{i+1}]) \bigcup V_t(G[B_i, B_{i+1}]),$ and for every integer $1 \leq i < n$, $V_i = V(G[A_i, A_{i+1}]) \bigcup V_t(G[B_i, B_{i+1}]) \bigcup V_h(G[B_i, B_{i+1}]).$ (ii) for every $1 \leq i \leq n$, $Vw \in A_i$ and $ws \in B_i \Rightarrow vs \notin A_i,$ (iii) for every $1 \leq i \leq n$, $Vw \in A_i$ and $ws \in B_i \Rightarrow vs \notin B_i$.

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2. A Set-family Corresponding to A Hoarded Graph

We introduced the definition of a cumulative graph in our previous paper [4]. The main motivation for this definition was to specify a particular class of graphs that would correspond to a n-set-family. It is natural to ask for which class of graphs there is a set-family corresponding to any graph of that class. To answer this question, we give the following definition.

Definition 1. A n-hoarded graph $G = (V, A, B)$ with pairwise disjoint families $V = \{V_i\}_{i \in I_n}, A = \{A_i\}_{i \in I_n}$ and $B = \{B_i\}_{i \in I_n}$ is an acyclic digraph $G = (\cup V, \cup (A \cup B))$ which satisfies the following conditions:

1. For every $2 \leq i \leq n$, the endpoints of every arc in $A_i$ belong to $V_i$ while tails of every arc in $B_i$ belong to $V_i$ and the set of heads of all arcs in $B_i$ equals to $V_{i-1}$.
2. If a vertex in $V_i$ precedes that in $V_j$ on some directed path in $G$, then $i \geq j$.
3. If $u_1, u_2, \ldots, u_m$ with $m \geq 3$ is a directed path in $G$ every arc of which belongs to $A_i$ for some $2 \leq i \leq n$, then $u_1 u_m \in A_i$.
4. For every $2 \leq i \leq n$, $vw \in A_i$ and $ws \in B_i \Rightarrow vs \in B_i$.

For every distinct pair $u, v$ of vertices in some $V_i$ with $1 \leq i \leq n$, there exists a vertex $w$ such that $w$ is the last vertex of some directed path with the first vertex $u$ but not that of any directed path with the first vertex $v$.

In the paper [4], we have shown the steps to obtain the $(n + 1)$-cumulative graph induced by a n-set-family. Now we introduce the steps to be taken to get the $(n - 1)$-set-family corresponding to a $n$-hoarded graph $G = (V, A, B)$.

Step 1 We set $\mathcal{F} = V_n$.

Step 2 We perform the following steps from $i = n$ to $2$.

Step 2.1 We substitute the set $\cup V_h(G[i], A_i); v)$ for each vertex $v$ occurring in $\mathcal{F}$.

Step 2.2 We substitute the set $V_h(G[i], B_i); v)$ for each vertex $v$ occurring in $\mathcal{F}$.

After performing the above steps, the resulting $\mathcal{F}$ is the set-family corresponding to the hoarded graph $G$.

Example 2. Let $G = (V, A, B)$ be a 4-hoarded graph with $V = \{V_i\}_{i \in I_4}, A = \{A_i\}_{i \in I_4}$ and $B = \{B_i\}_{i \in I_4}$ where

$V_1 = \{v_1, \ldots, v_6\}, V_2 = \{v_7, \ldots, v_{10}\},$

$V_3 = \{v_{11}, \ldots, v_{14}\}, V_4 = \{v_{15}, v_{16}, v_{17}\},$

$A_2 = \{v_8 v_7, v_9 v_7, v_{10} v_9\},$

$A_3 = \{v_{13} v_{11}, v_{14} v_{11}, v_{14} v_{12}\},$

$A_4 = \{v_{16} v_{15}, v_{17} v_{16}\},$

$B_2 = \{v_8 v_1, v_9 v_3, v_8 v_4, v_9 v_1, v_9 v_2, v_9 v_3, v_9 v_6, v_{10} v_5\},$

$B_3 = \{v_{11} v_7, v_{12} v_{10}, v_{13} v_9, v_{14} v_{18}\},$

$B_4 = \{v_{16} v_{11}, v_{16} v_{13}, v_{17} v_{12}, v_{17} v_{14}\}$

as Figure 1.

