Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (2) (2024), 342 – 355
DOI : 10.15672/hujms.1186239

Research Article

Homological objects of min-pure exact sequences

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Abstract

In a recent paper, Mao has studied min-pure injective modules to investigate the existence of min-injective covers. A min-pure injective module is one that is injective relative only to min-pure exact sequences. In this paper, we study the notion of min-pure projective modules which is the projective objects of min-pure exact sequences. Various ring characterizations and examples of both classes of modules are obtained. Along this way, we give conditions which guarantee that each min-pure projective module is either injective or projective. Also, the rings whose injective objects are min-pure projective are considered. The commutative rings over which all injective modules are min-pure projective are exactly quasi-Frobenius. Finally, we are interested with the rings all of its modules are min-pure projective. We obtain that a ring R is two-sided Köthe if all right R-modules are min-pure projective. Also, a commutative ring over which all modules are min-pure projective is quasi-Frobenius serial. As consequence, over a commutative indecomposable ring with $J(R)^2 = 0$, it is proven that all R-modules are min-pure projective if and only if R is either a field or a quasi-Frobenius ring of composition length 2.

Mathematics Subject Classification (2020). 16E10, 16E30, 16D40, 13C10, 16D50, 13C11

Keywords. (min-)purity, Köthe rings, universally mininjective rings, quasi-Frobenius rings

1. Introduction

Throughout, R will stand for associative ring with identity, and R-modules will be unitary modules unless otherwise specified. A_R ($_RA$) stands for any module A considered as a right (left) R-module.

As a generalization of injectivity, the concept of min-injectivity is introduced by Harada (see [22]). $_RA$ is called *min-injective* provided that $Ext^1_R(R/S, A) = 0$ for any minimal left ideal S. A_R is called *min-flat* provided that $\mathrm{Tor}^R_1(A, R/S) = 0$ for any minimal left ideal S (see [29]). By the natural equivalence $\mathrm{Ext}^1_R(R/S, A^+) \cong \mathrm{Tor}^R_1(A, R/S)^+$ for any

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Received: 05.11.2022; Accepted: 25.04.2023

minimal left ideal S, we can conclude a right module A is min-flat provided that A^+ is min-injective.

Min-injective rings and min-injective modules are the most important and most studied subjects of homological algebra along with module and ring theory. The main reason for this is that min-injective rings are naturally occurring in characterizing quasi-Frobenius (QF) rings. The importance of finite (quasi-Frobenius) rings has increased with the study of the rings of algebraic coding theory (see [21, 24, 38]).

In [28], Mao introduced the concept of min-purity and min-pure injectivity, to give further homologic characterizations of min-injective modules and to investigate the existence of min-injective covers. In the literature, purity has a considerable impact on module-ring theory, and several crucial generalizations of this notion are given since it was firstly introduced (see, [1, 5, 6, 8, 10, 31, 35, 36]). In accordance with the terminology of Mao [28], a sequence $0 \to D \to E \to F \to 0$ of right R-modules is called *min-pure exact* if $\mathrm{Hom}(R/aR, E) \to \mathrm{Hom}(R/aR, F) \to 0$ is epic for any $a \in R$ such that Ra is simple. A_R is called *min-pure injective* provided that A has injective property relative to all min-pure exact sequences. So far, min-purity, min-injectivity, min-pure injectivity and their homological objects are studied by many authors (see [22, 28, 29, 33]).

Motivated by min-pure injective modules, in this article, we first introduce the homological objects which are flat and projective relative to the min-pure exact sequences. We shall call A_R is *min-pure projective* provided that A_R is projective relative to min-pure exact sequences. Also, ${}_RA$ is called *min-pure flat* if ${}_RA$ is flat relative to min-pure exact sequences. Naturally, flat left modules are min-pure flat, and projective right modules are min-pure projective, but not conversely (see Example 2.2(2)).

