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Almost all about Rus-Hicks-Rhoades maps in quasi-metric spaces

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Abstract

Let (X, d) be a quasi-metric space. A Rus-Hicks-Rhoades (RHR) map $f : X \to X$ is the one satisfying $d(fx, f^2x) \leq \alpha d(x, fx)$ for every $x \in X$, where $\alpha \in [0, 1)$. In our previous work [37], we collected various fixed-point theorems closely related to RHR maps. In the present article, we collect almost all the things we know about RHR maps and their examples. Moreover, we derive new classes of generalized RHR maps and fixed point theorems on them. Consequently, many of the known results in metric fixed point theory are improved and reproved in an easy way.

Keywords: metric fixed point theory, quasi-metric space, fixed point, stationary point, maximal element.
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1. Introduction

The metric fixed point theory originates from Banach in 1922 on the study of the Banach contraction $f : X \to X$ on a complete metric space (X, d) satisfying

$$d(fx, fy) \leq \alpha\, d(x, y) \text{ with } \alpha \in [0, 1)$$

for any $x,\ y \in X$. Since then there have appeared several hundreds of contractive type conditions and almost one thousand spaces extending or modifying the complete metric spaces.

One of such extended contractive type conditions was due to Rus [46] in 1973 and Hicks-Rhoades [13] in 1979 as follows:

$$d(fx, f^2x) \leq \alpha\, d(x, fx) \quad \text{for every } x \in X,$$

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where $\alpha \in [0,1)$. Such f is called a *Rus-Hicks-Rhoades map* or an *RHR map*, and it has a large number of closely related mapping classes; see [37-39]. An RHR map was later called a *graphic contraction* or an *orbital contraction* by Rus [47] or Berinde-Păcurar [4], respectively.

Recall that the Banach contraction has thousands of related works, but its extended RHR maps have only a few works and examples. For example, Suzuki extended the RHR map to its so-called τ-distance version [49] and introduced particular types of RHR maps [50]. Many other authors called them the Suzuki type, followed his method, and obtained results with quite complicated proofs. In the present article, we show that Suzuki type maps in [50] and its many known variants are simply RHR maps.

Our main aim in this article is to collect almost all facts we know about the RHR maps. Moreover, we derive new classes of generalized RHR maps and fixed point theorems on them. Consequently we give simple proofs for such known results in previous works by many authors.

We organize this article as follows: Section 2 is for preliminaries and a basic fixed point theorem for quasi-metric spaces. In Section 3, we introduce a general equivalent formulation of the Covitz-Nadler fixed point theorem [9] based on our new 2023 Metatheorem in Ordered Fixed Point Theory [36]. One of the equivalent forms of the Covitz-Nadler theorem gives a general fixed point theorem for the RHR maps. In Section 4, we derive new versions of the Banach contraction principle or the RHR fixed point theorem for quasi-metric spaces. Section 5 devotes to show various types of known examples of weakly Picard operators or RHR type maps.

In Section 6, we recall some of previously known theorems related to the RHR maps. Section 7 devotes to introduce new examples of recently appeared RHR maps. We show that many of them have assumed redundant facts. In Section 8, we indicate how to get generalized forms of known RHR type theorems. Finally, Section 9 devotes some conclusion.

2. Basic Fixed Point Principle

We recall the following:

Definition 2.1. A *quasi-metric* on a non-empty set X is a function $d : X \times X \to \mathbb{R}^+ = [0,\infty)$ verifying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $d(x,y) = d(y,x) = 0 \Longleftrightarrow x = y$;
(b) (triangle inequality) $d(x,z) \le d(x,y) + d(y,z)$.

A *metric* in a set X is a quasi-metric satisfying that for all $x, y \in X$,

(c) (symmetry) $d(x,y) = d(y,x)$.

The convergence and completeness in a quasi-metric space (X,d) are defined as follows:

Definition 2.2. ([18]) Let (X,d) be a quasi-metric space and $T : X \to X$ a selfmap.

(1) A sequence $\{x_n\}$ in X *converges* to $x \in X$ if $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0$. [If $x_n = x$ for all $n \in \mathbb{N}$, then $d(x,x) = 0$. Therefore, we need (a) (self-distance) in Definition 2.1.]

(2) A sequence $\{x_n\}$ in (X,d) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $m > n > N$

(3) The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, Tx, \cdots, T^n x, \cdots\}.$$

(4) The space X is said to be *T-orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X.

(5) A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} T^n x = x_0 \implies \lim_{n\to\infty} T^{n+1}x = Tx_0$$

for any sequence $\{T^n x\}$ of X.

Note that every complete metric space is T-orbitally complete for all maps $T : X \to X$. There exists a T-orbitally complete metric space but it is not complete. Moreover, there exists an orbitally continuous map but it is not continuous.

The following is the main result of our previous article [39]:

Theorem 2.3. *Let T be a selfmap of a quasi-metric space (X, d) which is T-orbitally complete. Suppose $\varphi : X \to [0, \infty)$ is a function.*

(1) *If there exists a point $x \in X$ satisfying*

$$d(Tx, T^2x) \leq \varphi(x) - \varphi(Tx),$$

then $\{T^n(x)\}$ is a right-Cauchy sequence converging to an $x_0 \in X$.

(2) *T is orbitally continuous at x_0 if and only if x_0 is a fixed point of T.*

3. A Form of Our New 2023 Metatheorem

Let (X, d) be a metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\},\ \sup\{d(b, A) : b \in B\}\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$. Then H is called a generalized Hausdorff metric since it may have infinite values.

Recently, as a basis of Ordered Fixed Point Theory [36], we obtained the new 2023 Metatheorem and the following more general equivalent formulations of Nadler's fixed point theorem [27] in 1970 established by Covitz-Nadler [9] in 1970.

