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Pseudo-spectrum and the numerical range for Ricci tensor on the oscillator group of dimension four

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Abstract

The field of functional analysis presents a very interesting part of pure mathematics, but also applied mathematics such as the theory of approximations and the resolution of operational equations, the spectra of operators, pseudospectrum and their numerical range which are essential techniques for researchers in several fields of science and technology. In this work, we will give the notions of the numerical range of a matrix and some properties, and study it for curvature tensor $R_{ij} = R(\partial_i, \partial_j)$, also for Ricci tensor ρ on the oscillator group (G, g_a) of dimension four and we will give examples of each matrix with the use of Matlab.

1. Introduction

Eigenvalues have been the subject of study and research for more than a century and a half. They are used in practice in many cases such as the resolution of differential equations, or partial differential equations. Since the advent of computers, the calculation of eigenvalues through computer-implemented algorithms has become more popular. This made it possible to obtain matrix spectra much more simply. It is this practical use which raised some problems on these eigenvalues.

Indeed, the calculation of the eigenvalues via a computer generates rounding errors, which can give values very far from the theoretical eigenvalues. The study of the pseudo-spectrum allows us to study the behavior of these values (rounded) and thus to control errors. The pseudo spectrum is a fairly recent concept and has been discussed a lot to answer practical problems in particular. In 1974, Henry Landau created the epsilon approximate eigenvalues.

Jim Varah, in 1977, invented the epsilon-spectrum. He was interested in the stability of matrix invariant subspaces in the context of numerical solutions to eigenvalue problems of non-Hermitian matrices. It was in the 1980s that Sergei Godunov and the Novosibirsk group introduced the notion of a figured spectrum (spectral portrait). In 1988, the mathematician Nick Trefethen invented the epsilon approximation of eigenvalues. His work is rooted in observations concerning unstable eigenvalues of the spectrum of matrices for differential equations.

Finally, in 1990, Diederich Hinrichsen and Tony Pritchard brought the notion of spectral value set. The notion of the numerical range was introduced by Otto Toeplitz [1] in 1918 for complex matrices, in 1919 F. Hausdorff [2] proved that the numerical range of a complex matrix is convex, in years 1929 and 1932 A. Winter [3] and M. H. Stone [4] studied the relations between the numerical range and the convex hull of the spectrum of a bounded linear operator in a Hilbert space. In 2020, R. Derkaoui and A. Smail published an article entitled: "Generalized Spectrum and Numerical Range of Matrix the Lorentzian Oscillator Group of Dimension Four". In this article, they give the spectrum, pseudo-spectrum and numerical rang of matrix of the metric g_a the lorentzian oscillator

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group of dimension four (G, g_a) with illustrative axample (See [5]). This metric g_a is explicitly given by

$$g_a = adx_1^2 + 2ax_3dx_1dx_2 + (1+ax_3^2)dx_2^2 + dx_3^2 + 2dx_1dx_4 + 2x_3dx_2dx_4 + adx_4^2,$$

with $-1 < a < 1$. The matrix of the metric g_a is given by

$$A_a = \begin{pmatrix} a & ax_3 & 0 & 1 \\ ax_3 & 1+ax_3^2 & 0 & x_3 \\ 0 & 0 & 1 & 0 \\ 1 & x_3 & 0 & a \end{pmatrix}.$$

Proposition 1 *[5] The numerical rang of matrix A_a check the following relation:*

$$\left|\frac{x^*A_ax}{x^*x}\right| \leq (1+|a|)(1+|x_3|) + \left|ax_3^2\right|.$$

The pseudospectrum of a normal matrix A consists of circles of radius around each eigenvalue. For nonnormal matrices, the pseudospectrum takes different forms in the complex plane. In [6] The pseudospectrum of thirteen highly non-normal matrices is presented.

In this work, we will study the spectrum, pseudo-spectrum and the numerical range of the curvature tensor $R_{ij} = R(\partial_i, \partial_j)$, also for Ricci tensor ρ and we will give examples of each matrix with the use of Matlab.

2. Preliminaries

Pseudospectra of matrices and numerical range

We shall generally let A denote a matrix in $\mathbb{C}^{n\times n}$. We are able to motivate what the idea of pseudospectra is by what we can observe throught applied mathematics, "is A singuler" isn't robut because of an arbitry small perturbation the answer will change but it is better to "Is $\|A^{-1}\|$ large". Now, to define the eigenvalue we need the condition of matrix singularity. To know if " z is an eigenvalue of A" is the same as to ask "is $zI - A$ singular "therefore, the property of being an eigenvalue of a matrix isn't robust then to ask better "is $\|(zI-A)^{-1}\|$ large"

Definition 1 (The norm of resolvent) *[7] Let $M \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$, the $\varepsilon-$ pseudospectrum of the matrix M is the set of complex numbers such that the norm of the resolvent is very large, i.e,*

$$\sigma_\varepsilon = \left\{z \in \mathbb{C} : \left\|(zI-M)^{-1}\right\| > \varepsilon^{-1}\right\}.$$

Definition 2 (The perturbation theory) *[7] Let $M \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$, the pseudospectrum of matrix M is a perturbation of a spectrum, i.e. it is the set of all eigenvalues of neighboring matrices of matrix M, i.e,*

$$\sigma_\varepsilon = \left\{z \in \mathbb{C} : z \in \sigma(M+E) \text{ for some } E \in \mathbb{C}^{n\times n} \text{ with } \|E\| < \varepsilon\right\}.$$

Proposition 2 *[7] For a normal matrix, the ε-pseudospectrum is simply the union of open ε-balls with center eigenvalues and radius ε.*

Definition 3 *[8] Let T be an operator in $B(H)$ (i.e bounded linear operator on a Hilbert space H), the numerical range of T is the set $W(T)$ of complex numbers defined by*

$$W(T) = \{\langle Tx, x\rangle \ : x \in H, \ \|x\| = 1\}.$$

Definition 4 *[7] The (2-norm) numerical range of a matrix $M \in \mathbb{C}^{n\times n}$ is the set*

$$W(M) = \left\{\frac{z^*Mz}{z^*z}, \ z \in \mathbb{C}^n, \|z\| \neq 0\right\},$$

is defined to be where z^* denotes the conjugate transpose of the vector z.

