CHAOS Theory and Applications in Applied Sciences and Engineering

e-ISSN: 2687-4539
RESEARCH ARTICLE
Vol.4 / No.4 / 2022 / pp.205-211
Special Issue (EDIESCA 2022)
https://doi.org/10.51537/chaos.1185892

Multi-scroll Systems Synchronization on Strongly Connected Digraphs

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ABSTRACT In this paper, we study the synchronization problem in complex dynamic networks of Piece Wise Linear (PWL) systems. PWL systems exhibit multi-scrolls and belong to a special class of Unstable Dissipative Systems (UDS). We consider strongly connected digraphs and linear diffusive couplings. The synchronization regions are computed using the concept of disagreement vectors, generalized algebraic connectivity of the network topology, and Lyapunov functions, which provide lower bounds on the coupling gain of the network. Then, different combinations of linear diffusive coupling are explored by changing the observed and measured variables to illustrate the contribution of our results. The theoretical results are validated by numerical simulations.

KEYWORDS
Synchronization;
Complex networks
Digraphs
Multi-Scroll attractors
Unstable dissipative systems.

INTRODUCTION

In the last decade, the study of synchronization phenomena in a group of coupled Piece-Wise Linear (PWL) in the context of nonlinear systems theory has attracted considerable attention due to its wide application in fields such as physics, biology, and engineering, among others (Muñoz-Pacheco *et al.* 2012; Anzo-Hernández *et al.* 2019; Carbajal-Gómez and Sánchez-López 2019; Ruiz-Silva *et al.* 2021; Echenausía-Monroy *et al.* 2021; Ruiz-Silva *et al.* 2022).

One way to analyze these kinds of interconnected systems is to model them as complex networks whose nodes are the individual dynamical systems and the coupling is represented by a static graph. One of the most important aspects in the study of complex networks and their emergent behaviour is the structural analysis of the topology and dynamical properties of their nodes, to determine the conditions under which a set of interconnected dynamical systems achieve stable collective behaviour (Boccaletti *et al.* 2006; Wu 2007; Ávila-Martínez and Barajas-Ramírez 2018, 2021; Ávila-Martínez 2022). In this context, the term *synchronization* refers to the collective phenomenon in which two or more elements exhibit temporally coordinated dynamical behaviour (Boccaletti *et al.* 2002; Pikovsky *et al.* 2002).

A starting point for the study of synchronization in complex networks, whether a PWL or other non-linear system, is the assumption that nodes are identical, links are static, and coupling is diffusive. The diffusive condition is a basic assumption in this type of problem because it is a requirement that occurs naturally in many real-world networks and is a relatively soft condition on the structure of the network model (Chen *et al.* 2014). On the other hand, to achieve synchronization in a complex network, it is possible to consider different properties of the network links, such as unidirectional couplings (Anzo-Hernández *et al.* 2019; Posadas-Castillo *et al.* 2014), bidirectional or symmetric couplings (Ruiz-Silva *et al.* 2022; Soriano-Sánchez *et al.* 2016), connections with weights (Ruiz-Silva *et al.* 2021; Ontañón-García *et al.* 2021) or changes in the nature of the coupling functions (Echenausía-Monroy *et al.* 2021; Mishra *et al.* 2022). All these properties are reflected in the stability analysis of the synchronized behaviour, and some of them simplify it.

In this paper, we focus on the synchronization problem for a complex network under a fixed communication structure, where the dynamics of each node belongs to a class of affine linear systems. Traditionally, this problem can be approached by studying the system stability of the error around the synchronization solution using the λ_2 criterion (Chen *et al.* 2014), or the master stability function method (Pecora and Carroll 1998; Huang *et al.* 2009).

The method proposed in this paper is essentially compatible

Manuscript received: 7 October 2022,
Revised: 12 November 2022,
Accepted: 12 November 2022.

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with the λ_2 criterion since Lyapunov stability theory is used for the node dynamics and synchronization of the complex network. However, it presents a different point of view since the analysis is performed using the stability region of the nodes, the concept of disagreement vectors, and the generalized algebraic connectivity of the network topology (Li *et al.* 2010; Yu *et al.* 2010). First, we focus on the individual dynamics of the nodes and the internal coupling matrix, since unbounded stability regions must be determined to simplify the analysis of network synchronization. Then, using the disagreement vectors, we analyze the effect of the network topology on the stability regions of the nodes, which can be adjusted by the strength of the network coupling. Moreover, it is important to mention that an advantageous feature of this approach is that it can be used in bidirectional or unidirectional topologies as long as they represent strongly connected structures.

The rest of the document is structured as follows: We introduce first the multi-scroll system, the network model, and some helpful graph theory results. Then, we analyze the synchronized behaviour of strongly connected digraphs using the Lyapunov stability theory. We later present a case of study, followed by some numerical simulations illustrating our results. In the end, we discuss some conclusions.