Theorem H. *Let (X, d) be a complete metric space and $0 < h < 1$. Then the following equivalent statements hold:*

(α) *For a multimap $T : X \to \mathrm{Cl}(X)$, there exists an element $v \in X$ such that $H(Tv, Tw) > hd(v, w)$ for any $w \in X\backslash\{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X\backslash\{fx\}$, there exists a $y \in X\backslash\{x\}$ satisfying $d(fx, fy) \leq hd(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $d(fx, f^2x) \leq hd(x, fx)$ for all $x \in X\backslash\{fx\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $T : X \to \mathrm{Cl}(X)$ such that, for any $x \in X\backslash Tx$, there exists $y \in X\backslash\{x\}$ satisfying $H(Tx, Ty) \leq hd(x, y)$. Then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = Tv$ for all $T \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : X \to \mathrm{Cl}(X)$ satisfying $H(Tx, Ty) \leq hd(x, y)$ for all $x \in X$ and any $y \in Tx\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = Tv$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in X\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $H(Tx, Tz) \leq hd(x, z)$ for some $T : X \to \mathrm{Cl}(X)$, then there exists a $v \in X \cap Y = Y$.*

Remark 3.1. (1) When $\mathfrak{F}$ is a singleton, (β)-(ϵ) are denoted by $(\beta 1)$-$(\epsilon 1)$, respectively, They are also logically equivalent to (α)-(η).

(2) Moreover, $(\delta 1)$ and $(\epsilon 1)$ implies the well-known theorems of Nadler [27] and Covitz-Nadler [9].

(3) Note that all ten statements in Theorem H are equivalent to the Covitz-Nadler theorem [9] in 1970 and Theorem H gives its elementary proof.

The following $(\gamma 1)$ of Theorem H is the basis in the present article:

Theorem H$(\gamma 1)$**.** *Let X be a complete metric space and $0 < h < 1$.*

$(\gamma 1)$ *If a map $f : X \to X$ satisfies $d(fx, f^2x) \leq hd(x, fx)$ for all $x \in X$, then f has a fixed element $v \in X$, that is, $v = fv$.*

Consequently, all ten statements are close relatives of Theorems of Rus [46] and Hicks-Rhoades [13]. This is rather surprising and all of them also extends the Banach contraction principle in 1922.

The origin of the RHR maps was given as follows by Rus [46] in 1971:

Corollary 3.2. (Rus) *Let f be a continuous selfmap of a complete metric space (X, d) satisfying*

$$d(fx, f^2x) \leq \alpha d(x, fx) \quad \text{for every } x \in X,$$

where $0 < \alpha < 1$. Then f has a fixed point.

The following is given by Hicks and Rhoades [13] in 1979 independently to Rus [46] in 1971:

Corollary 3.3. (Hicks-Rhoades) *Let T be a nonexpansive selfmap of a complete metric space (X, d) satisfying*

$$d(Tx, T^2x) \leq \alpha d(x, Tx) \quad \text{for all } x \in X,$$

where $0 < \alpha < 1$. Then T has a fixed point.

Note that Theorem H$(\gamma 1)$ $\Longrightarrow$ Theorem 3.2 $\Longrightarrow$ Theorem 3.3. Therefore, the continuity in Theorem 3.2 and the nonexpansivity in Theorem 3.3 are redundant.

From now on, *all RHR selfmaps on a complete metric space have a fixed point* by Theorem H$(\gamma 1)$. Our main aim in the present article is to find maps satisfying Theorem H$(\gamma 1)$ and their generaliations.

4. A Generalization of the RHR Theorem

Let us consider an RHR map T on a quasi-metric space (X, d). Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ by $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, T^2x_{n-1}) &\leq \alpha d(x_{n-1}, Tx_{n-1}) \\ &= \alpha d(x_{n-1}, x_n) \leq \cdots \leq \alpha^n d(x_0, x_1). \end{aligned}$$

For all $m \geq n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (\alpha^{n+1} + \cdots + \alpha^{m-1}) d(x_0, x_1) = \frac{\alpha^n - \alpha^m}{1-\alpha} d(x_0, x_1) \\ &< \frac{\alpha^n}{1-\alpha} d(x_0, x_1) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is Cauchy (or T-orbitally Cauchy). Suppose X is complete (or T-orbitally complete). Then there exists $z \in X$ such that $x_n \to z$. If T is continuous (or T-orbitally continuous) at $z \in X$, then

$x_{n+1} = Tx_n \to Tz$ and hence $Tz = z$. Conversely, if z is a fixed point of T, then it is orbitally continuous at z.

This is why there are too many generalizations of the Banach contraction principle. We state only one of them:

Theorem 4.1. *Let T be an RHR map on a quasi-metric space (X, d).*

(i) *If X is T-orbitally complete, then, for each $x_0 \in X$, there exists a point $z \in X$ such that*

$$\lim_{n\to\infty} T^n(x_0) = z$$

and

$$d(T^n(x_0), z) \le \frac{\alpha^n}{1-\alpha} d(x_0, Tx_0), \quad n = 1, 2, \cdots .$$

(ii) *z is a fixed point of T if and only if T is orbitally continuous at z.*

The following is our version of the Banach contraction principle:

Corollary 4.2. *Let (X, d) be a quasi-metric space and let $T : X \to X$ be a contraction, that is,*

$$d(Tx, Ty) \le \alpha d(x, y) \text{ for every } x, y \in X,$$

with $0 < \alpha < 1$. If (X, d) is T-orbitally complete, then T has a unique fixed point $x_0 \in X$. Moreover, for each $x \in X$,

$$\lim_{n\to\infty} T^n(x) = x_0$$

and, in fact, for each $x \in X$,

$$d(T^n x, x_0) \le \frac{\alpha^n}{1-\alpha} d(x, Tx), \quad n = 1, 2, \cdots .$$

Almost all text-books or monographs on topology or fixed point theory do not mention on quasi-metric spaces relative to the Banach principle.

From now on, $\mathrm{Fix}(T)$ denotes the set of fixed points of a map T.

5. Weakly Picard Operators

In 2022, Berinde and Păcurar [4] stated that the Picard iteration with a contraction T on a complete metric space (X, d), i.e., the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, $n \ge 0$, for some $x_0 \in X$, is an approximate fixed point sequence with respect to T. The map T is called a *weakly Picard operator*, see for example [3], if

(p1) $\mathrm{Fix}(T) \neq \emptyset$;
(p2) the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \dots$ converges to some $p \in \mathrm{Fix}(T)$, for any $x_0 \in X$.