Proposition 3 *[7] Let $M_1 \in \mathbb{C}^{n\times n}, M_2 \in \mathbb{C}^{n\times n}$, $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$ then:*

1)
$$W(M_1+M_2) \subset W(M_1)+W(M_2).$$

2)
$$W(\mu_1 M_1) = \mu_1 W(M_1).$$

3)
$$W(\mu_2 M_1 + \mu_3 I_n) = \mu_2 W(M_1) + \mu_3.$$

4)
$$W(M_1^*) = \{\bar{z}, z \in W(M_1)\}.$$

Theorem 1 *[7] The numerical range of matrix A is no empty bounded and convex set.*

Proposition 4 *[7] Let $M \in \mathbb{C}^{n\times n}$,*
$$\sigma(M) \subset W(M).$$

Proof 1 *Let $\lambda \in \sigma(M)$ and $x \in H$ such that $\|x\| = 1$ then, $Mx = \lambda x$ and then $\langle (M-\lambda)x, x\rangle = 0$ then, $\langle Mx, x\rangle = \lambda$ then $\lambda \in W(M)$*

Theorem 2 *[7] Let $M \in C^{n\times n}$ then,*
$$\sigma_\varepsilon(M) \subseteq W(M) + \Delta_\varepsilon.$$
Where $\Delta_\varepsilon$is the closed disk of center 0 and radius ε.

The components of the curvature tensor $R_{ij} = R(\partial_i, \partial_j) = \nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial i}$, $i, j \in \{1,..,4\}$ on the oscillator group (G, g_a), relative to the local coordinate system, are given by (See [9]):

$$R_{12} = \begin{pmatrix} \frac{a^2x_3}{4} & \frac{a^2x_3^2+a}{4} & 0 & \frac{ax_3}{4} \\ -\frac{a^2}{4} & -\frac{ax_3}{4} & 0 & -\frac{a}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_{13} = \begin{pmatrix} 0 & 0 & \frac{a}{4} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a^2}{4} & -\frac{a^2x_3}{4} & 0 & -\frac{a}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{23} = \begin{pmatrix} 0 & 0 & ax_3 & 0 \\ 0 & 0 & -\frac{3a}{4} & 0 \\ -\frac{a^2x_3}{4} & \frac{3a-a^2x_3^2}{4} & 0 & -\frac{ax_3}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_{34} = \begin{pmatrix} 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{4} & \frac{ax_3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, . \tag{1}$$

$$R_{24} = \begin{pmatrix} -\frac{ax_3}{4} & -\frac{a^2x_3+1}{4} & 0 & -\frac{x_3}{4} \\ \frac{a}{4} & \frac{ax_3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let ρ be the Ricci tensor on the oscillator group (G, g_a). The components $\rho_{ij} = \rho(\partial_i, \partial_j)$ with respect to the local coordinate system are then given by (See [9]):

$$\rho = \begin{pmatrix} \frac{1}{2}a^2 & \frac{1}{2}a^2x_3 & 0 & \frac{1}{2}a \\ \frac{1}{2}a^2x_3 & \frac{1}{2}a\left(ax_3^2-1\right) & 0 & \frac{1}{2}ax_3 \\ 0 & 0 & -\frac{1}{2}a & 0 \\ \frac{1}{2}a & \frac{1}{2}ax_3 & 0 & \frac{1}{2} \end{pmatrix}. \tag{2}$$

3. The main result

3.1. Spectrum of R_{12}, R_{13}, R_{23}, R_{34} and R_{24}

In this section, we find spectrum and the numerical range for curvature tensor $R_{ij} = R(\partial_i, \partial_j)$ on the oscillator group (G, g_a) of dimension four.

Proposition 5 *The eigenvalues of matrice R_{12}, R_{13}, R_{23}, R_{34} and R_{24} defined in (1) are given by*
1-For R_{12} :if

$$\frac{2-\sqrt{7}}{3} \leq a \leq 0,$$

or

$$0 \leq a < 1 \text{ and } x_3 \in \left[\frac{-\sqrt{a}}{\sqrt{-3a^2+4a+1}}, \frac{\sqrt{a}}{\sqrt{-3a^2+4a+1}}\right],$$

then

$$\sigma(R_{12}) = \left\{ \begin{array}{l} \frac{1}{2}a\sqrt{-\frac{3}{16}a^2x_3^2+\frac{1}{8}ax_3^2-\frac{1}{4}a+\frac{1}{16}x_3^2}-\frac{1}{8},\ ax_3+\frac{1}{8}a^2x_3, \\ \frac{1}{8}a^2x_3-\frac{1}{2}a\sqrt{-\frac{3}{16}a^2x_3^2+\frac{1}{8}ax_3^2-\frac{1}{4}a+\frac{1}{16}x_3^2}-\frac{1}{8}ax_3,\ 0 \end{array} \right\}.$$