PRELIMINARIES

Multi-Scroll System

It is known that the generation of attractors with multiple scrolls depends on both the stability of the generated equilibrium points and the type of switching function implemented (Echenausía-Monroy *et al.* 2020). It is possible to analyze the stability of the equilibrium points of this type of systems using the Unstable Dissipative Systems (UDS) theory, which describes a variety of three-dimensional systems with dissipative and conservative components. The coexistence of both components leads to the appearance of the so-called attractors with multi-scrolls (Campos-Cantón *et al.* 2010, 2012; Campos-Cantón 2016).

As in previous works (Gilardi-Velázquez *et al.* 2017; Echenausía-Monroy and Huerta-Cuellar 2020), we consider that each dynamical system is defined by a class of affine linear systems given by:

$$\dot{x}_i = Ax_i + B(x_i), \tag{1}$$

where $x_i = [x_{i1},\ x_{i2},\ x_{i3}]^T \in \mathbb{R}^3$ is the state vector of the i-th system, the constant matrix, $A = \{a_{ij}\} \in \mathbb{R}^{3\times 3}$, is the linear operator of the system, and $B = [b_1,\ b_2,\ b_3]^T \in \mathbb{R}^3$ is a vector with real entries. It should be noted that the behavior of the system (1) is determined by the eigen-spectrum of the matrix A, which can produce a variety of combinations and thus different dynamic behaviors.

The class of affine linear systems considered here are UDS of type 1, i.e., the eigenvalues associated with the linear operator A correspond to a hyperbolic saddle point where one eigenvalue is real negative and the other two are complex conjugate with a positive real component. Moreover, the sum of these values must be less than zero (Campos-Cantón *et al.* 2010, 2012; Campos-Cantón 2016). If the affine linear system given by Eq. (1) satisfies the UDS I definition with $B = 0$, then it is possible to generate an attractor with multi-scrolls by constructing a commutation law, in this case a PWL function. The purpose of the commutation function is to generate as many equilibrium points as desired and to control their visitation, which is achieved by coexisting a large number of unstable single-spiral trajectories (Echenausía-Monroy and Huerta-Cuellar 2020; Echenausía-Monroy *et al.* 2020).

Next, before we present the concept of a complex network, we introduce some preliminaries of algebraic graph theory.

Algebraic Graph Theory

A *directed graph* (in short, a *digraph*) of order N, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$ is a set of elements called nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of ordered pair of nodes. For $i, j \in \mathcal{V}$ the ordered pair $(j, i) \in \mathcal{E}$ denotes an edge that starts on node j and ends in node i. The neighbourhood of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. In $\mathcal{G}$ a *directed path* of length m from node i to j is a sequence of edges with distinct nodes n_k, with $k = 1, 2, \ldots, m$, such that $(i, n_1), (n_1, n_2), \ldots, (n_m, j) \in \mathcal{E}$. A graph $\mathcal{G}$ is *strongly connected* if there exists a directed path connecting every nodes pair. A digraph $\mathcal{G}$ is called *weighted* if for every edge $(j, i) \in \mathcal{E}$ there is an associated *weight* $w_{ij} > 0$.

The Laplacian matrix of a weighted digraph $\mathcal{G}$ is a zero row sum non-negative matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ defined as:

$$l_{ij} := \begin{cases} -w_{ij} & \text{if } (j, i) \in \mathcal{E}, \\ \sum_{i=1, j\neq i}^{N} w_{ij} & \text{if } i = j. \end{cases}$$

Now, we present some results related to matrix Laplacians.

Lemma 1. *(Li 2015) Suppose that $\mathcal{G}$ is strongly connected. Then, there is a positive left eigenvector $z = [z_1, \cdots, z_N]^T \in \mathbb{R}^N$ of L associated with the zero eigenvalue and $\hat{L} := ZL + L^T Z \geq 0$, where $Z = Diag(z_1, \cdots, z_N)$.*

Lemma 2. *(Li 2015) For a strongly connected graph $\mathcal{G}$ with Laplacian L, define its generalized algebraic connectivity as*

$$\alpha := \min_{z^T x = 0, x \neq 0} \left\{ \frac{x^T \hat{L} x}{x^T Z x} \right\}, \tag{2}$$

where z and Z are defined as in Lemma 1. Then, $\alpha > 0$.

Lemma 3. *(Yu et al. 2010) The generalized algebraic connectivity of a strongly connected digraph $\mathcal{G}$ can be computed by the following:*

$$\begin{aligned} &\mathit{max} && \mu, \\ &\mathit{subject\ to} && Q^T \left(\frac{1}{2}\hat{L} - \mu Z \right) Q \geq 0, \end{aligned} \tag{3}$$

where $Q = \begin{pmatrix} \mathcal{I}_{N-1} \\ -\hat{z}^T / z_N \end{pmatrix} \in \mathbb{R}^{N\times(N-1)}$ *and* $\hat{z} = [z_1, \ldots, z_{N-1}]^T \in \mathbb{R}^{N-1}$.