If T is a weakly Picard operator and $\mathrm{Fix}(T) = \{p\}$, then T is called a *Picard operator*.

Berinde and Păcurar [4] gave a list of Picard operators due to previous authors as follows:

(1) Banach (1922)
(2) Kannan (1969)
(3) Ćirić-Reich-Rus (1971)

(4) Bianchini (1972)
(5) Chatterjea (1972)
(6) Zamfirescu (1972)
(7) Maia (1968)
(8) Ćirić (1971)

Note that all results due to (1)–(8) are RHR maps. See also our previous work [37].

One of the most interesting generalizations of the Banach contraction principle was given by Hardy-Rogers [12] in 1973:

Theorem 4.1. (Hardy-Rogers) *Let (X, d) be a complete metric space. The map $T : X \to X$ is called an interpolative Hardy-Rogers type contraction if there exist positive reals $\alpha, \beta, \gamma, \delta > 0$, with $\alpha + \beta + \gamma + \delta < 1$, such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \frac{\delta}{2}(d(x, Ty) + d(y, Tx))$$

for each $x, y \in X \setminus \mathrm{Fix}(T)$. Then, the map T has a unique fixed point in X.

Note that, for $y = Tx$, the Hardy-Rogers contraction condition implies

$$d(Tx, T^2x) \leq (\alpha + \beta)d(x, Tx) + \gamma d(Tx, T^2x) + \frac{\delta}{2}(d(x, Tx) + d(Tx, T^2x)).$$

Then $d(Tx, T^2x) \leq c\, d(x, Tx)$ for some $c \in (0, 1)$, that is, T is an RHR map.

Therefore all maps in this section are RHR selfmaps on a complete metric space and have a fixed point by Theorem H(γ1).

6. Known Basic Theorems

In this section, we introduce some of previously known theorems related to the RHR maps. Some of them extend known theorems for RHR maps.

6.1. Park [30] in 1980

Park gave a Banach type fixed point theorem with respect to contraction pairs of selfmaps on complete metric spaces in [36]:

Theorem C. *Let g and h be selfmaps of a metric space X. If there exists a sequence $\{u_i : i \in \omega\}$ in X such that*

$$u_{2i+1} = gu_{2i}, \quad u_{2i+2} = hu_{2i+1} \quad \text{for } i \in \omega$$

and $\overline{\{u_i\}}$ is complete, and if there exists a $\lambda \in [0, 1)$ such that

$$d(gx, hy) \leq \lambda d(x, y) \tag{1}$$

holds for any distinct $x, y \in \overline{\{u_i\}}$ satisfying either $x = hy$ or $y = gx$, then either

(i) *g or h has a fixed point in $\{u_i\}$, or*

(ii) *$\{u_i\}$ converges to some $\xi \in X$, and*

$$d(u_i, \xi) \leq \frac{\lambda^i}{1 - \lambda} d(u_0, u_i) \quad \text{for } i > 0.$$

Further, if one of g or h is continuous at ξ and (1) holds for any distinct $x, y \in \overline{\{u_i\}}$, then ξ is a common fixed point of g and h.

This seems to be artificial, but this was made to include as many as existing results of similar nature. In fact, by putting $h = g$ and $y = gx$, (1) becomes $d(gx, g^2x) \leq \lambda\, d(x, gx)$. Hence g is an RHR map.

6.2. Park [33] in 1983

We proved the following generalization of the Caristi fixed point theorem:

Proposition 7. *Let A be a set, X a complete metric space, $f, g : A \to X$ such that*

(i) *f is surjective, and*
(ii) *there exists a lower semicontinuous function $\phi : X \to [0, \infty)$ satisfying*

$$d(fx, gx) \leq \phi(fx) - \phi(gx) \text{ for each } x \in A.$$

Then f and g have a coincidence point.

Later in 1988, Park and Rhoades [42] showed that the first four theorems of Wang-Li-Gao-Iséki [52] are all consequences of the above Proposition. These are for expansion type surjections $f : X \to X$ with a real $a > 1$ satisfying

$$d(fx, f^2x) \geq ad(x, fx) \text{ for each } x \in X.$$

In this case f is not necessarily continuous. Such map f can be called an *anti* RHR map and may deserve to be studied. In fact, such type of maps was studied by Shobkolaei et al. [48] in 2013.

6.3. Suzuki [49] in 2001

Suzuki gave a result on an RHR type map as follows:

Theorem 1. (Suzuki) *Let X be a complete metric space and let T be a mapping from X into itself. Suppose that there exist a τ-distance p on X and $r \in [0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$ for all $x \in X$. Assume that either of the following holds:*

(i) *If $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$, then $Ty = y$;*
(ii) *If $\{x_n\}$ and $\{Tx_n\}$ converge to y, then $Ty = y$;*
(iii) *T is continuous.*

Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0) = x_0$.

As we have seen in Theorem H(γ1), (i)–(iii) are not necessary when p is a metric.

7. New Examples of RHR maps

In this section, we introduce new examples of recently appeared RHR maps which can be applied Theorem H(γ1). The numbers attached to Definitions, Theorems, or Corollaries appeared in this section are the same ones in the original sources.

We note that many results were given under some redundant assumptions. In this section, the present author's opinion is expressed in *slant type sentences.*

7.1. Ćirić [7] in 1974

The non-unique fixed point theorem of Ćirić [7] is as follows:

Theorem 1. ([16]) *Let T be an orbitally continuous selfmap on the T-orbitally complete standard metric space (X, d). If there is $k \in [0, 1)$ such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y),$$

for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T.

For $y = Tx$, the condition implies $\min\{d(Tx, T^2x), d(x, Tx)\} \leq kd(x, Tx)$. Hence T is an RHR map (which seems to be the oldest.)