2-For R_{13} :if

$$-1 < a \leq 0,$$

then

$$\sigma(R_{13}) = \left\{\frac{1}{4}\sqrt{-a^3},\ -\frac{1}{4}\sqrt{-a^3},\ 0\right\}.$$

3-For R_{23} :if

$$-1 < a < 0, \text{ and } x_3 \in \left]-\infty, -\frac{3}{\sqrt{-a}}\right] \cup \left[\frac{3}{\sqrt{-a}}, \infty\right[,$$

then

$$\sigma(R_{23}) = \left\{\frac{1}{4}a\sqrt{-ax_3^2-9}, -\frac{1}{4}a\sqrt{-ax_3^2-9}, 0\right\}.$$

3-For R_{34} :if

$$-1 < a \leq 0,$$

then

$$\sigma(R_{34}) = \left\{\frac{1}{4}\sqrt{-a}, -\frac{1}{4}\sqrt{-a}, 0\right\}.$$

3-For R_{24} :if

$$x_3 \in \left[\frac{a^3-\sqrt{a^6+4a^3}}{2a^2}, \frac{a^3+\sqrt{a^6+4a^3}}{2a^2}\right], \text{ and } a \neq 0$$

then

$$\sigma(R_{24}) = \left\{\frac{1}{4}\sqrt{-a^3x_3+a^2x_3^2-a}, -\frac{1}{4}\sqrt{-a^3x_3+a^2x_3^2-a}, 0\right\}.$$

3.1.1. Numerical range of R_{12}, R_{13}, R_{23}, R_{34} and R_{24}

Theorem 3 *Let R_{12}, R_{13}, R_{23}, R_{34} and R_{24} be the matrices defined in (1),*
1- If $\eta \in W(R_{12})$ then

$$|\eta| \leq x_3^2 + \frac{3}{2}|x_3| + \frac{1}{2}. \tag{3}$$

2- If $\eta \in W(R_{13})$ then

$$|\eta| \leq \frac{1}{4}(1+|x_3|) + \frac{1}{2}. \tag{4}$$

3- If $\eta \in W(R_{23})$ then

$$|\eta| \leq \frac{x_3^2}{4} + \frac{3}{2}|x_3| + \frac{3}{2}. \tag{5}$$

4- If $\eta \in W(R_{34})$ then

$$|\eta| \leq \frac{|x_3|}{4} + \frac{3}{4}. \tag{6}$$

5- If $\eta \in W(R_{24})$ then

$$|\eta| \leq \frac{x_3^2}{4} + \frac{3|x_3|}{4} + \frac{3}{4}. \tag{7}$$

Proof 2 *1-Let $\eta \in W(R_{12})$, so $\exists\, z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{C}^4$ with $z_i = \rho_i e^{i\theta_i}, i = \overline{1,4}$ such that $z \neq 0$, such as*

$$\eta = \frac{z^* R_{12} z}{\sum_{i=1}^{4} |z_i|^2},$$

so

$$\eta = \frac{\frac{1}{4}a(z_2\overline{z_1}(ax_3^2+1) - x_3(\rho_2^2 - a\rho_1^2) - z_4\overline{z_2} + z_4\overline{z_1}x_3 - a\overline{z_2}z_1)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2}.$$

We have

$$\frac{|z_j|^2}{\sum_{i=1}^{4} |z_i|^2} \leq 1,\ \forall j \in \{1,2,3,4\} \tag{8}$$

because if we have the opposite, i.e, we pose $\exists j \in \{1,2,3,4\}$, such as

$$\frac{|z_j|^2}{\sum_{i=1}^{4} |z_i|^2} > 1,$$

so, for example ($j = 1$) we find

$$|z_2|^2 + |z_3|^2 < 0,$$

and this is a contradiction.

So from (8) we find

$$|\eta| \leq \frac{|ax_3|}{2} + \frac{|a|}{2} + \frac{a^2}{4}\left(x_3^2 + |x_3|\right).$$

We proved (3).

2-For $W(R_{13})$, we find

$$\eta = \frac{z^* R_{12} z}{\sum_{i=1}^{4} |z_i|^2} = \frac{\frac{1}{4}a\overline{z_1}z_3 - \frac{1}{4}a^2 z_1\overline{z_3} - \frac{1}{4}a\overline{z_3}z_4 - \frac{1}{4}a^2 x_3 z_2\overline{z_3}}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2},$$

so from (8), we find

$$|\eta| \leq \frac{a^2}{4}(1+|x_3|) + \frac{|a|}{2}.$$

We proved (4).

3-Now for $W(R_{23})$, we obtain

$$\eta = \frac{z^* R_{23} z}{\sum_{i=1}^{4} |z_i|^2} = \frac{\overline{z_3}z_2\left(\frac{3}{4}a - \frac{1}{4}a^2x_3^2\right) - z_3\left(\frac{3}{4}a\overline{z_2} - a\overline{z_1}x_3\right) - \frac{1}{4}a^2\overline{z_3}x_3z_1 - \frac{1}{4}a\overline{z_3}x_3z_4}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2},$$

so from (8), we find

$$|\eta| \leq \frac{3}{2}|x_3| + \frac{x_3^2}{4} + \frac{3}{2}.$$

We proved (5).

4-Next, for $W(R_{34})$, *we obtain*

$$\eta = \frac{z^* R_{34} z}{\sum_{i=1}^{4} |z_i|^2} = \frac{\frac{a}{4}\overline{z_3}z_1 + \frac{ax_3}{4}z_2\overline{z_3} - \frac{1}{4}\overline{z_1}z_3 + \frac{1}{4}\overline{z_3}z_4}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2},$$

so from (8), *we find*

$$|\eta| \leq \frac{|x_3|}{4} + \frac{3}{4}.$$

We proved (6).