The Complex Dynamical Network Models

A complex dynamic network is defined as a set of interconnected systems, being each system a fundamental entity whose dynamics depend on the nature of the network (Chen *et al.* 2014). The interaction structure or network topology is modeled by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of fundamental units, and an edge $(i, j) \in \mathcal{E}$ depicts the interaction between nodes i and j. Therefore, the state describing the dynamic network are as follows:

$$\dot{x}_i = f(x_i) - c \sum_{j\in\mathcal{N}_i} w_{ij}\Gamma(x_i - x_j), \qquad i \in \mathcal{V}, \tag{4}$$

where $x_i = [x_{i1},\ x_{i2},\ x_{i3}]^T \in \mathbb{R}^3$ is the state vector of node i, the function $f(x_i) = Ax_i + B(x_i)$ can be derived from Eq. (1), and determines the dynamics of an isolated multi-scroll system.

The constant $c > 0$ denotes the uniform coupling strength of the network. Let $\Gamma = Diag(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^{3\times 3}$ be a constant matrix describing the internal coupling between the systems in the network, constructed as follows: if $\gamma_k > 0$ indicates that the ith node and the jth node are coupled by their kth state variable, otherwise $\gamma_k = 0$. The value $w_{ij} > 0$ is the weight of the ij-th edge, portraying the external coupling. The network described by Eq. (4) can be rewritten in terms of the matrix Laplacian entries as follows:

$$\dot{x}_i = Ax_i + B(x_i) - c\sum_{j\in\mathcal{N}_i} \ell_{ij}\Gamma x_j, \qquad i \in \mathcal{V}, \tag{5}$$

which in vector form is given by

$$\dot{x} = \Big((\mathcal{I}_N \otimes A) - c(L\otimes\Gamma)\Big)x + \tilde{B}(x), \tag{6}$$

where $x = [x_1^T,\ x_2^T, \cdots,\ x_N^T]^T \in \mathbb{R}^{3N}$, $\tilde{B}(x) = [B(x_1)^T, \cdots,\ B(x_N)^T]^T \in \mathbb{R}^{3N}$, $\mathcal{I}_N$ is the identity matrix of size $N\times N$, and $\otimes$ denotes the Kronecker product. It is worth noting that the network model describes all kinds of topologies, where they can consider connection patterns with uniform weights or non-uniform connections.

SYNCHRONIZATION PROBLEM AND MAIN RESULTS

One of the most-studied collective behaviors for a set of interconnected systems is the synchronization phenomenon, which emerges when the dynamics of the systems correlate over time (see (Chen *et al.* 2014; Boccaletti *et al.* 2002; Pecora and Carroll 1998; Arenas *et al.* 2008) and references therein). Although there are several definitions of synchronization in dynamic networks, this study focuses on *complete synchronization*. Mathematically, this is defined as follows:

Definition 1. *(Chen et al. 2014) It is said that the dynamic network (4) achieves complete asymptotic synchronization when*

$$\lim_{t\to\infty}\|x_i - x_j\| = 0, \qquad i,j\in\mathcal{V}, \tag{7}$$

where $\|\cdot\|$ is the Euclidean norm of a vector.

The goal of this paper is to find sufficient conditions for the nodes in the network to achieve complete synchronization, i.e., to ensure that Eq. (7) is satisfied regardless of the initial conditions. Since the linear operator A, the constant vector $B(\cdot)$, and the matrix Γ have a particular form, synchronization must be achieved by suitably designing the coupling strength, taking into account the structural properties of the network.

Stability Analysis on Strongly Connected Digraphs

Inspired by (R. Olfati-Saber and R. M. Murray 2004; Li *et al.* 2010), we introduce *disagreement functions* to perform stability analysis of the synchronous behavior of the network (6).

Let $z \in \mathbb{R}^N$ be defined as in Lemma 1 such that $z^T\mathbf{1} = 1$, where $\mathbf{1}\in\mathbb{R}^N$ denotes the vector where all entries are ones. Thus, the *disagreement vector* is defined as:

$$\delta := \Big((\mathcal{I}_N - \mathbf{1}z^T)\otimes\mathcal{I}_3\Big)x, \tag{8}$$

where $\delta = [\delta_1^T,\ \delta_2^T,\ \ldots, \delta_N^T] \in \mathbb{R}^{3N}$ satisfies the condition $(z^T\otimes I_3)\delta := \mathbf{0}$. It is important to emphasize that $\delta_i = x_i - \sum_{k=1}^N z_k x_k$ and $\delta_i - \delta_j = x_i - x_j$. Thus, by the Definition 1, the synchronization state is reached if and only if $\delta\to\mathbf{0}$ is $t\to 0$. Also, it can be proved that δ evolves according to the development given by *disagreement dynamics*:

$$\dot{\delta} = \Big[(\mathcal{I}_N\otimes A) - c(L\otimes\Gamma)\Big]\delta + \Big[(\mathcal{I}_N - \mathbf{1}z^T)\otimes\mathcal{I}_3\Big]\tilde{B}(x). \tag{9}$$