7.2. Suzuki [50] in 2008

Suzuki generalized the Banach contraction principle as follows:

Theorem 2. ([50]) *Let (X, d) be a complete metric space and T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases} \tag{2}$$

Assume there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

This means T is an RHR map and a Picard operator, hence Theorem 2 follows from Theorem H(γ1). From now T is called the Suzuki type as its many followers used it.

Suzuki noted that the following theorem says that $\theta(r)$ is the best constant for every $r \in [0, 1)$.

Theorem 3. ([50]) *Define a function θ as in Theorem 2. Then for each $r \in [0, 1)$, there exist a complete metric space (X, d) and a mapping T on X such that T does not have a fixed point and*

$$\theta(r)d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$.

Note that $\theta(r)d(x, Tx) < d(x, y)$ means T can not have a fixed point $x = y = Tx$.

Kikkawa-Suzuki [22] in 2008 stated the following is in Suzuki [50]:

Corollary 1. ([22]) *For a metric space (X, d), the following are equivalent:*

(i) *X is complete.*

(ii) *Every mapping T on X satisfying the following has a fixed point:*
- *There exists $r \in [0, 1)$ such that $\theta(r)d(x, Tx) \le d(x, y)$ implies $d(Tx, Ty) \le rd(x, y)$ for all $x, y \in X$.*

(iii) *There exists $r \in (0, 1)$ such that every mapping T on X satisfying the following has a fixed point:*
- $\frac{1}{10000}d(x, Tx) \le d(x, y)$ *implies* $d(Tx, Ty) \le rd(x, y)$ *for all* $x, y \in X$.

This can be extended as follows:

(i) $\Longleftrightarrow$ Theorem H(γ1) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Recall that there are hundreds of equivalent conditions for metric completeness, e.g. Kirk [24], Park [34], Cobzaş [8] as typical examples. Comments for them are given by Park and Rhoades [41] in 1986.

7.3. Kikkawa and Suzuki [22] in 2008

In this paper, it is noted that $d(Tx, Ty) \le rd(x, y)$ in the original Suzuki type map in [50] can be replaced by one of the following:

$$d(Tx, Ty) \le r \max\{d(x, Tx), d(y, Ty)\},$$
$$d(Tx, Ty) \le \max\{\alpha d(x, Tx), \beta d(y, Ty)\}, \ \alpha, \beta \in [0, 1).$$

Similarly, it can be replaced by the Hardy-Rogers condition.

They also generalized Kannan mappings as follows:

Theorem 1. ([22]) *Let T be a mapping on complete metric space (X, d) and let φ be a non-increasing function from $[0, 1)$ into $(1/2, 1]$ defined by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases} \tag{3}$$

Let $\alpha \in [0, 1/2)$ and $r = \alpha/(1 - \alpha) \in [0, 1)$. Suppose that

$$\varphi(r)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le \alpha d(x, Tx) + \alpha d(y, Ty)$$

for all $x, y \in X$. Then, T has a unique fixed point z, and $\lim_n T^n x = z$ holds for every $x \in X$.

This is also an example of RHR maps. In fact, they gave various examples of RHR type maps in spaces more general than metric spaces.

7.4. Enjouji, Nakanishi, and Suzuki [10] in 2009

In order to observe the condition of Kannan mappings, the authors prove a generalization of Kannan's fixed point theorem. Their theorem involves constants and they obtain the best constants to ensure a fixed point. They consider "$\alpha d(x, Tx) + \beta d(y, Ty)$" instead of "$\alpha d(x, Tx) + \alpha d(y, Ty)$."

Let $\Delta = \{(\alpha, \beta) : \alpha \ge 0, \beta \ge 0, \alpha + \beta < 1\}$. Define a nonincreasing function $\psi : \Delta \to (1/2, 1]$. Let T be a map on a complete metric space such that there exists $\alpha, \beta \in \Delta$ satisfying

$$\psi(\alpha, \beta)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le \alpha d(x, Tx) + \beta d(y, Ty)$$

for all $x, y \in X$.

Note that T is an RHR map and that the condition $\psi(\alpha, \beta)d(x, Tx) < d(x, y)$ in Theorem 4,1 implies nonexistence of a fixed point of T.

7.5. Nakanishi and Suzuki [28] in 2010

In this paper, in order to observe the condition of Kannan maps more deeply, they prove a generalization of Kannan's fixed point theorem.

Moreover, the authors assumed something like $\theta(\alpha)d(x, Tx) < d(x, y)$, which is false for $x = y = Tx$. Hence, from the beginning, such T can not have a fixed point.

Further, for $(\alpha, \beta) \in \Delta = [0, 1)^2$ and a function $\varphi : \Delta \to (1/2, 1]$, let T be a selfmap of a complete metric space (X, d) satisfying

$$\varphi(\alpha, \beta)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le \max\{\alpha d(x, Tx), \beta d(y, Ty)\},$$

for all $x, y \in X$.

Note that T is an RHR map and that the condition $\varphi(\alpha, \beta)d(x, Tx) < d(x, y)$ does not hold for $x = y = Tx$.

7.6. Altun and Erduran [1] in 2011

The authors present a fixed-point theorem for a single-valued map in a complete metric space using implicit relation, which is a generalization of several previously stated results including that of Suzuki [50] in 2008.

The aim of this paper is to generalize the above results using the implicit relation technique in such a way that

$$F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx) \le 0,$$

for $x, y \in X$, where $F : [0,\infty)^6 \to \mathbb{R}$ is a function as given as follows:

Let Ψ be the set of all continuous functions $F : [0,\infty)^6 \to \mathbb{R}$ satisfying the following conditions:

F1: $F(t_1,\dots,t_6)$ is nonincreasing in variables $t_2,\dots,t_6$,

F2: there exists $r \in [0,1)$, such that $F(u,v,v,u,u+v,0) \le 0$ or $F(u,v,0,u+v,u,v) \le 0$ or $F(u,v,v,v,v,v) \le 0$ implies $u \le rv$,

F3: $F(u,0,0,u,u,0) > 0$, for all $u > 0$.