5-Finally, for $W(R_{34})$, *we find*

$$\eta = \frac{z^* R_{24} z}{\sum_{i=1}^{4} |z_i|^2} = \frac{z_4\left(\frac{1}{4}\overline{z_2} + \frac{1}{4}\overline{z_1}x_3\right) + z_1\left(\frac{1}{4}a\overline{z_2} - \frac{1}{4}a\overline{z_1}x_3\right) - z_2\left(\overline{z_1}\left(\frac{1}{4}a^2x_3^2 + \frac{1}{4}\right) - \frac{1}{4}a\overline{z_2}x_3\right)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2},$$

so from (8), *we find*

$$|\eta| \leq \frac{3}{4} + \frac{3|x_3|}{4} + \frac{x_3^2}{4}.$$

We proved (7).

3.2. The numerical range for Ricci tensor ρ

In this section, we will give we find spectrum, pseudos pectrum and the numerical range for Ricci tensor ρ on the oscillator group (G, g_a) of dimension four.

Proposition 6 *The eigenvalues of matrix* ρ *defined in* (2) *are given by, if*

$$-1 < a \leq 0,$$

or

$$0 < a < 1 \text{ and } x_3 \in \left[\frac{-2a^2+2a-2-\sqrt{-16a(a^2-1)}}{2a^2}, \frac{-2a^2+2a-2-\sqrt{-16a(a^2-1)}}{2a^2}\right],$$

then

$$\sigma(\rho) = \left\{ \begin{array}{l} \lambda_1 = \frac{1}{4}a^2x_3^2 - \frac{1}{4}a - \frac{1}{4}Q + \frac{1}{4}a^2 + \frac{1}{4}, \\ \lambda_2 = \frac{1}{4}a^2x_3^2 - \frac{1}{4}a + \frac{1}{4}Q + \frac{1}{4}a^2 + \frac{1}{4}, \\ \lambda_3 = -\frac{1}{2},\ \lambda_4 = 0 \end{array} \right\},$$

with

$$Q = \sqrt{a^4x_3^4 + 2a^4x_3^2 + a^4 - 2a^3x_3^2 + 2a^3 + 2a^2x_3^2 + 3a^2 + 2a + 1}.$$

3.2.1. Pseudo spectrum of ρ

Proposition 7 *Since* ρ *is symmetrical therefore* ρ *is normal, therefore pseudo-spectrum noted by* $\Lambda_\varepsilon(\rho)$ *is given by:*

$$\Lambda_\varepsilon(\rho) = \{z \in \mathbb{C} : |z - \lambda_i| \leq \varepsilon,\ \lambda_i \in \sigma(\rho)\},\ \text{with } i \in \{1, \ldots, 4\}.$$

3.2.2. Numerical range of ρ

Theorem 4 *The numerical range of matrix* ρ *defined in* (2), *is given by:*

1.

$$W(\rho) \subseteq \left[-2|x_3| - 3,\ x_3^2 + 2|x_3| + 3\right],\ \text{if: } 0 \leq a < 1, \tag{9}$$

2.

$$W(\rho) \subseteq \left[2|x_3| - 1,\ x_3^2 + 2|x_3| + 5\right],\ \text{if: } -1 < a \leq 0, \tag{10}$$

Proof 3 *Let $z \in \mathbb{C}^4$ such that $z \neq 0$, we put $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_3 \end{pmatrix}$, with $z_i = \rho_i e^{i\theta_i}$, $i = \overline{1,4}$. So*

$$\frac{z^*\rho z}{z^*z} = \frac{S_1 + S_2}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2},$$

such as

$$S_1 = \frac{1}{2}\rho_1^2 a^2 + \frac{1}{2}\rho_2^2 a^2(ax_3^2 - 1) - \frac{1}{2}\rho_3^2 a + \frac{1}{2}\rho_4^2,$$

and

$$S_2 = \frac{1}{2}a^2 x_3(z_1\overline{z_2} + z_2\overline{z_1}) + \frac{1}{2}a(z_1\overline{z_4} + z_4\overline{z_1}) + \frac{1}{2}ax_3(z_4\overline{z_2} + z_2\overline{z_4}).$$

we have

$$-1 \leq \frac{z_i\overline{z_j} + z_j\overline{z_i}}{\sum_{i=1}^{4}|z_i|^2} = \frac{2\rho_i\rho_j\cos(\theta_i - \theta_j)}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2} \leq 1, \quad \forall i, j \in \{1,2,3,4\}, \tag{11}$$

Moreover we have

$$\frac{\frac{1}{2}\rho_i}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2} \leq 1, \quad \forall i = \overline{1,4}. \tag{12}$$

1. If $0 \leq a < 1$: So from(11), (12), if : $x_3 \geq 0$ then

$$-2x_3 - 3 \leq \frac{z^*\rho z}{z^*z} \leq x_3^2 + 2x_3 + 3,$$

if on the contrary $x_3 < 0$ then

$$2x_3 - 3 \leq \frac{z^*\rho z}{z^*z} \leq x_3^2 - 2x_3 + 3.$$

We proved (9).

2. If $-1 < a \leq 0$: So from (11), (12), if : $x_3 \geq 0$ we find

$$-2x_3 - 1 \leq \frac{z^*\rho z}{z^*z} \leq x_3^2 + 2x_3 + 5,$$

if on the contrary $x_3 < 0$ then

$$2x_3 - 1 \leq \frac{z^*\rho z}{z^*z} \leq x_3^2 - 2x_3 + 5.$$

We proved (10).