To show the stability for each of the disagreement vectors, the following assumptions are required for the remainder of this paper:

Assumption 1. *For each configuration of the matrix $\Gamma = Diag(\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}^{3\times 3}$ with $\Gamma > 0$, there exist constants $d > 0$ and $\eta > 0$ such that*

$$A + A^T - d\Gamma \le -\eta\mathcal{I}_3, \tag{10}$$

where $\mathcal{I}_3$ is the identity matrix of size 3×3.

Assumption 2. *Let us assume that there are known or unknown non-negative constants $\beta_{ij}\ge 0$, so that*

$$\|B(x_i) - B(x_j)\| \le \beta_{ij}\|x_i - x_j\|, \tag{11}$$

with $i\neq j$, for $i,j = 1,2,\cdots,N$.

Under these assumptions, we establish the following result:

Theorem 1. *Suppose that the Assumption 1 holds and that the dynamic network described by Eq. (6) is strongly connected. If the coupling strength c satisfies the condition*

$$c \ge \frac{d^*}{\alpha}, \tag{12}$$

where d^* is a non-positive constant and α is the generalized algebraic Fielder's connectivity of $\mathcal{G}$. Then the disagreement dynamics is asymptotically stable at the equilibrium, or equivalently $\delta_i \to 0$, for any $i = 1,2,\cdots,N$. Consequently, the complex dynamical network (6) achieves synchronization.

Proof. Define the Lyapunov function candidate as:

$$V(\delta) := \frac{1}{2}\delta^T(Z\otimes\mathcal{I}_3)\delta, \tag{13}$$

with the positive matrix $Z = Diag(z_1, z_2,\cdots, z_N) > 0$ defined as in Lemma 1.

The time derivative of Eq. (13) along the trajectories of (9) yields:

$$\dot{V}(\delta) = \delta^T(Z\otimes\mathcal{I}_3)\dot{\delta} = U(\delta) + W(\delta,\tilde{B}(x)), \tag{14}$$

with

$$U(\delta) := \frac{1}{2}\delta^T\Big[\Big(Z\otimes(A+A^T)\Big) - c\,(\hat{L}\otimes\Gamma)\Big]\delta$$
$$W(\delta,\tilde{B}(x)) := \delta^T\Big[Z(\mathcal{I}_N - \mathbf{1}z^T)\otimes\mathcal{I}_3\Big]\tilde{B}(x).$$

Using Lemma 1 and Lemma 2 in $U(\delta)$ we obtain

$$\begin{aligned} U(\delta) &\le \frac{1}{2}\delta^T\Big[\Big(Z\otimes(A+A^T)\Big) - c\alpha\,(Z\otimes\Gamma)\Big]\delta \\ &= \frac{1}{2}\delta^T\Big[\Big(Z\otimes(A+A^T - c\alpha\Gamma)\Big)\Big]\delta \\ &= \frac{1}{2}\sum_{i=1}^N z_i\delta_i^T\Big(A+A^T - c\alpha\Gamma\Big)\delta_i, \end{aligned} \tag{15}$$

where α is the generalized algebraic connectivity of the graph $\mathcal{G}$. For $W(\delta, \tilde{B}(x))$ it is also true that

$$\begin{aligned} W(\delta, \tilde{B}(x)) &= \sum_{i=1}^{N} z_i \delta_i^T \left(B(x_i) - \sum_{k=1}^{N} z_k B(x_k) \right) \\ &= \sum_{i=1}^{N} z_i \delta_i^T \left(B(x_i) - B(\bar{x}) + B(\bar{x}) - \sum_{k=1}^{N} z_k B(x_k) \right) \\ &= \sum_{i=1}^{N} z_i \delta_i^T \left(B(x_i) - B(\bar{x}) \right) \\ &\quad + \left(B(\bar{x}) - \sum_{k=1}^{N} z_k B(x_k) \right)^T \sum_{i=1}^{N} z_i \delta_i \\ &= \sum_{i=1}^{N} z_i \delta_i^T \left(B(x_i) - B(\bar{x}) \right), \end{aligned} \tag{16}$$

where $\bar{x} := \sum_{k=1}^{N} z_k x_k$, and we use the fact that $\sum_{i=1}^{N} z_i \delta_i = \mathbf{0}$.