From this, the authors showed $d(Tx,T^2x) \le d(x,Tx)$, that is, T is a variant of the RHR maps.

7.7. Khojasteh, Abbas and Costache [19] in 2014

In this paper, the next theorem was shown:

Theorem 2. *Let (X,d) be a complete metric space and T a mapping from X into itself satisfying the following condition:*

$$d(Tx,Ty) \le \frac{d(y,Tx)+d(x,Ty)}{d(x,Tx)+d(y,Ty)+1}d(x,y)$$

for all $x,\ y \in X$. Then

(i) *T has at least one fixed point $z \in X$,*

(ii) *$\{T^n x\}$ converges to a fixed point for all $x \in X$, and*

(iii) *if z and w are distinct fixed points of T, then $d(z,w) \ge 1/2$.*

For $y = Tx$, we have

$$d(Tx,T^2x) \le \frac{d(x,T^2x)}{d(x,Tx)+d(Tx,T^2x)+1}d(x,Tx).$$

From this Amen et al. showed that T is an RHR map.

7.8. Karapinar [15] in 2018

Abstract : We revisited the well-known fixed point theorem of Kannan under the aspect of interpolation. By using the interpolation notion, we propose a new Kannan type contraction to maximize the rate of convergence.

Definition 2.1. Let (X,d) be a metric space. We say that a selfmap $T : X \to X$ is an *interpolative Kannan type contraction*, if there exist a constant $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \le \lambda[d(x,Tx)]^\alpha[d(y,Ty)]^{1-\alpha}$$

for all $x, y \in X$ with $x \ne Tx$.

Theorem 2.2. *Let (X,d) be a complete metric space and T be an interpolative Kannan type contraction. Then T has a unique fixed point in X.*

By putting $y = Tx$, $[d(Tx,T^2x)]^\alpha \le k[d(x,Tx)]^\alpha$, that is,

$$d'(Tx,T^2x) \le kd'(x,Tx) \text{ with } d^\alpha = d'.$$

Therefore, T is an RHR map. The uniqueness in Theorem 2.2 is removed in the next article.

7.9. Karapinar, Agarwal, and Aydi [17] in 2018

Abstract: By giving a counter-example, we point out a gap in the paper by Karapinar [15] in 2018 where the given fixed point may be not unique and we present the corrected version. We also state the celebrated fixed point theorem of Reich-Rus-Ćirić in the framework of complete partial metric spaces, by taking the interpolation theory into account. Some examples are provided where the main result in papers by Reich (1971, 1971, 1972) is not applicable.

As a correction of Theorem 1 (that is, Theorem 2.2 in Karapinar [15] in 2018), the authors should state

Theorem 2. *Let (X, ρ) be a complete metric space. A self-map $T : X \to X$ possesses a fixed point in X, if there exist constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$\rho(T\zeta, T\eta) \leq \lambda[\rho(\zeta, T\zeta)]^{\alpha} \cdot [\rho(\eta, T\eta)]^{1-\alpha}$$

for all $\zeta, \eta \in X \backslash \mathrm{Fix}(T)$.

The following theorem was proved by Reich, Rus and Ćirić (1971–1979) independently to combine and improve both Banach and Kannan fixed point theorems.

Theorem 3. *In the framework of a complete metric space (X, ρ), if $T : X \to X$ forms a Reich-Rus-Ćirić contraction map, i.e.,*

$$\rho(T\zeta, T\eta) \leq \lambda[\rho(\zeta, \eta) + \rho(\zeta, T\zeta) + \rho(\eta, T\eta)]$$

for all $\zeta, \eta \in X$, where $\lambda \in [0, 1/3)$, then T possesses a unique fixed point.

Notice that several variations of Reich contractions can be stated. We may state the following:

$$\rho(T\zeta, T\eta) \leq a\rho(\zeta, \eta) + b\rho(\zeta, T\zeta) + c\rho(\eta, T\eta),$$

where $a, b, c \in (0, \infty)$ such that $0 \leq a + b + c < 1$.

In this paper, we shall investigate the validity of the interpolation approach for Reich contractions in the context of partial metric spaces that was introduced by Matthews (1994).

The main contribution of the paper to ensure the existence of fixed points for interpolative Reich-Rus-Ćirić type contraction mappings on partial metric spaces.

In this paper, we can find some RHR maps.

7.10. Karapinar, Alqahtani, and Aydi [18] in 2018

The authors introduced the following notion of interpolative Hardy-Rogers type contraction.

Theorem 1.4. *Let (X, d) be a complete metric space. The mapping $T : X \to X$ is called an interpolative Hardy-Rogers type contraction if there exist $\lambda \in [0, 1)$ and positive reals $\beta, \alpha, \gamma > 0$, with $\beta + \alpha + \gamma < 1$, such that*

$$d(Tx, Ty) \leq \lambda([d(x, y)]^{\beta} \cdot [d(x, Tx)]^{\alpha} \cdot [d(y, Ty)]^{\gamma} \cdot [1/2(d(x, Ty) + d(y, Tx))]^{1-\alpha-\beta-\gamma}$$

for each $x,\ y \in X \setminus \mathrm{Fix}(T)$. Then the mapping T has a fixed point in X.

Note that

Case 1: $d(x, Tx) \leq d(Tx, T^2x)$ *implies* $[d(Tx, T^2x)]^{\alpha+\beta} \leq \lambda[d(x, Tx)]^{\alpha+\beta}$.

Case 2: $(Tx, T^2x) \leq d(x, Tx)$ *implies* $d(Tx, T^2x) \leq \lambda d(x, Tx)$.

Hence, T is an RHR map.

7.11. Aouiney and Aliouche [2] in 2021

Abstract: We prove unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed. Examples are provided to illustrate the validity of their results and we give some remarks about some previous papers. Afterwards, they apply their result to study the possibility of optimally controlling the solution of an ordinary differential equation via dynamic programming.