4. Numerical Examples

Example 4.1 *For $x_3 = 0$ and $a = \frac{1}{2}$, the matrix R_{12} is*

$$R_{12} = \begin{pmatrix} 0 & \frac{1}{8} & 0 & 0 \\ -\frac{1}{16} & 0 & 0 & -\frac{1}{8} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$\frac{z^*R_{12}z}{z^*z} = \frac{\frac{1}{8}(\frac{1}{2}\rho_1\rho_2\cos(\theta_2 - \theta_1) + i\frac{3}{2}\rho_1\rho_2\sin(\theta_1 - \theta_2) - \rho_2\rho_4\cos(\theta_4 - \theta_2) - i\rho_2\rho_4\sin(\theta_4 - \theta_2))}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2}.$$

The real part of $\frac{z^*R_{12}z}{z^*z}$ is

$$Re(\frac{z^*R_{12}z}{z^*z}) = \frac{\frac{1}{16}\rho_1\rho_2\cos(\theta_2 - \theta_1) - \frac{1}{8}\rho_4\rho_2\cos(\theta_4 - \theta_2)}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2}.$$

We have

$$\frac{-1}{10} \leq \frac{\frac{1}{16}\rho_1\rho_2\cos(\theta_2-\theta_1)-\frac{1}{8}\rho_4\rho_2\cos(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} \leq \frac{1}{10},$$

to prouve that we will suppose the contrast and we show that isn't true. It means,

$$\frac{\frac{1}{16}\rho_1\rho_2\cos(\theta_2-\theta_1)-\frac{1}{8}\rho_4\rho_2\cos(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} > \frac{1}{10},$$

so

$$\rho_3^2+\rho_4^2+\frac{11}{8}\rho_1\rho_2+(\rho_1\cos\theta_2-\rho_2\cos\theta_1)^2+(\rho_1\sin\theta_2-\rho_2\sin\theta_1)^2<0,$$

this is a contradiction. We show that $\frac{1}{10} \notin Re(W(R_{12}))$, *since if we have the opposite, i.e.* $\exists z \in \mathbb{C}^4$, *such as*

$$\frac{\frac{1}{16}\rho_1\rho_2\cos(\theta_2-\theta_1)-\frac{1}{8}\rho_4\rho_2\cos(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} = \frac{1}{10},$$

i.e.

$$\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2-\frac{5}{8}\rho_1\rho_2\cos(\theta_2-\theta_1)+\frac{5}{4}\rho_2\rho_4\cos(\theta_4-\theta_2)=0,$$

implies

$$\begin{gathered}(\tfrac{1}{2}\rho_1\cos\theta_1-\tfrac{5}{8}\rho_2\cos\theta_2)^2+(\tfrac{1}{2}\rho_1\sin\theta_1-\tfrac{5}{8}\rho_2\sin\theta_2)^2+(\tfrac{5}{\sqrt{39}}\rho_4\cos\theta_4+\tfrac{\sqrt{39}}{8}\rho_2\cos\theta_2)^2\\ +(\tfrac{5}{\sqrt{39}}\rho_4\sin\theta_4+\tfrac{\sqrt{39}}{8}\rho_2\sin\theta_2)^2+\tfrac{3}{4}\rho_1^2+\rho_3^2+\tfrac{14}{39}\rho_4^2=0.\end{gathered}$$

The only condition which verify the equation is just when $\rho_1=\rho_2=\rho_3=\rho_4=0$ *and that is not possible because* $z \neq 0$. *Then,* $\frac{1}{10} \notin Re(W(R_{12}))$. *Now we show that*

$$\frac{\frac{1}{16}\rho_1\rho_2\cos(\theta_2-\theta_1)-\frac{1}{8}\rho_4\rho_2\cos(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} \geq -\frac{1}{10},$$

for proving that we need to prove that the contract isn't true, it means,

$$\frac{\frac{1}{16}\rho_1\rho_2\cos(\theta_2-\theta_1)-\frac{1}{8}\rho_4\rho_2\cos(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} < -\frac{1}{10},$$

and we get

$$\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2+\frac{5}{8}\rho_1\rho_2\cos(\theta_2-\theta_1)-\frac{5}{4}\rho_2\rho_4\cos(\theta_4-\theta_2)<0,$$

then

$$\begin{gathered}(\tfrac{1}{2}\rho_1\cos\theta_1+\tfrac{5}{8}\rho_2\cos\theta_2)^2+(\tfrac{1}{2}\rho_1\sin\theta_1+\tfrac{5}{8}\rho_2\sin\theta_2)^2+(\tfrac{5}{\sqrt{39}}\rho_4\cos\theta_4-\tfrac{\sqrt{39}}{8}\rho_2\cos\theta_2)^2\\ +(\tfrac{5}{\sqrt{39}}\rho_4\sin\theta_4+\tfrac{\sqrt{39}}{8}\rho_2\sin\theta_2)^2+\tfrac{3}{4}\rho_1^2+\rho_3^2+\tfrac{14}{39}\rho_4^2<0,\end{gathered}$$

that is impossible. We prove that

$$\forall z \in \mathbb{C}^4,\ \ \frac{\frac{1}{16}\rho_1\rho_2\cos(\theta_2-\theta_1)-\frac{1}{8}\rho_4\rho_2\cos(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} \neq -\frac{1}{10},$$

by the absurd

$$\exists z \in \mathbb{C}^4,\ \ \frac{\frac{1}{16}\rho_1\rho_2\cos(\theta_2-\theta_1)-\frac{1}{8}\rho_4\rho_2\cos(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} = -\frac{1}{10},$$