Under Assumption 2, it follows that

$$\begin{aligned} \left\| \sum_{i=1}^{N} z_i \delta_i^T \left(B(x_i) - B(\bar{x}) \right) \right\| &\leq \sum_{i=1}^{N} z_i \|\delta_i\| \, \|B(x_i) - B(\bar{x})\| \\ &\leq \sum_{i=1}^{N} \beta z_i \|\delta_i\| \, \|x_i - \bar{x}\| \\ &\leq \sum_{i=1}^{N} \beta z_i \|\delta_i\|^2, \end{aligned} \tag{17}$$

with $\beta > 0$ is the largest Lipschitz constant of the function $B(\cdot)$.

Substitute Eqs. (15) and (17) into Eq. (14), we get:

$$\begin{aligned} \dot{V} &\leq \frac{1}{2} \sum_{i=1}^{N} z_i \delta_i^T \left(A + A^T - c\alpha\Gamma \right) \delta_i + \sum_{i=1}^{N} \beta z_i \delta_i^T \delta_i \\ &= \sum_{i=1}^{N} z_i \delta_i^T \left(\frac{1}{2}(A + A^T - c\alpha\Gamma) + \beta \mathcal{I}_3 \right) \delta_i. \end{aligned} \tag{18}$$

Let $d = c\alpha$, then under the Assumption 1 it follows that

$$A + A^T - c\alpha\Gamma \leq -\eta \mathcal{I}_3, \tag{19}$$

with $d^* \leq d = c\alpha$ and $\eta > 0$. Since $d^* > 0$ and $\alpha > 0$, we solve for c from the inequality $d^* \leq c\alpha$ and we have the condition (12). Therefore, the inequality (18) can be rewritten as

$$\dot{V}(\delta) \leq \sum_{i=1}^{N} z_i \left(\beta - \frac{\eta}{2} \right) \|\delta_i\|^2. \tag{20}$$

Note that the right-hand side of the previous inequality is a quadratic function and $z_i > 0$ for $i = 1, 2, \cdots, N$. Thus, if we choose $\eta > 2\beta$, it follows that $\dot{V}(\delta) < 0$. Consequently, $\delta_i \to \mathbf{0}$ as $t \to \infty$, *i.e.* the network (6) asymptotically synchronizes. □

It should be emphasized that the Assumption 1 provides a bound on the stability of the linear operator A, while the Assumption 2 indicates that the vector B around zero is a fading perturbation. Moreover, the value of α can be computed as in Lemma 3. Up to this point, the Theorem 1 gives such a value for the coupling strength c that $\|\delta_i\| \to 0$ as $t \to \infty$. Thus, there is a certain range for the coupling strength in which the synchronization of the digraph is guaranteed. Notice that other values that can lead to synchronization of the network are not excluded.

A CASE OF STUDY

Consider a multi-scroll system whose dynamics is described by Eq. (1). In particular, take the following dynamic system:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -a & -a \end{bmatrix}}_{A} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ ab(x_i) \end{bmatrix}}_{B(x_i)}, \tag{21}$$

where $x_i = [x_{i1}, x_{i2}, x_{i3}]^T \in \mathbb{R}^3$ is the state vector, a is a dynamical parameter modifying the Lyapunov exponent, the order and the magnitude of the attractor (Echenausía-Monroy *et al.* 2018), and $b(\cdot) : \mathbb{R}^3 \mapsto \mathbb{R}$ is a PWL function given as:

$$b(x_i) = \begin{cases} -6 & \text{if } x_i \in \mathcal{D}_1 = \{\mathbf{x_i} \mid x_{i1} < -5\}, \\ -4 & \text{if } x_i \in \mathcal{D}_2 = \{\mathbf{x_i} \mid -5 \leq x_{i1} < -3\}, \\ -2 & \text{if } x_i \in \mathcal{D}_3 = \{\mathbf{x_i} \mid -3 \leq x_{i1} < -1\}, \\ 0 & \text{if } x_i \in \mathcal{D}_4 = \{\mathbf{x_i} \mid -1 \leq x_{i1} < 1\}, \\ 2 & \text{if } x_i \in \mathcal{D}_5 = \{\mathbf{x_i} \mid 1 \leq x_{i1} < 3\}, \\ 4 & \text{if } x_i \in \mathcal{D}_6 = \{\mathbf{x_i} \mid 3 \leq x_{i1} < 5\}, \\ 6 & \text{if } x_i \in \mathcal{D}_7 = \{\mathbf{x_i} \mid x_{i1} \geq 5\}, \end{cases} \tag{22}$$

where $\mathcal{D} = \{\mathcal{D}_1, \cdots, \mathcal{D}_7\}$ is a finite partition of the phase space. As mentioned before, the parameter a in Eq. (21) determines the system's equilibrium points stability, and must satisfy the UDS I conditions (Campos-Cantón *et al.* 2010; Anzo-Hernández *et al.* 2018). To achieve this, a can only take values from the set $a \in (0, 1)$ and thus, generating the same number of scrolls as equilibrium points in the system.