The main result of Khojasteh et al. [19] is extended to the following example and others:

Theorem 4. *Let (X, d) be a complete metric space and T a mapping from X into itself satisfying the following condition*

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} \max\{d(x, Tx), d(y, Ty)\}.$$

for all $x, y \in X$. Then

a) T has a unique fixed point $z \in X$,

b) for any sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} d(Ty_n, y_n) = 0$, we have $\lim_{n\to\infty} d(y_n, z) = 0$, and

c) T is continuous at z.

Some applications are added. Note that T is an RHR map and that c) seems to hold for any $x \in X$.

7.12. Yeşilkaya [53] in 2021

Abstract: In this paper, we obtain a fixed point theorem ω-ψ-interpolative Hardy-Rogers contractive of Suzuki type mappings. In the following, we present an example to illustrate the new theorem is applicable. Subsequently, some results are given. These results generalize several new results present in the literature.

Describe using Ψ the set of all nondecreasing self-mappings ψ on $[0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$. Regard that for $\psi \in \Psi$, we have $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$.

Corollary 2.5. *Let (X, d) be a complete metric space, $\lambda \in [0, 1)$, and T be a self mapping on X such that*

$$\tfrac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies}$$

$$d(Tx, Ty) \leq \lambda[d(x, y)]^{\beta} \cdot [d(x, Tx)]^{\alpha} \cdot [d(y, Ty)]^{\gamma} \cdot [\tfrac{1}{2}(d(x, Ty) + d(y, Tx))]^{1-\alpha-\beta-\gamma})$$

for each $x.y \in X \backslash \mathrm{Fix}(T)$, where $\psi \in \Psi$ and positive real $\beta, \alpha, \gamma > 0$, with $\beta + \alpha + \gamma < 1$. Then T posses a fixed point.

Note that T is an RHR map as in Theorem 1.4 of Karapinar et al. [18] in 2018.

7.13. Chandra, Joshi, and Joshi [5] in 2022

Let (M, d) be a metric space, and $T : M \to M$. Then for all $x, y \in M$, we denote

$$m(Tx, Ty) = ad(x, y) + b \max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)],$$

where a, b and c are non-negative reals such that $a + b + 2c = r$ with $r \in [0, 1)$. Now, we consider the following generalized contractive condition

$$\theta(r) \min\{d(x, Tx), d(x, Ty)\} \leq d(x, y) \text{ implies } d(Tx, Ty) \leq m(Tx, Ty).$$

(Here, θ is as in Suzuki [50] in 2008.) It is remarkable that this condition is a generalization of the condition (22) and several other conditions mentioned in Transaction Paper of Rhoades [43].

Note that T is an RHR map.

7.14. Romaguerra [45] in 2022

For a selfmap F of a quasi-metric space (X, ρ), $c \in (0,1)$ and $x,\ y \in X$, one of the following conditions hold:

$$\rho(x, Fx) \leq 2\rho(x, y) \Longrightarrow \rho(Fx, Fy) \leq c\rho(x, y),$$

$$\min \rho(x, Fx), \rho(y, Fy) \leq 2\rho(x, y) \Longrightarrow \rho(Fx, Fy) \leq c\rho(x, y),$$

$$\min\{\rho(x, Fx), \rho(y, Fy), \rho(Fy, y)\} \leq 2\rho(x, y) \Longrightarrow \rho(Fx, Fy) \leq c\rho(x, y).$$

Note that such F's are RHR maps.

7.15. Fierro and Pizarro [11] in 2023

Let (X, d) be a complete metric space. The authors state that the following corollary is an equivalent version of the main result of Hicks and Rhoades [13]:

Corollary 3.3. *Let $\xi : X \to X$ be a function and $k \in [0,1)$. Suppose there exists $x_0 \in X$ such that, for all $x \in O(x_0, \xi)$, $d(\xi(x), \xi^2(x)) \leq kd(x, \xi(x))$. Then, there exists $x^* \in X$ such that the following two conditions hold:*

(i) $\lim_{n\to\infty} d(x^*, \xi^n(x_0)) = 0$ *and*
(ii) $d(x^*, \xi^n(x_0)) \leq \frac{k^n}{1-k} d(x_0, \xi(x_0))$, *for all* $n \in \mathbb{N}$.

Moreover, $x^* = \xi(x^*)$, if and only if, the function $x \in X \mapsto d(x, \xi(x)) \in \mathbb{R}$ is (x_0, ξ)-orbitally lower semicontinuous at x^*.

This is almost same to our Theorem 2.3 in Section 2. It is possible to unify this with Theorem 2.3.

7.16. Pant and Khantwa [29] in 2023

Abstract: We present some new existence results for single and multivalued mappings in metric spaces on very general settings. Some illustrative examples are presented to validate our theorems. Finally, we discuss an application to the Volterra-type integral inclusions.

Let $\psi : [0, \infty) \to [0, \infty)$ be upper semi-continuous from the right such that $\psi(t) < t$ for all $t > 0$.

Theorem 2.2. *Suppose (E, ρ) is a metric space. Let $f : E \to E$ a mapping such that for some $v_0 \in E$,*

$$\frac{1}{2}\rho(x, f(x)) \leq \rho(x, y) \Longrightarrow \rho(f(x), f(y)) \leq \psi(N(x, y)) \text{ for all } x, y \in \overline{O(v_0, f)} \text{ with } x \neq y,$$

where $N(x, y) = \max\{\rho(x, y), \rho(x, f(x)), \rho(y, f(y)), \frac{1}{2}[\rho(y, f(x)) + \rho(x, f(y))]$.

If E is f-orbitally complete then the sequence of iterations $(f^n(v_0))$ is Cauchy in E and converges to the unique fixed point of f in $\overline{O(v_0, f)}$.

Easily we can show $\rho(f(x), f^2(x)) \leq \psi(\rho(x, f(x))$ and f is an RHR map.

8. Generalizations of the RHR maps

From the late 1970's to the early 1990's we had engaged to study the metric fixed point theory. Especially, in 1979-1993, we made eleven joint papers with Billy E. Rhoades. In [35], we collected briefly the contents of all of the joint papers. As he recalled "Our collaboration has ceased only because our research interests have moved in different directions."