The only condition which verify the equation is just when $\rho_1=\rho_2=\rho_3=\rho_4=0$ *and that is not possible because* $z \neq 0$. *Then,* $-\frac{1}{10} \notin Re(W(R_{12}))$. *So we conclude, for* $x_3=0$ *and* $a=\frac{1}{2}$, *on obtains*

$$Re(\frac{z^*R_{12}z}{z^*z}) \in]-\frac{1}{10},\frac{1}{10}[.$$

Now for imaginary part of $\frac{z^*R_{12}z}{z^*z}$,

$$Im(\frac{z^*R_{12}z}{z^*z}) = \frac{3}{16}\rho_1\rho_2\sin(\theta_1-\theta_2) - \frac{1}{8}\rho_2\rho_4\sin(\theta_4-\theta_2),$$

We have

$$\frac{\frac{3}{16}\rho_1\rho_2\sin(\theta_1-\theta_2) - \frac{1}{8}\rho_2\rho_4\sin(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} \leq \frac{3}{20},$$

for proving that we need to prove that the contract isn't true, it means,

$$\frac{\frac{3}{16}\rho_1\rho_2\sin(\theta_1-\theta_2) - \frac{1}{8}\rho_2\rho_4\sin(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} > \frac{3}{20},$$

i.e.

$$(\tfrac{5}{\sqrt{39}}\rho_1\cos\theta_1 + \tfrac{\sqrt{39}}{8}\rho_2\sin\theta_2)^2 + (\tfrac{5}{\sqrt{39}}\rho_1\sin\theta_1 - \tfrac{\sqrt{39}}{8}\rho_2\cos\theta_2)^2 + (\tfrac{5}{8}\rho_2\cos\theta_2 + \tfrac{2}{3}\rho_4\sin\theta_4)^2 \\ + (\tfrac{5}{8}\rho_2\sin\theta_2 - \tfrac{2}{3}\rho_4\cos\theta_4)^2 + \tfrac{14}{39}\rho_1^2 + \rho_3^2 + \tfrac{5}{9}\rho_4^2 < 0,$$

so this is a contradiction. We show that

$$\frac{\frac{3}{16}\rho_1\rho_2\sin(\theta_1-\theta_2) - \frac{1}{8}\rho_2\rho_4\sin(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} \neq \frac{3}{20},$$

because if we have the opposite

$$\frac{\frac{3}{16}\rho_1\rho_2\sin(\theta_1-\theta_2) - \frac{1}{8}\rho_2\rho_4\sin(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} = \frac{3}{20},$$

then

$$(\tfrac{5}{\sqrt{39}}\rho_1\cos\theta_1 + \tfrac{\sqrt{39}}{8}\rho_2\sin\theta_2)^2 + (\tfrac{5}{\sqrt{39}}\rho_1\sin\theta_1 - \tfrac{\sqrt{39}}{8}\rho_2\cos\theta_2)^2 + (\tfrac{5}{8}\rho_2\cos\theta_2 + \tfrac{2}{3}\rho_4\sin\theta_4)^2 \\ + (\tfrac{5}{8}\rho_2\sin\theta_2 - \tfrac{2}{3}\rho_4\cos\theta_4)^2 + \tfrac{14}{39}\rho_1^2 + \rho_3^2 + \tfrac{5}{9}\rho_4^2 = 0.$$

So this is a contradiction because $z \neq 0$. *So* $\frac{3}{20} \notin Im(W(R_{12}))$. *We have*

$$\frac{\frac{1}{16}\rho_1\rho_2\sin(\theta_2-\theta_1) - \frac{1}{8}\rho_4\rho_2\sin(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} \geq -\frac{3}{20},$$

because, if we have the opposite

$$\exists z \in \mathbb{C}^4,\ \ \frac{\frac{3}{16}\rho_1\rho_2\sin(\theta_1-\theta_2) - \frac{1}{8}\rho_2\rho_4\sin(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} < -\frac{3}{20},$$

we obtain

$$\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2 + \frac{5}{8}\rho_1\rho_2\cos(\theta_2-\theta_1) - \frac{5}{4}\rho_2\rho_4\cos(\theta_4-\theta_2) < 0,$$

imply

$$(\tfrac{1}{2}\rho_1\cos\theta_1 + \tfrac{5}{8}\rho_2\cos\theta_2)^2 + (\tfrac{1}{2}\rho_1\sin\theta_1 + \tfrac{5}{8}\rho_2\sin\theta_2)^2 + (\tfrac{5}{\sqrt{39}}\rho_4\cos\theta_4 - \tfrac{\sqrt{39}}{8}\rho_2\cos\theta_2)^2 \\ +(\tfrac{5}{\sqrt{39}}\rho_4\sin\theta_4 + \tfrac{\sqrt{39}}{8}\rho_2\sin\theta_2)^2 + \tfrac{3}{4}\rho_1^2 + \rho_3^2 + \tfrac{14}{39}\rho_4^2 < 0,$$

this is a contradiction. We prove that

$$\forall z \in \mathbb{C}^4,\ \ \frac{\frac{3}{16}\rho_1\rho_2\sin(\theta_1-\theta_2) - \frac{1}{8}\rho_2\rho_4\sin(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} \neq -\frac{3}{20},$$

by the absurd

$$\exists z \in \mathbb{C}^4,\ \ \frac{\frac{3}{16}\rho_1\rho_2\sin(\theta_1-\theta_2) - \frac{1}{8}\rho_2\rho_4\sin(\theta_4-\theta_2)}{\rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2} = -\frac{3}{20},$$