To illustrate that Eqs. (21)-(22) form a multi-scroll system, take $a = 0.6$. Hence, the matrix A has a negative real eigenvalue and two complex conjugate eigenvalues whose sum is negative, i.e.[5]

$$\sigma(A) = \{-0.794,\ 0.097 \pm 0.863\mathbf{i}\}, \text{ and } \sum_{i=1}^{3} \sigma_i = -0.6. \tag{23}$$

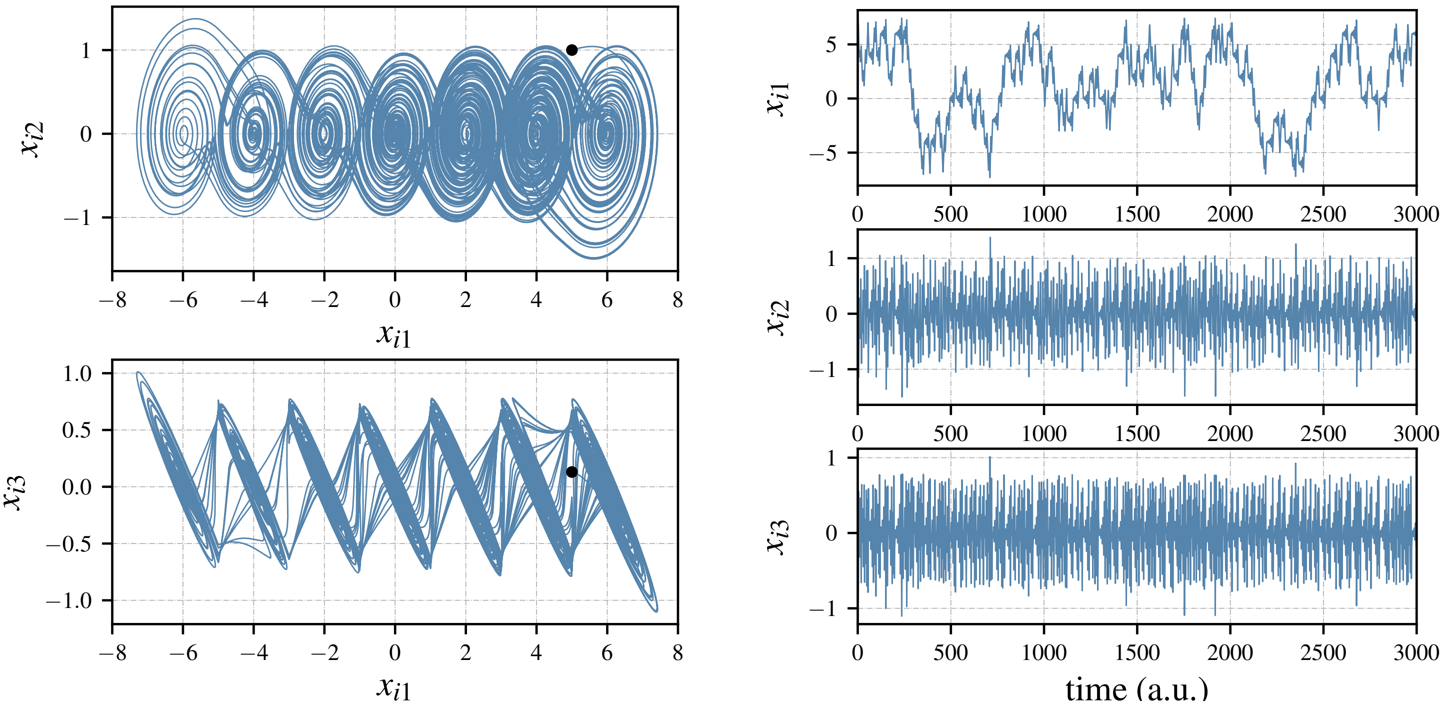
Under these conditions, system described by Eq. (21) is a UDS type I system. Figure 1 shows its state trajectories with an initial condition $x_i^0 = [5,\ 1,\ 0.13]^T$. In Figure 1(a) we show the projection of the multi-scroll attractor onto the planes $(x_{i1} - x_{i2})$ and $(x_{i1} - x_{i3})$. Figure 1(b) corresponds to the temporal behaviour of the states x_{i1}, x_{i2} and x_{i3} with arbitrary units (a.u.) time.

Dynamical Network

For ease of illustration, consider a network of N identical multi-scroll systems with dynamics described by Eq. (21), with linear and diffusive couplings. Thus, we describe the dynamic network by Eq. (5) and $\beta_{ij} = \beta > 0$, for all $i, j \in \mathcal{V}$, in Assumption 2.