8.1. Rhoades [44] in 2007

Billy E. Rhoades recalled the contents of his historical Transactions paper [43] and gave comments on our works as follows ([44], pp.12–13):

"Sehie Park also observed that fixed point theorems for many contractive definitions used the same proof technique. In 1980 [30] he proved the following two theorems, where $O(u) := \{u, Tu, T^2u, \dots\}$.

Theorem 3.1. ([30]) *Let T be a selfmap of a metric space (X, d). If there exists a point $u \in X$ and a $\lambda \in [0, 1)$ such that $\overline{O}(u)$ complete and*

$$(*) \qquad d(Tx, Ty) \le \lambda d(x, y)$$

holds for any $x, y = Tx$ in $O(u)$, then $\{T^n u\}$ converges to some $\xi \in X$, and

$$d(T^i u, \xi) \le \frac{\lambda^i}{1-\lambda} d(u, fu) \quad for \quad i \ge 1.$$

Further, if f is orbitally continuous at ξ or if $(*)$ holds for any $x, y \in \overline{O}(u)$, then ξ is fixed point of T.

Theorem 3.2. ([30]) *Let T be a selfmap of a metric space (X, d). If*
(i) *there exists a point $u \in X$ such that the orbit $O(u)$ has a cluster point $\xi \in X$,*
(ii) *T is orbitally continuous at ξ and $T\xi$, and*
(iii) *T satisfies*

$$d(Tx, Ty) < d(x, y)$$

for each $x, y = Tx \in \overline{O}(u)$, $x \ne y$,
then ξ is a fixed point of T.

These theorems contain as special cases a number of papers involving contractive conditions not covered by my Transaction paper."

And then Billy E. Rhoades added an example of an application of Theorem 3.1, not previously published. He continues as follows:

"In 1980 Sehie Park [31] constructed a table of contractive conditions of Meir-Keeler type, which extended the list in my Transactions paper."

Let f be a selfmap of a metric space (X, d). Given $x \in X$, let $O(x) = \{f^n x : n \in \mathbb{N}\}$ and $\overline{O(x)}$ be its closure. A point $x \in X$ is said to be *regular* for f if $\operatorname{diam} O(x) < \infty$. Given $x, y \in X$, let

$m(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$,

$\delta(x, y) = \operatorname{diam}\{O(x) \cup O(y)\}$ whenever x and y are regular.

We list contractive type conditions to be considered.

(A) For any $x, y \in X, x \ne y$,
  (Ad) $d(fx, fy) < d(x, y)$. [Edelstein]
  (Am) $d(fx, fy) < m(x, y)$. [Rhoades]
  (Aδ) if x and y are regular, $d(fx, fy) < \delta(x, y)$.

(B) Given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$,
  (Bd) $\varepsilon \le d(x, y) < \varepsilon + \delta$ implies $d(fx, fy) < \varepsilon$. (Meir-Keeler)
  (Bm) $\varepsilon \le m(x, y) < \varepsilon + \delta$ implies $d(fx, fy) < \varepsilon$.
  (Bδ) $\varepsilon \le \delta(x, y) < \varepsilon + \delta$ implies $d(fx, fy) < \varepsilon$.

(C) Given $\varepsilon > 0$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for any $x, y \in X$,
  (Cd) $\varepsilon \le d(x, y) < \varepsilon + \delta_0$ implies $d(fx, fy) \le \varepsilon_0$.

(Cm) $\varepsilon \le m(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \le \varepsilon_0$.
(Cδ) $\varepsilon \le \delta(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \le \varepsilon_0$. (Hegedüs-Szilágyi).

(D) There exists a nondecreasing right continuous function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) < t$ for $t > 0$ and, for any $x, y \in X$,
(Dd) $d(fx,fy) \le \phi(d(x,y))$. (Browder)
(Dm) $d(fx,fy) \le \phi(m(x,y))$. (Danes)
(Dδ) $d(fx,fy) \le \phi(\delta(x,y))$ if x, y are regular. (Kasahara)

(E) There exists $\alpha \in [0,1)$ such that for any $x, y \in X$,
(Ed) $d(fx,fy) \le \alpha d(x,y)$. (Banach)
(Em) $d(fx,fy) \le \alpha m(x,y)$. (Ćirić, Massa)
(Eδ) $d(fx,fy) \le \alpha\delta(x,y)$ if x, y are regular. (Hegedüs)

Then we have the following diagram:

$$\begin{array}{ccccccccc} (\mathrm{Ad}) & \Leftarrow & (\mathrm{Bd}) & \Leftarrow & (\mathrm{Cd}) & \Leftarrow & (\mathrm{Dd}) & \Leftarrow & (\mathrm{Ed}) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (\mathrm{Am}) & \Leftarrow & (\mathrm{Bm}) & \Leftarrow & (\mathrm{Cm}) & \Leftarrow & (\mathrm{Dm}) & \Leftarrow & (\mathrm{Em}) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (\mathrm{A}\delta) & \Leftarrow & (\mathrm{B}\delta) & \Leftarrow & (\mathrm{C}\delta) & \Leftarrow & (\mathrm{D}\delta) & \Leftarrow & (\mathrm{E}\delta) \end{array}$$

Let $M(X)$ denote the set of all metrics on X that are topologically equivalent to d for a given metric space (X,d).

Theorem 3.3. ([31]) *Let f be a continuous compact selfmap of a metric space X satisfying* (Aδ). *Then f has a unique fixed point, and furthermore, for any $\alpha \in (0,1)$ there exists a metric ρ in $M(X)$ relative to which f satisfies* (Ed) *with the Lipschitz constant α.*

Theorem 3.4(Cδ). ([31]) *Let f be a selfmap of a metric space X. Suppose there exists a regular point $u \in X$ such that* (1) *$O(u)$ has a regular cluster point $p \in X$, and* (2) *the condition* (Cδ) *holds on $O(u) \cup O(p)$. Then f has a unique fixed point p in $O(u)$ and $f^n u \to p$.*

Theorem 3.5(Cδ). ([31]) *Let f be a selfmap of a complete metric space X. If* (Cδ) *holds for all regular points $x, y \in X$, then f has a unique fixed point $p \in X$, and $f^n x \to p$ for any regular point $x \in X$.*

Jungck first gave a fixed point theorem for commuting selfmaps f and g of a complete metric space X satisfying the conditions $gX \subset fX$, f is continuous, and

(Ed)$'$ $d(gx,gy) \le \alpha d(fx,fy)$, $\alpha \in [0,1)$.