we find $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$, *and that is not possible because* $z \neq 0$. *Then,* $-\frac{3}{20} \notin Im(W(R_{12}))$. *So we conclude, for* $x_3 = 0$ *and* $a = \frac{1}{2}$, *we obtain*

$$Im(\frac{z^* R_{12} z}{z^* z}) \in]-\frac{3}{20}, \frac{3}{20}[.$$

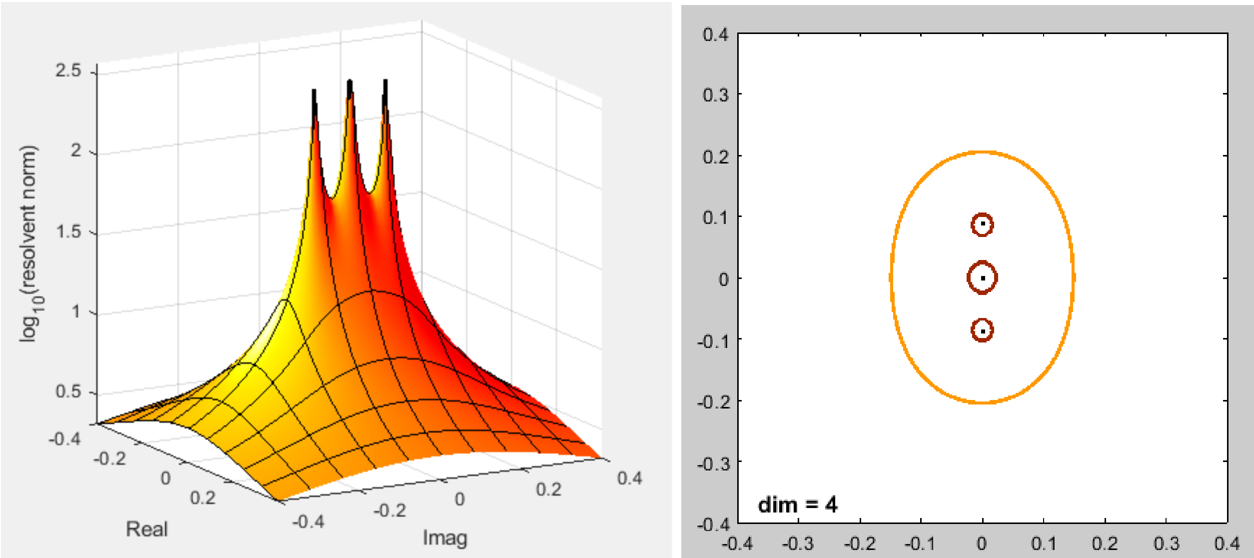

Figure 1: *Spectrum and Pseudospectrum of the curvature tensor* R_{12} *for* $x_3 = 0$ *and* $a = \frac{1}{2}$ *on the oscillator group of dimension four.*

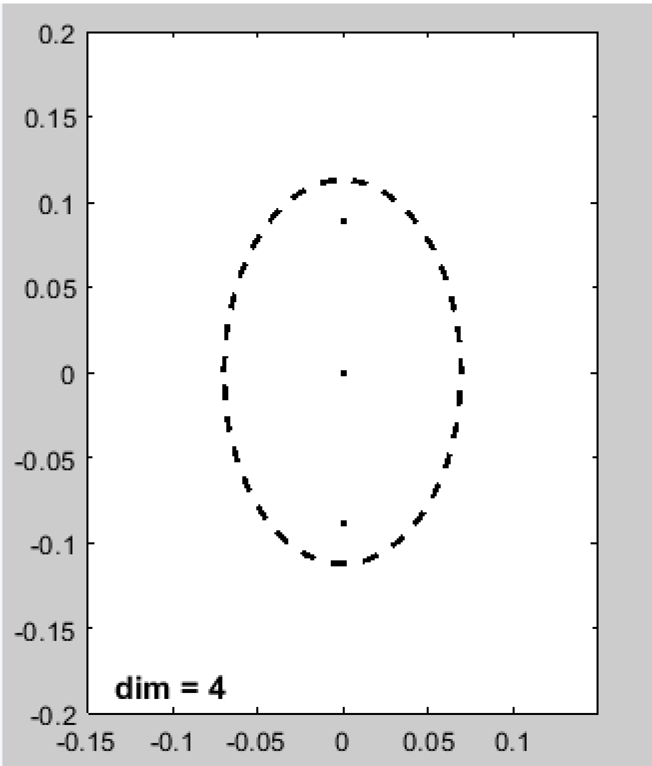

Figure 2: *Numerical range of the curvature tensor* R_{12} *for* $x_3 = 0$ *and* $a = \frac{1}{2}$ *on the oscillator group of dimension four.*

Example 4.2 *For $x_3 = 0$ and $a = \frac{1}{2}$, we find*

$$W(\rho_0^{0.5}) = \left[\frac{-1}{4}, \frac{5}{8}\right],$$

indeed, we have

$$W(\rho_0^{0.5}) = \begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{1}{4} \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Let $\lambda \in W(\rho_0^{0.5})$, so $\exists z \in \mathbb{C}^4$such as

$$\lambda = \frac{\frac{1}{8}\rho_1^2 - \frac{1}{4}\rho_2^2 - \frac{1}{4}\rho_3^2 + \frac{1}{2}\rho_4^2 + \frac{1}{4}(z_1\overline{z_4} + z_4\overline{z_1})}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2}.$$

On the other hand we have

$$\lambda = \frac{5}{8} + \frac{-\frac{1}{2}\rho_1^2 - \frac{1}{8}\rho_4^2 + \frac{1}{2}\rho_1\rho_4\cos(\theta_1 - \theta_4) - \frac{7}{8}\rho_2^2 - \frac{7}{8}\rho_3^2}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2}.$$