Figure 1. An example of a hoarded graph.

We first set $\mathcal{F} = V_4 = \{v_{15}, v_{16}, v_{17}\}$. For $i = 4$, we write $\mathcal{F} = \{v_{15}, v_{16} \cup v_{15}, v_{17} \cup v_{16} \cup v_{15}\}$ since $V_4 = v_{15} \cup \emptyset = v_{15}$.

$V_4(G[i, A_4]; v_{16}) = v_{16} \cup \{v_{15}\} = v_{16} \cup v_{15}$

$V_4(G[i, A_4]; v_{17}) = v_{17} \cup \{v_{15}\} = v_{17} \cup v_{15}$

And since

$V_4(G[i, B_4]; v_{15}) = \emptyset,$

$V_4(G[i, B_4]; v_{16}) = \{v_{11}, v_{13}\},$

$V_4(G[i, B_4]; v_{17}) = \{v_{12}, v_{14}\}$

we get $\mathcal{F} = \{\emptyset, \{v_{11}, v_{13}\}, \{v_{11}, v_{12}, v_{13}, v_{14}\}\}$. Then by performing Step 2 for $n = 3$, we get

$V_3(G[i, A_3]; v_{11}) = v_{11} \cup \emptyset = v_{11},$

$V_3(G[i, A_3]; v_{12}) = v_{12} \cup \emptyset = v_{12},$

$V_3(G[i, A_3]; v_{13}) = v_{13} \cup \{v_{11}\} = v_{13} \cup v_{11},$

$V_3(G[i, A_3]; v_{14}) = v_{14} \cup \{v_{11}, v_{12}\}$

$= v_{14} \cup v_{12} \cup v_{11}.$

So, we obtain

$\mathcal{F} = \{\emptyset, v_{11}, v_{13} \cup v_{11}, \{v_{11}, v_{12}, v_{13}, v_{14}\} \}.$

Then we write

$V_3(G[i, B_3]; v_{11}) = \{v_7\},$

$V_3(G[i, B_3]; v_{12}) = \{v_{10}\},$

$V_3(G[i, B_3]; v_{13}) = \{v_9\},$

$V_3(G[i, B_3]; v_{14}) = \{v_8\}$
which yield
\[ F = \{\emptyset, \{v_7\}, \{v_7, v_9\}, \{v_7, v_9, v_{10}\}\}, \]

Continuing Step 2, we rewrite
\[ F = \{\emptyset, \{v_7\}, \{v_7, v_9\}, \{v_7, v_9, v_{10}\}\}, \]

because
\[ V_h(G[1, 2]; v_7) = v_7 \cup \emptyset = v_7, \]
\[ V_h(G[2, 2]; v_8) = v_8 \cup \bigcup \{v_7\} = v_8 \cup v_7, \]
\[ V_h(G[2, 2]; v_9) = v_9 \cup \bigcup \{v_7\} = v_9 \cup v_7, \]
\[ V_h(G[2, 2]; v_{10}) = v_{10} \cup \bigcup \{v_9\} = v_{10} \cup v_9 \cup v_7. \]

In the sequel, we find as
\[ V_h(G[1, 2]; v_7) = \emptyset, \]
\[ V_h(G[2, 2]; v_8) = \{v_1, v_3, v_4\}, \]
\[ V_h(G[2, 2]; v_9) = \{v_1, v_2, v_3, v_6\}, \]
\[ V_h(G[2, 2]; v_{10}) = \{v_5\} \]

and hence we finally get
\[ F = \{\emptyset, \emptyset, \{v_1, v_2, v_3, v_6\}\}, \]
\[ \{\emptyset, \{v_1, v_3, v_4, v_5\}, \emptyset, \{v_1, v_2, v_3, v_6\}\}, \]
\[ \emptyset, \{v_1, v_3, v_4\}, \{v_1, v_3, v_4, v_5\}\}. \]

3. Topological Hoarded Graphs

We first introduce the definition of topological hoarded graph:

**Definition 3.** A 2-hoarded graph \( G = (V, \mathcal{A}, \mathcal{B}) \) with \( V = \{V_1, V_2\}, \mathcal{A} = \{A_2\} \) and \( \mathcal{B} = \{B_2\} \) is called a topological hoarded graph and denoted by \( G = (V_1, V_2, A_2, B_2) \) if it satisfies the following conditions:

1. There exists a vertex in \( V_2 \) that is the tail of no arc in \( G \).
2. For every vertex \( v \) in \( V_1 \), there exists a vertex \( u \in V_2 \) in which a directed path from itself to \( v \) exists.
3. For any subset of mutually two non-semiconnected vertices in \( V_2 \), there exists a vertex \( v \) in \( V_2 \) such that \( G \) contains a dipath from \( v \) to \( v \) for each vertex \( s \) in \( S \).