In section 2, we give some preliminary properties of min-pure projective and min-pure flat modules. After giving various equivalent conditions of min-purity, absolutely min-purity of modules are described via min-purity. Moreover, it is shown that R is left universally mininjective if and only if all min-pure projective (resp. flat) right R-modules are projective (resp. flat). Also, we show that R is left FS if and only if injective dimensions of min-pure injective right R-modules ≤ 1 if and only if flat dimension of min-pure flat left R-modules ≤ 1. Finally, projective dimensions of min-pure projective right R-modules ≤ 1 equivalent to that for any $a \in R$ such that Ra is simple, aR is projective.

In section 3, we consider the covering and enveloping properties of min-pure injective and min-pure projective modules. We show that all min-pure injective modules have an injective cover, and if R is left min-coherent, then all min-pure projective right R-modules have a projective preenvelope. Also, we get that all right modules have a min-pure projective precover and min-pure injective envelope.

In section 4, we focused on the rings whose all injective modules are min-pure projective. Along the way, being R is quasi-Frobenius equivalent to that R is right CF and every injective right R-module is min-pure projective. For a commutative ring R we prove that all injective R-modules are min-pure projective if and only if R is quasi-Frobenius. Moreover, it is shown that R is semisimple if and only if every min-pure projective (resp. injective) right R-module is injective (resp. projective). Finally, we focused on the rings whose all R-modules are min-pure projective (resp. injective). For this purpose, we prove that R is a two-sided Köthe ring provided that every right R-module is min-pure projective (resp. injective). Consequently, for a commutative indecomposable ring with $J(R)^2 = 0$, it is shown that R is either a field or a quasi-Frobenius ring of composition length 2 if and only if all R-modules are min-pure projective.

For future research, we close the paper by giving some questions that are partially answered inside the paper.

2. Min-pure projective and min-pure flat modules

A sequence $0 \to D \to E \to F \to 0$ of right R-modules is called *min-pure exact* if for any $a \in R$ such that Ra is simple, $0 \to D \otimes (R/Ra) \to E \otimes (R/Ra) \to F \otimes (R/Ra) \to 0$ is exact. Moreover, A_R is called *min-pure injective* provided that A_R is injective relative to every min-pure sequence (see [28]). Motivated by min-pure injective modules, we introduce the homological objects which are projective and flat with respect to the min-pure sequences.

Definition 2.1. **(a)** A_R is called *min-pure projective* if for all min-pure sequences $0 \to D \to E \to F \to 0$ of right R-modules, the induced map $\beta : \mathrm{Hom}(A, E) \to \mathrm{Hom}(A, F)$ is an epimorphism.

(b) ${}_RA$ is called *min-pure flat* if for all min-pure sequences $0 \to D \to E \to F \to 0$ of right R-modules, the induced map $\alpha : D \otimes A \to E \otimes A$ is a monomorphism.

Example 2.2. (1) For any $a \in R$ such that Ra is simple, R/aR is min-pure projective and R/Ra is min-pure flat.

(2) Any projective right module is min-pure projective and any flat left module is min-pure flat. However, in general case, the converses need not be true. Consider the ring $R := \mathbb{Z}/p^2\mathbb{Z}$ for some prime integer p. R/pR is a min-pure flat and min-pure projective R-module since pR is simple ideal. Whereas the module R/pR is not flat, otherwise R/pR would be projective by [29, Corollary 3.3]. But the epimorphism $R \to R/pR \to 0$ does not split.

By the following theorem, further equivalent conditions of min-pure flatness are given.

Theorem 2.3. *Let $\mathscr{F} = \{R/Ra \mid$ for any $a \in R$ such that Ra is simple$\}$. The following are equivalent for ${}_RA$:*

(1) *A is min-pure flat;*
(2) *A^+ is min-pure injective;*
(3) *$A \cong E/D$ where E is in Add($\mathscr{F} \cup \{{}_RR\}$) and D is pure in E;*
(4) *A can be written as a direct limit of finite direct sums of modules from $\mathscr{F} \cup \{{}_RR\}$.*
Also, when R is commutative, above statements are equivalent to:
(5) *$\mathrm{Hom}(A, D)$ is min-pure injective, for any injective R-module D;*
(6) *$A \otimes C$ is min-pure flat, for any flat R-module C.*