Similarly, we can consider other conditions ( )$'$ just imitating (Ed)$'$.

In 1999, Liu [26] stated as follows:

" On the other hand, the following open questions were raised by Park [31]:

1. Are there other counterexamples of the implications between various conditions in (Ed)-(Aδ)?
2. Are there any extensions of Theorem 3.4 (Cδ) to the conditions (Bm) and (Bδ)?"

From his Summary: "We answer two fixed-point questions of Park by constructing ten nontrivial examples and prove some fixed-point theorems for general contractive type mappings which, in turn, generalize, improve, and unify some results due to Fisher, Hegedüs, Hegedüs and Szilágyi, Hikida, Kasahara, Park, Park and Rhoades, and others."

8.2. Park and Rhoades [40] in 1980

In this paper we established several fixed point theorems involving hypotheses weak enough to include a number of known theorems as special cases.

Let f be a selfmap of a topological space X. The orbit $O(x)$ of $x \in X$ under f is defined by $O(x) = \{x, f(x), f^2(x), \dots\}$. A function $G : X \to [0, \infty)$ is said to be f-orbitally lower semicontinuous at a point $p \in X$ if, for every $x_0 \in X$, $x_{n_k} \to p$ implies $G(p) \leq \liminf_k G(x_{n_k})$ where $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, which is defined by $x_{n+1} = f(x_n)$, i.e. $\{x_n\}_{n=1}^{\infty} = O(x_0)$.

Theorem 1. *Let d be a nonnegative real valued function defined on $X \times X$ such that $d(x, y) = d(y, x)$ and $d(x, y) = 0$ iff $x = y$. If there exists a point $u \in X$ such that $\lim_n d(f^{n+1}(u), f^n(u)) = 0$, and if $\{f^n(u)\}$ has a convergent subsequence with limit $p \in X$, then p is a fixed point of f iff $G(x) = d(x, f(x))$ is f-orbitally lower semicontinuous at p.*

Theorem 2. *Let f be a selfmap of a metric space (X, d) satisfying:*

(i) *$\delta(O(x)) < \infty$ for each $x \in X$, where δ denotes the diameter.*

(ii) *There exists a $u \in X$ such that $O(u)$ has a cluster point $p \in X$.*

(iii) *There exists a function $\varphi : [0, \infty) \to [0, \infty)$ which is nondecreasing, continuous from the right and satisfies $\phi(t) < t$ for each $t > 0$ and the inequality*

$$d(f(x), f^2(x)) \leq \varphi(\delta(O(x) \cup O(f(y)))) \quad \text{for each} \quad x, y \in X.$$

Then p is the unique fixed point of f and $\lim_n f^n(u) = p$.

These results extend works of Pal-Maiti, Park, Hegedüs and Daneš. A 2-metric space version of Theorem 2 is added in [40].

Here we add new information related to the above theorems:

Kirk-Saliga [25] in 2001 and Chen-Cho-Yang [6] in 2002 introduced the following concept: We say that $\varphi : M \to \mathbb{R}$ is *lower semicontinuous from above* if given any sequence $\{x_n\}$ in M, the conditions $\lim_n x_n = x$ and $\{\varphi(x_n)\} \downarrow r \Rightarrow \varphi(x) \leq r$.

This concept can be applied to improve Theorem 2.

8.3. Generalized RHR maps

In the previous subsections, some of the results can be extended to the RHR type maps. Actually, in Subsection 8.1, Theorem 3.1 is for RHR maps. Moreover, if $y = fx$, $m(x, y)$ and $\delta(x, y)$ become

$m(x, fx) = \max\{d(x, fx), d(fx, f^2x), d(x, f^2x)\}$,

$\delta(x, fx) = \operatorname{diam} O(x)$ whenever x is regular.

Then Theorems 3.3–3.5 reduce as follows:

Theorem 3.3.$'$ *Let f be a continuous compact selfmap of a metric space X satisfying*

(Aδ') *if x is regular, $d(fx, f^2x) < \delta(x, fx) = \operatorname{diam} O(x)$.*

Then f has a unique fixed point, and furthermore, for any $\alpha \in (0, 1)$ there exists a metric ϱ in $M(X)$ relative to which f satisfies (Ed) *with the Lipschitz constant α.*

Theorem 3.4(Cδ'). *Let f be a selfmap of a metric space X. Suppose there exists a regular point $u \in X$ such that*

(1) *$O(u)$ has a regular cluster point $p \in X$, and*
(2) *the following condition holds on $x \in O(u) \cup O(p)$:*

($C\delta'$) $\varepsilon < \delta(x, fx) < \varepsilon + \delta_0$ *implies* $d(fx, f^2x) \leq \varepsilon_0$.

Then f *has a unique fixed point* p *in* $O(u)$ *and* $f^n u \to p$.

Theorem 3.5($C\delta'$). *Let* f *be a selfmap of a complete metric space* X. *If* ($C\delta'$) *holds for all regular points* $x \in X$, *then* f *has a unique fixed point* $p \in X$, *and* $f^n x \to p$ *for any regular point* $x \in X$.

9. Conclusion

The RHR maps properly include the Banach contraction and many of its extensions or modifications. In this article we collected many new results related to the RHR maps as a continuation of our previous [37]-[39]. Especially, we showed that the Suzuki type maps [50] in 2008 and its many modifications are RHR maps. Hence they need not their traditional complicated proofs of the Suzuki type maps.

Since there are thousands of results related to the Banach contraction, we anticipate many new results related to the RHR maps and their generalizations. We are encouraging the appearances of useful results on the class of generalized RHR maps, but not for artificial spaces or contractive conditions.

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