For any non-semiconnected pair \( u, v \) of vertices in \( V_2 \), if \( G \) contains pairs of directed paths with the first vertices \( u, v \) and the same last vertex \( v \), then there exists a vertex \( v \in V_2 \) such that \( G \) contains pairs of \( v \)-headed arcs with the tails \( u, v \) on these directed paths.

**Theorem 4.** If \( G = (X, Y, A, B) \) be a topological hoarded graph, then \( X \) equipped with the 1-set-family \( \tau \) corresponding to \( G \) is a topological space.

**Proof.** Let us first show that \( \tau \) contains the empty set. From Definition 3(3), there exists a vertex \( y \) in \( Y \) such that \( y \) is not the tail of any arc in \( G \). When we first perform Step 1 to obtain 1-set-family \( \tau \) corresponding to \( G \), we get \( \tau = Y \). In Step 2.1, we write
\[ y \cup \bigcup V_h(G[A]; y) = y \cup \emptyset = y \]
instead of \( y \) in \( \tau \) since \( y \) is not the tail of any arc in \( G \).

In Step 2.2, since \( y \) is not the tail of any arc in \( G \), we replace \( y \) in \( \tau \) with
\[ V_h(G[B]; y) = \emptyset \]
which means that \( \tau \) contains \( \emptyset \).

Now we show that \( \tau \) contains the set \( X \). Assume that \( X \notin \tau \). It implies that \( X \neq V_h(G[B]; y) \) for every occurrence \( y \) in \( \tau \) obtained by applying Step 2.1. Then for every occurrence \( y \) in \( \tau \) obtained by applying Step 2.1, there exists a point \( x \in X \) such that \( x \notin V_h(G[B]; y) \) which contradicts Definition 3(3). So \( X \in \tau \).

Given a subfamily \( \{U_i\}_{i \in I} \) of \( \tau \), let’s show that \( \tau \) contains \( U_{i \in I} U_i \). If \( U_{i = 0} \emptyset \) for a particular \( i_0 \in I \), then \( U_{i = 0} U_{i = 1} \) for every \( i \in I \). If \( \emptyset \) is a subset of \( U_{i = 1} U_{i = 1} \) for every distinct indices \( i, j \in J \). For each \( i \in J \), \( U_i \) corresponding some vertex \( v_i \in Y \) is obtained by applying Step 2.1 and Step 2.2. From Definition 3(3), there exists a vertex \( w \) in \( Y \) such that \( G \) contains a dipath from \( w \) to \( v_i \) for every \( i \in J \). Therefore, \( U_{i = 1} U_{i = 1} W \) is a subset of \( \tau \).

Let \( U \) and \( V \) be members of \( \tau \). Finally, if we show that \( U \cap V \in \tau \), then we complete the proof. If \( U \) does not intersect \( V \), then \( U \cap V = \emptyset \in \tau \). If \( U \subseteq V \) or \( V \subseteq U \), then it is clear that \( U \cap V = U \in \tau \) or \( U \cap V = V \in \tau \). In the other case, \( U \) and \( V \) corresponding some vertices \( u, v \in Y \), respectively, are obtained by performing Step 2.1 and Step 2.2. Since \( U \cap V \neq \emptyset \) and \( U \not= V \) and \( V \not= U \), \( G \) contains pairs of directed paths with the first vertices \( u, v \) and the same last vertex \( w \) in \( X \) that corresponds to each point \( p \in U \cap V \). From Definition 3(3), there exists a vertex \( w \) in \( Y \) such that \( G \) contains pairs of \( w \)-headed arcs with the tails \( u, v \) on these directed paths. Just after performing Step 2.1 and Step 2.2, we obtain an \( U \cap V = W \) in \( \tau \).