Theorem 1 must satisfy the Assumption 1 and satisfy the inequality (12). Note that in order to obtain an appropriate value for the coupling gain in Eq. (12), we need to compute d as shown in Assumption 1, and Eq. (10) imposes a Hurwitz condition over the

[5] Here $\mathbf{i}$ stands for the imaginary unit.



(a) (b)

Figure 1 Attractor and states behaviour generated by Eqs. (21)-(22) with $a = 0.6$.

matrix $A + A^T - d\Gamma$. Therefore, in this section we are interested in finding a method to design the internal coupling matrix Γ and the external coupling gain c to achieve synchronization. Inspired by the λ_2 criterion (Chen *et al.* 2014) and the Master Stability Function (MSF) (Pecora and Carroll 1998; Huang *et al.* 2009), we compute synchronizability regions over an $(a - d)$-plane from which we can choose a particular matrix Γ and values for d and hence for c.

From the Eq. (10), described in Assumption 1, and the linear operator A defined in Eq. (21) we obtain the following matrix:

$$M := A + A^T - d\Gamma = \begin{bmatrix} -d\gamma_1 & 1 & -a \\ 1 & -d\gamma_2 & 1-a \\ -a & 1-a & -(2a + d\gamma_3) \end{bmatrix}, \quad (24)$$

with characteristic polynomial

$$p(M) := \sigma^3 + \kappa_2\sigma^2 + \kappa_1\sigma + \kappa_0, \quad (25)$$

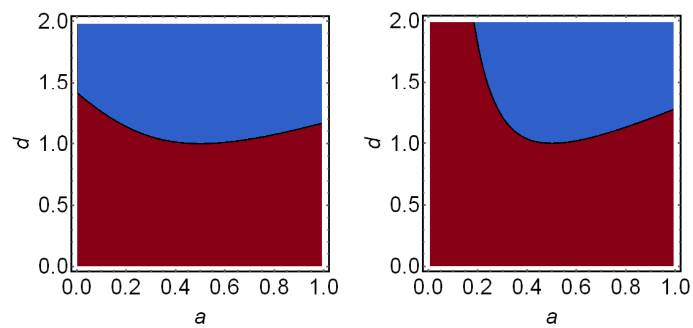
where

$$\begin{aligned} \kappa_2 &= 2a + d(\gamma_1 + \gamma_2 + \gamma_3), \\ \kappa_1 &= d^2(\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3) + 2ad(\gamma_1 + \gamma_2) \\ &\quad -2\left(a + (1-a)^2\right), \\ \kappa_0 &= d^3\gamma_1\gamma_2\gamma_3 + 2ad\gamma_1(1 + d\gamma_2) - a^2(2 + d\gamma_1) \\ &\quad -d(\gamma_1 + \gamma_3). \end{aligned}$$

Note that M is symmetric and therefore all its eigenvalues are real. Denote by σ_k, with $k \in \mathcal{K} := \{1, 2, 3\}$, the eigenvalues of the matrix M. For all $k \in \mathcal{K}$, $\sigma_k < 0$ holds if and only if M satisfies the Routh-Hurwitz stability criterion, namely

$$\kappa_2 > 0,\ \kappa_1 > 0,\ \kappa_0 > 0, \text{ and } \kappa_2\kappa_1 - \kappa_0 > 0. \quad (26)$$

Recall that a is in $(0, 1)$, so two different values of a can lead to different multi-scroll systems with different parameters for $p(M)$. Therefore, a particular matrix Γ and a particular value for d may not be appropriate for every choice. To accommodate a variety of multi-scroll systems, the proposed method is to choose an internal coupling matrix Γ and numerically solve the inequalities of Eq. (26) as a function of a and d. The result is a synchronizability region in the $(a - d)$ plane. In Figure 2 we show some examples of this; The blue regions indicate values for which inequalities in Eq. (26) hold.



(a) (b)

Figure 2 Synchronizability region (blue) of the matrix M subject to parameters a and d with: (a) $\Gamma = Diag(1, 1, 1)$ and, (b) $\Gamma = Diag(1, 1, 0)$.

Remark 1. *Although there are up to eight different combinations of the values for γ_1, γ_2, and γ_3, a quick examination of the inequalities from Eq. (26) shows that six of them cannot satisfy them. Synchronization can be achieved only if $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 \geq 0$.*

Remark 2. *In Assumption 1, notice that, by the min-max theorem (Allaire and Kaber 2007), we can choose* $\eta = \min_{k \in \mathcal{K}} \{|\sigma_k|\}$.