Proof. (1) $\Rightarrow$ (2). Let ${}_RA$ be min-pure flat and $0 \to D \to E \to F \to 0$ a min-pure sequence of right R-modules. So, $0 \to D \otimes A \to E \otimes A$ is monic, whence $(E \otimes A)^+ \to (D \otimes A)^+ \to 0$ is epic. This implies that $\mathrm{Hom}(E, A^+) \to \mathrm{Hom}(D, A^+) \to 0$ is also epic and so A^+ is min-pure injective.

(2) $\Rightarrow$ (3). Assume that A^+ is min-pure injective. By [30, Proposition 1.2], there exist an $\mathscr{F}$-pure sequence $0 \to D \to E \to A \to 0$ where E is in Add($\mathscr{F} \cup \{{}_RR\}$). Also, by the isomorphism used in (2) $\Leftrightarrow$ (7) from the Lemma 2.4, the sequence $0 \to A^+ \to E^+ \to D^+ \to 0$ would be min-pure. Since A^+ is min-pure injective, $0 \to A^+ \to E^+ \to D^+ \to 0$ splits and so the sequence $0 \to D \to E \to A \to 0$ is pure.

(3) $\Rightarrow$ (4). Easily follows by [37, Theorem 34.2].

(4) $\Rightarrow$ (1). Let $0 \to D \to E \to F \to 0$ be a min-pure sequence of right R-modules and $\{F_\alpha\}_{\alpha \in \Lambda}$ is a finite family of right R-modules such that for each $\alpha \in \Lambda$, $A = \varinjlim F_\alpha$, where F_α's is a finite direct sums of modules from $\mathscr{F} \cup \{{}_RR\}$. Since F_α is min-pure flat for each $\alpha \in \Lambda$, $0 \to D \otimes F_\alpha \to E \otimes F_\alpha \to F \otimes F_\alpha \to 0$ is exact. So by [37, Theorem 24.11], the sequence $0 \to D \otimes \varinjlim F_\alpha \to E \otimes \varinjlim F_\alpha \to F \otimes \varinjlim F_\alpha \to 0$ is exact. Therefore, A is min-pure flat.

(1) $\Rightarrow$ (5). Let D be an injective R-module. If we consider the splitting map $0 \to D \to \prod R^+$, we would have the map $0 \to \mathrm{Hom}(A, D) \to \mathrm{Hom}(A, \prod R^+)$ which is also

splits. Being A^+ is min-pure injective by (1) together with the isomorphisms $\prod A^+ \cong \mathrm{Hom}(A, \prod R^+)$ implies that $\prod A^+$ is min-pure injective. This gives the min-pure injectivity of $\mathrm{Hom}(A, D)$.

(5) $\Rightarrow$ (6). Assume that C is any flat R-module. Since $(A \otimes C)^+$ is isomorphic to $\mathrm{Hom}(A, C^+)$, it is min-pure injective by (5) and by the injectivity of C^+. This gives the min-pure flatness of $A \otimes C$.

(6) $\Rightarrow$ (1) straightforward by putting $C = R$. □

Now, we are ready to give further characterizations of min-purity.

Lemma 2.4. *Let $0 \to D \to E \to F \to 0$ be a sequence of right R-modules. The following are equivalent:*