We pose

$$\begin{aligned} \beta &= -\frac{1}{2}\rho_1^2 - \frac{1}{8}\rho_4^2 + \frac{1}{2}\rho_1\rho_4\cos(\theta_1 - \theta_4) \\ &= -((\frac{1}{\sqrt{2}}\rho_1\cos\theta_1 - \frac{1}{2\sqrt{2}}\rho_4\cos\theta_4)^2 + (\frac{1}{\sqrt{2}}\rho_1\sin\theta_1 - \frac{1}{2\sqrt{2}}\rho_4\sin\theta_4)^2) \leq 0. \end{aligned}$$

So

$$\lambda = \frac{5}{8} + \frac{\beta - \frac{7}{8}\rho_2^2 - \frac{7}{8}\rho_3^2}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2} \leq \frac{5}{8},$$

moreover we have, if $\rho_2 = \rho_3 = 0$, $\rho_4 = 2\rho_1$ and $\theta_4 = \theta_1 + 2k\pi$, $k \in \mathbb{Z}$ we find $\lambda = \frac{5}{8}$.Then $\lambda \in W(\rho_0^{0.5})$. Since

$$\lambda = -\frac{1}{4} + \frac{\delta + \frac{1}{4}\rho_1^2 + \frac{1}{4}\rho_4^2}{\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2},$$

with

$$\begin{aligned} \delta &= \frac{1}{8}\rho_1^2 + \frac{1}{2}\rho_4^2 + \frac{1}{2}\rho_1\rho_4\cos(\theta_1 - \theta_4) \\ &= (\frac{1}{2\sqrt{2}}\rho_1\cos\theta_1 - \frac{1}{\sqrt{2}}\rho_4\cos\theta_4)^2 + (\frac{1}{2\sqrt{2}}\rho_1\sin\theta_1 - \frac{1}{\sqrt{2}}\rho_4\sin\theta_4)^2 \geq 0, \end{aligned}$$

then

$$\lambda \geq -\frac{1}{4}.$$

On the other hand we have, if $\rho_1 = \rho_4 = 0$, and $\rho_2, \rho_3 \in \mathbb{R}_+^*$. We find $\lambda = -\frac{1}{4}$. So $-\frac{1}{4} \in W(\rho_0^{0.5})$.

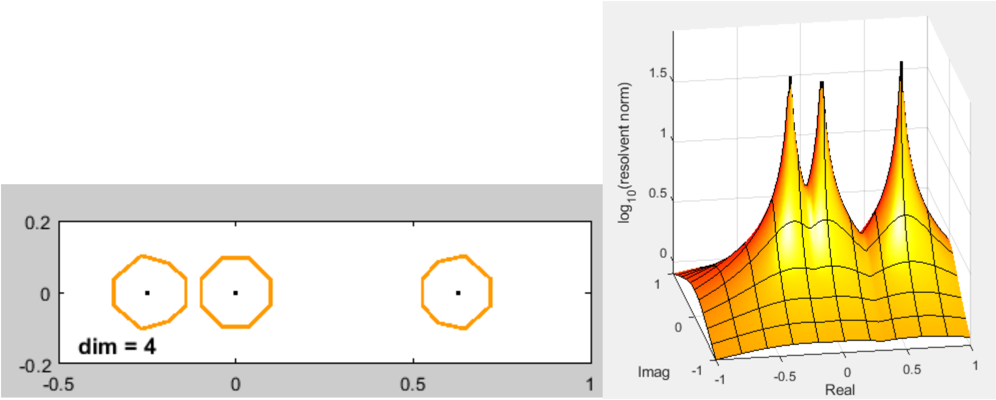

Figure 3: *Spectrum and Pseudospectrum of the Ricci tensor* $\rho_0^{0.5}$ *for* $x_3 = 0$ *and* $a = \frac{1}{2}$ *on the oscillator group of dimension four.*

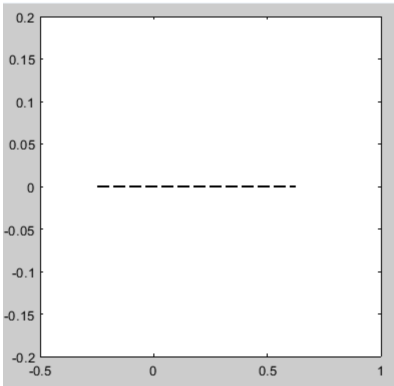

Figure 4: *Numerical range of the Ricci tensor* $\rho_0^{0.5}$ *for* $x_3 = 0$ *and* $a = \frac{1}{2}$ *on the oscillator group of dimension four.*

5. Conclusion

The numerical range of an operator is a fairly recent concept and has been discussed a lot to respond to practical problems in particular. In this work, we presented some basic definitions and properties of the numerical range of a matrix since it is of great importance in the field of mathematics and physics. We found in this article the numerical range of curvature tensor $R_{ij} = R(\partial_i, \partial_j)$ and Ricci tensor ρ on the oscillator group (G, g_a) of dimension four. We illustrated this with two examples, the first for $R_{ij} = R(\partial_i, \partial_j)$ and the second for ρ, and we used programming on Matlab. There is still a lot to learn about the behavior of the numerical range in practice, and no doubt that future problems on the subject will arise for us in different fields.

Declaration of Competing Interest

The author(s), declares that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Bendehiba Menad: Data creation, Draft preparation, Writing, Reviewing.
Rafik Derkaoui: Methodology, Writing, Editing, Investigation.

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