NUMERICAL ILLUSTRATION

Let the inner coupling matrix $\Gamma = Diag(1,1,1) \in \mathbb{R}^3$, and let the topology of the network be as shown in Figure 3, whose elements satisfy the conditions of a strongly connected graph. Therefore, the Laplacian matrix and its left eigenvector are given by:

$$L := \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 6 & -2 & 0 & -2 & -2 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 \\ -1 & 0 & 0 & 0 & -5 & 6 \end{bmatrix} \text{ and } z = \begin{bmatrix} 0.168 \\ 0.237 \\ 0.168 \\ 0.118 \\ 0.118 \\ 0.188 \end{bmatrix}.$$

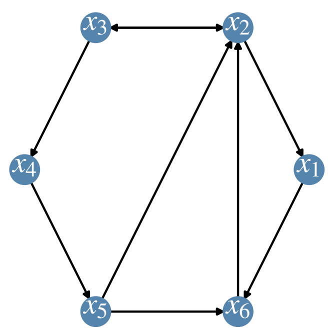


Figure 3 A strongly connected digraph of order $N = 6$.

As mentioned in the previous section, for each configuration of the internal coupling matrix and each value of the parameter a, there are critical values d such that $A + A^T - d\Gamma$ is a negative definite matrix. For this example, it is possible to choose the value of d using Figure 2(a), let $d = 1.5$ hold, which is valid for all $a \in (0,1)$. Then the generalized algebraic connectivity for the graph shown in Figure 3 is $\alpha \approx 0.7017$. Thus, to ensure synchronization in the nodes, the coupling strength must satisfy $c > d/\alpha \approx 2.13$ according to the Theorem 1.

To illustrate the above in more detail, Figure 4 shows the time series of coupled systems (5) with randomly chosen initial conditions. In the numerical simulations, the Figure 4(a) corresponds to the time series of the network state with parameter $a = 0.45$ and $\Gamma = Diag(1,1,1)$; while the Figure 4(b) corresponds to the time series of states for a network with $a = 0.6$ and $\Gamma = Diag(1,1,0)$. In both simulations, it is assumed that for $t < 1000$ $(a.u.)$ the nodes are decoupled, so that each solution evolves its own attractor. While for $t > 1000$ $(a.u.)$ the nodes are connected in a network structure with a coupling strength $c = 2.14$. Moreover, it can be observed how the trajectories of all nodes collapse in the three states, i.e., the nodes achieve complete synchronization.

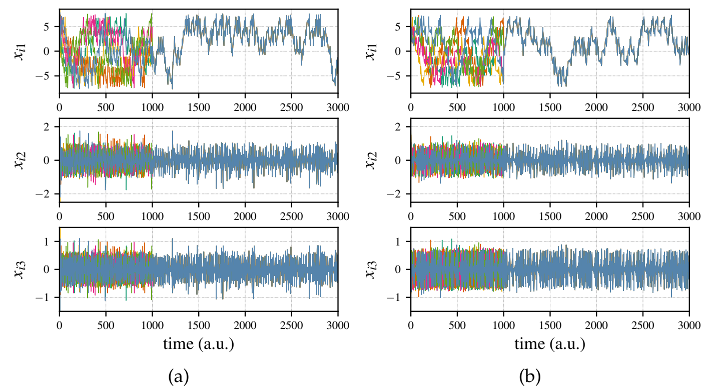


Figure 4 Numerical simulation of the system from Eq. (5) with: (a) $a = 0.45$ and $\Gamma = Diag(1,1,1)$, (b) $a = 0.6$ and $\Gamma = Diag(1,1,0)$.

CONCLUSIONS

This paper studies the synchronization problem in a complex network where each node belongs to a class of PWL systems. The network's topology is directed and strongly connected with linear and diffusive couplings. Using graph theory and Lyapunov stability theory, we established synchronization conditions utilising the notion of disagreement vectors and generalized algebraic connectivity for digraphs. We then use our main result and the Routh-Hurwitz criterion to determine synchronizability regions for a given affine system, namely a UDS type-I system. For a given inner coupling matrix and a directed network topology, we compute the synchronizability regions as a function of a system parameter and a ratio between the generalized connectivity and the coupling strength. In this way, we determine minimum values for the coupling strength that allow synchronization. An advantageous feature of our approach is its flexibility in network structures. Although our main result is related to strongly connected digraphs, it is also suitable for undirected graphs.

In future work, we will further investigate the synchronization of UDS Type-I systems and provide a general method for computing synchronizability regions. We will also consider networks of systems with a different number of scrolls, including the effects of the performance parameters associated with the nonlinear functions on their electronic implementation.

Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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How to cite this article: Ávila-Martínez, E. J., Echenausía-Monroy, J. L., and Ruiz-Silva, A. Multi-scroll Systems Synchronization on Strongly Connected Digraphs. *Chaos Theory and Applications*, 4(4), 205-211, 2022.