(1) $0 \to D \to E \to F \to 0$ *is min-pure;*
(2) $\mathrm{Hom}(R/aR, E) \to \mathrm{Hom}(R/aR, F) \to 0$ *is epic for any $a \in R$ such that Ra is simple;*
(3) $\mathrm{Hom}(A, E) \to \mathrm{Hom}(A, F) \to 0$ *is epic for any min-pure projective R-module A_R;*
(4) $\mathrm{Hom}(E, A) \to \mathrm{Hom}(D, A) \to 0$ *is epic for any min-pure injective R-module A_R;*
(5) $\mathrm{Hom}(R/Ra, E^+) \to \mathrm{Hom}(R/Ra, D^+) \to 0$ *is epic for $a \in R$ such that Ra is simple;*
(6) $0 \to D \otimes B \to E \otimes B$ *is monic for any min-pure flat R-module ${}_RB$;*
(7) $0 \to R/aR \otimes F^+ \to R/aR \otimes E^+$ *is monic for any $a \in R$ such that Ra is simple. Also, if R is commutative or two sided mininjective, then the above are equivalent to:*
(8) $\mathrm{Hom}(R/aR, E) \to \mathrm{Hom}(R/aR, F) \to 0$ *is epic for any $a \in R$ such that aR is simple.*

Proof. (1) $\Leftrightarrow$ (2) follows by [28, Lemma 2.1].

(1) $\Leftrightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (6) are obvious.

(4) $\Rightarrow$ (1). Let $a \in R$ such that Ra is simple. Min-pure flatness of R/Ra implies the min-pure injectivity of $(R/Ra)^+$ by Theorem 2.3. Thus by (4), the induced sequence $0 \to \mathrm{Hom}(F, (R/Ra)^+) \to \mathrm{Hom}(E, (R/Ra)^+) \to \mathrm{Hom}(D, (R/Ra)^+) \to 0$ can be obtained, and that gives the sequence $0 \to (F \otimes R/Ra)^+ \to (E \otimes R/Ra)^+ \to (D \otimes R/Ra)^+ \to 0$. So (1) follows by the exactness of $0 \to D \otimes R/Ra \to E \otimes R/Ra \to F \otimes R/Ra \to 0$.

(1) $\Leftrightarrow$ (5). Let $a \in R$ such that Ra is simple. Then the right exactness of $0 \to D \otimes (R/Ra) \to E \otimes (R/Ra) \to F \otimes (R/Ra) \to 0$ is equivalent to the left exactness of $0 \to (F \otimes (R/Ra))^+ \to (E \otimes (R/Ra))^+ \to (D \otimes (R/Ra))^+ \to 0$, equivalently $0 \to \mathrm{Hom}(R/Ra, F^+) \to \mathrm{Hom}(R/Ra, E^+) \to \mathrm{Hom}(R/Ra, D^+) \to 0$ is exact. Now, (1) $\Leftrightarrow$ (5) is obvious.

(6) $\Rightarrow$ (1) is obvious since every R/S is min-pure flat for any simple left ideal S.

(2) $\Leftrightarrow$ (7). Let $a \in R$ such that Ra is simple. Take into consideration the next diagram:

$$\begin{array}{cccccccc}
0 \to & R/aR \otimes F^+ & \to & R/aR \otimes E^+ & \to & R/aR \otimes D^+ & \to 0 \\
 & \mu\downarrow & & \delta\downarrow & & \lambda\downarrow & \\
0 \to & \mathrm{Hom}(R/aR, F)^+ & \to & \mathrm{Hom}(R/aR, E)^+ & \to & \mathrm{Hom}(R/aR, D)^+ & \to 0
\end{array}$$

By [12, Lemma 2], μ, δ and λ are isomorphisms. Thus exactness of the first row is equivalent to the exactness of the second row, and equivalently the map $\mathrm{Hom}(R/aR, E) \to \mathrm{Hom}(R/aR, F) \to 0$ is epic.

(2) $\Leftrightarrow$ (8). If R is commutative, it is easy.

Let R be left-right mininjective and $a \in R$. Then being aR is a minimal right ideal equivalent to that Ra is a minimal left ideal by [33, Theorem 1.14]. So in either cases (2) $\Leftrightarrow$ (8) follows. □

Remark 2.5. (1). Obviously purity implies the min-purity, but not conversely. Indeed, by [13, Example 3.1(ii)], there is an R-algebra S over a local Artinian ring R, such that the inclusion homomorphism $R \hookrightarrow S$ is cyclically pure, and so is min-pure. But $R \hookrightarrow S$ is not pure.