**Example 5.** Let \( G = (X, Y, A, B) \) be a topological hoarded graph where
\[
X = \{v_1, ..., v_6\}, Y = \{v_7, ..., v_{19}\},
\]
\[
A = \{v_8v_7, v_7v_6, v_{10}v_8, v_{10}v_9, v_11v_6, v_12v_9, v_{13}v_{10},
v_{13}v_{11}, v_{14}v_{10}, v_{14}v_{12}, v_{15}v_{11}, v_{15}v_{12}, v_{16}v_{13},
v_{16}v_{14}, v_{16}v_{15}, v_{17}v_{14}, v_{18}v_{16}, v_{18}v_{17}, v_{19}v_{18}\},
\]
\[
B = \{v_8v_1, v_9v_2, v_{11}v_3, v_{12}v_5, v_{17}v_6, v_{19}v_4\}
\]
as Figure 5.

\[
V_h(G[\cdot, A_2]; v_7) = v_7 \cup \emptyset = v_7,
\]
\[
V_h(G[\cdot, A_2]; v_8) = v_8 \cup \bigcup \{v_7\} = v_8 \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_9) = v_9 \cup \bigcup \{v_7\} = v_9 \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{10}) = v_{10} \cup \bigcup \{v_9, v_3\} = v_{10} \cup v_9 \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{11}) = v_{11} \cup \bigcup \{v_9, v_3\} = v_{11} \cup v_9 \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{12}) = v_{12} \cup \bigcup \{v_9\} = v_{12} \cup v_9 \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{13}) = v_{13} \cup \bigcup \{v_{10}, v_{11}\} = v_{13} \cup v_{10} \cup v_9 \cup v_8 \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{14}) = v_{14} \cup \bigcup \{v_{10}, v_{12}\} = v_{14} \cup v_{12} \cup v_9 \cup v_8 \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{15}) = v_{15} \cup \bigcup \{v_{11}, v_{12}\} = v_{15} \cup v_{12} \cup v_9 \cup v_8 \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{16}) = v_{16} \cup \bigcup \{v_{13}, v_{14}, v_{15}\} = v_{16} \cup \cdots \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{17}) = v_{17} \cup \bigcup \{v_{14}\} = v_{17} \cup \cdots \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{18}) = v_{18} \cup \bigcup \{v_{16}, v_{17}\} = v_{18} \cup \cdots \cup v_7,
\]
\[
V_h(G[\cdot, A_2]; v_{19}) = v_{19} \cup \bigcup \{v_{18}\} = v_{19} \cup \cdots \cup v_7.
\]

And since
\[
V_h(G[\cdot, B_2]; v_7) = \emptyset, V_h(G[\cdot, B_2]; v_9) = v_9, V_h(G[\cdot, B_2]; v_{13}) = \emptyset,
\]
\[
V_h(G[\cdot, B_2]; v_1) = v_1, V_h(G[\cdot, B_2]; v_{15}) = \emptyset,
\]
\[
V_h(G[\cdot, B_2]; v_3) = v_3, V_h(G[\cdot, B_2]; v_{16}) = \emptyset,
\]
\[
V_h(G[\cdot, B_2]; v_{10}) = \emptyset, V_h(G[\cdot, B_2]; v_{17}) = \{v_9\},
\]
\[
V_h(G[\cdot, B_2]; v_{11}) = \{v_3\}, V_h(G[\cdot, B_2]; v_{18}) = \emptyset,
\]
\[
V_h(G[\cdot, B_2]; v_{12}) = \{v_5\}, V_h(G[\cdot, B_2]; v_{19}) = \{v_4\},
\]
\[
V_h(G[\cdot, B_2]; v_{13}) = \emptyset,
\]
we get
\[
F = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}\},
\]
which can easily be proved to be a topology on \(X\).

4. Conclusion and Suggestions

We first give a concept of an \(n\)-hoarded graph to which there exists a \((n-1)\)-set family corresponding. We present the steps to be performed to get the corresponding \(n\)-set-family, and we have shown the results of these steps in an example. We then introduced the concept of a topological hoarded graph. Above all, we show that \(X\) equipped with the \(1\)-set-family \(\tau\) corresponding to a topological hoarded graph \(G = (X, Y, A, B)\) is a topological space. And finally, we have confirmed this fact with an example.
References