(2). By (1), every min-pure injectivity (resp. min-pure projectivity) of modules implies pure-injectivity (resp. pure-projectivity), but not conversely. Every Artinian R-module is well known as pure-injective. Hence the artinian ring R in [13, Example 3.1(ii)] is pure-injective. But it is not min-pure injective, otherwise the inclusion map $R \hookrightarrow S$ above splits.

(3). By (2) and the following corollary, we ensure that the existence of pure-projective module which is not min-pure projective.

A ring R is a *valuation ring* (commutative but not necessarily a domain) provided that all ideals of R are totally ordered by inclusion.

Corollary 2.6. *The next statements are equal for a ring R:*

(1) *All left modules are min-pure flat;*
(2) *All pure-projective right modules are min-pure projective;*
(3) *All pure-injective right modules are min-pure injective;*
(4) *All min-pure exact sequences of right modules are pure.*
Moreover, if R is commutative, $R_{\mathfrak{p}}$ is a valuation ring for every prime ideal $\mathfrak{p}$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Rightarrow$ (3) are easy.

(3) $\Rightarrow$ (1). For any left module A, pure-injectivity of A^+ implies its min-pure injectivity by (3). Thus by Theorem 2.3, we conclude that A is min-pure flat.

Since cyclically pure exact sequences are min-pure, the last statement follows by [13, Theorem 2.7]. □

Let $\mathscr{C}$ denotes the set $\mathscr{C} = \{R/aR \mid \text{for any } a \in R \text{ such that } Ra \text{ is simple}\}$. Note that min-pure= $\mathscr{C}$-pure= $\mathscr{C} \cup \{R_R\}$-pure. The following due to Warfield Jr.(see [36, Proposition 1, p.700]).

Lemma 2.7. *([30, Proposition 1.2]) For a module A_R we have:*

(1) *There exists a min-pure exact sequence $0 \to D \to E \to A \to 0$ where E is a direct sum of copies of modules in $\mathscr{C} \cup \{R_R\}$.*
(2) *The class of all min-pure projective right modules is exactly Add($\mathscr{C} \cup \{R_R\}$).*

We will call A_R is absolutely min-pure (similar to absolutely purity) provided that A is min-pure in every extension of it.

Proposition 2.8. *The next statements are equal for an R-module A_R:*

(1) *A_R is absolutely min-pure;*
(2) *All exact sequences starting with A are min-pure;*
(3) *$Ext^1(D, A) = 0$ for any min-pure projective R-module D_R;*
(4) *$Ext^1(R/aR, A) = 0$ for any $a \in R$ such that Ra is simple;*
(5) *There exists a min-pure sequence $0 \to A \to E \to C \to 0$ with E injective;*
(6) *For all min-pure injective R-modules D_R, all homomorphisms from A to D factors through an injective R-module.*
Also, if R is commutative, then the above conditions are equivalent to:
(7) *A is min-injective.*

Proof. (1) $\Leftrightarrow$ (2) is easy by definition.

(2) $\Rightarrow$ (5) is obvious, since we can embed A in an injective right R-module.

(5) $\Rightarrow$ (6). Let $f : A \to B$ be a homomorphism for any min-pure injective R-module B_R. Being $0 \to A \xrightarrow{i} E$ is min-pure, gives the existence of a map $g : E \to B$ such that $gi = f$, and this proves (6).

(6) ⇒ (2). Let $g : A \to D$ be any homomorphism with D min-pure injective and $\xi : 0 \to A \xrightarrow{i} K \to L \to 0$ be an exact sequence. So, there are a map $h : A \to E$ with E injective and a map $f : E \to D$ such that $fh = g$ by (6). By injectivity of E, there is a map $\alpha : K \to E$ such that $\alpha i = h$. So $g = f\alpha i$, whence ξ is min-pure by Lemma 2.4.

(1) ⇒ (3) and (3) ⇒ (4) follows from Lemma 2.7.

(4) ⇒ (6). We always have a sequence $\varepsilon : 0 \to A \xrightarrow{i} K \to L \to 0$ with K injective. Since by (4), $\mathrm{Ext}^1(R/aR, A) = 0$ for any $a \in R$ such that Ra is simple, $\mathrm{Hom}(R/aR, K) \to \mathrm{Hom}(R/aR, L) \to 0$ is epic. Thus ε is min-pure by Lemma 2.4, and so every homomorphism $A \to B$ with B min-pure injective factors through E.

(2) ⇔ (7) follows by [28, Proposition 2.3]. □

Recall by Puninski et al. [34] that, R is an *RD-ring* provided that purity and *RD*-purity coincides (this property is right-left symmetric). A serial ring and a regular ring are always *RD* (see [11, Theorem I.4] and [34, Remark 2.7]). By Puninski et al. [34, Proposition 4.5], a commutative *RD* ring is exactly an arithmetic ring, i.e., the rings with a distributive lattice of ideals.

Proposition 2.9. *The next statements hold for a ring R:*

(1) *If all min-pure sequences are pure, then R is an RD-ring.*
(2) *If R is commutative and all min-pure sequences are pure, then R is arithmetic and all min-injective R-modules are Absolutely pure.*
(3) *If R is commutative Noetherian ring such that all min-pure sequences are pure, then R is quasi-Frobenius arithmetic.*

Proof. (1). If we assume that every min-pure exact sequence is pure, then every *RD*-exact sequence is pure, whence R is an *RD*-ring.

(2). By (1) and [34, Proposition 4.5], R is arithmetic. Also, if A is min-injective, then $0 \to A \hookrightarrow E(A) \to E(A)/A \to 0$ is min-pure by Proposition 2.8. So, it is pure exact and this implies that A is Absolutely pure.

(3). Being arithmetic comes from (2). Again by (2) and Noetherianity of R, all min-injective R-modules are injective, whence R is Artinian by [2, Theorem 1] and the fact that simple injectives are min-injective. Thus R_R is pure-injective, whence R_R is min-pure injective by hypothesis. Hence by Theorem 4.6, R is quasi-Frobenius. □

Relationship between min-pure injective (resp. min-pure projective, min-pure flat) modules and injective (resp. projective, flat) modules is given below.

Corollary 2.10. *The next conditions are true for any ring R:*

(1) *Any min-pure injective absolutely min-pure right R-module is injective.*
(2) *Any min-flat min-pure projective right R-module is projective.*
Moreover, if R is commutative, then
(3) *Any min-pure injective min-injective R-module is injective.*
(4) *Any min-pure flat min-flat R-module is flat.*

Proof. (1). For any min-pure injective absolutely min-pure right R-module A, By Proposition 2.8, there is a min-pure sequence $0 \to A \to E \to B \to 0$ with E injective. Splitting of this sequence gives us the injectivity of A.

(2). For any min-pure projective min-flat right R-module A, we always have $0 \to K \to F \to A \to 0$ where F projective. Since A is min-flat, this exact sequence is min-pure. Splitting of this sequence gives us the projectivity of A.

(3) follows by Proposition 2.8.

(4). For any min-flat min-pure flat module A, A^+ is min-pure injective and min-injective by Theorem 2.3. This gives the injectivity of A^+ by (2), whence is flatness of A. □

Recall by [33] that, R is *left universally mininjective* ring if all left R-modules are min-injective, equivalently R is left min-injective and left PS. Now, we obtain new equivalent conditions of left universally mininjective rings via min-purity.

Proposition 2.11. *The next statements are equal for a ring R:*

(1) *R is left universally mininjective;*
(2) *Every exact sequences of right R-modules is min-pure;*
(3) *Every right R-module is absolutely min-pure;*
(4) *Every min-pure injective right R-module is injective;*
(5) *Every min-pure injective right R-module is absolutely min-pure;*
(6) *Every min-pure flat left module is flat;*
(7) *Every min-pure projective right R-module is projective.*

Proof. (1) $\Leftrightarrow$ (2) follows by [28, Theorem 4.3] and (3) $\Leftrightarrow$ (2) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (4). Hypothesis implies that any min-pure injective right R-module is a direct summand of an injective module, and so (2) follows.

(4) $\Rightarrow$ (6). For any min-pure flat left module A, A^+ is min-pure injective, whence is injective by (2). Therefore A would be flat.

(6) $\Rightarrow$ (1). Let M be a min-pure flat left R-module. We always have a sequence $\varepsilon : 0 \to D \to E \to A \to 0$ where E is projective. Flatness of M, gives the monic map $0 \to D \otimes M \to E \otimes M$, and so ε is min-pure by Lemma 2.4. Thus, any right R-module A is min-flat by [28, Proposition 2.2], whence R is left universally mininjective by [28, Theorem 4.3].

(1) $\Rightarrow$ (7). Since R is left universally mininjective, for any $a \in R$ such that Ra is simple, R/aR is min-flat by [28, Theorem 4.3], whence is projective by [29, Corollary 3.3]. If $\mathscr{C} = \{R/aR \mid \text{for any } a \in R \text{ such that } Ra \text{ is simple}\}$, any min-pure projective module contained in $\mathrm{Add}(\mathscr{C} \cup \{R_R\})$ by Lemma 2.7(2). Since any $R/aR \in \mathscr{C}$ is projective, (7) follows.

(7) $\Rightarrow$ (1). Since by (7), R/aR is projective for any minimal left ideal Ra, (1) follows by [29, Theorem 5.10]. □

The rings all of whose minimal left ideals are projective is called *left PS* [32]. Nonsingular rings, Semiprime rings and V-rings are left PS. A ring R is *left FS* [27], if every simple left ideal of R is flat.

Proposition 2.12. *The next statements are equal for a ring R:*

(1) *R is left FS;*
(2) *$Id(A) \leq 1$ for any min-pure injective module A_R;*
(3) *$Fd(A) \leq 1$ for any min-pure flat module ${}_RA$.*

Proof. (1) $\Rightarrow$ (2). By [28, Theorem 4.1], for any right R-module F, we have $0 \to D \to E \to F \to 0$ with E projective and D min-flat. This gives by [28, Proposition 2.2], for any min-pure injective right R-module A, $\mathrm{Ext}^2(F, A) \cong \mathrm{Ext}^1(D, A) = 0$. That is, $Id(A) \leq 1$.

(2) $\Rightarrow$ (3). For any min-pure flat R-module ${}_RA$, A^+ is min-pure injective by Theorem 2.3. By (2), for any R-module D_R, we have $\mathrm{Tor}_2(D, A)^+ \cong \mathrm{Ext}^2(D, A^+) = 0$. So, $\mathrm{Tor}_2(D, A) = 0$, and hence $fd(A) \leq 1$.

(3) $\Rightarrow$ (1). Since R/S is min-pure flat for any minimal left ideal S, flat dimension of R/S is ≤ 1. In this case S is flat and so R is left FS. □

Next we discuss the conditions related to min-pure projective modules which exactly characterizes left PS rings as follows.

Proposition 2.13. *The next statements are equal for a ring R:*

(1) *aR is projective for any $a \in R$ such that Ra is simple;*
(2) *$Pd(A) \leq 1$ for any min-pure projective module A_R;*

(3) *Absolutely min-pure left R-modules is closed under homomorphic images.*
Also, when R is commutative, above conditions are equal to:
(4) *R is PS.*

Proof. (1) $\Rightarrow$ (3). Let B be a submodule of an absolutely min-pure right R-module A. We shall show that A/B is absolutely min-pure. For any $a \in R$ such that Ra is simple, consider the induced exact sequence

$$\mathrm{Ext}^1(R/aR, A) \to \mathrm{Ext}^1(R/aR, A/B) \to \mathrm{Ext}^2(R/aR, B)$$

By Proposition 2.8, $\mathrm{Ext}^1(R/aR, A) = 0$. Consider $\mathrm{Ext}^2(R/aR, B) \cong \mathrm{Ext}^1(aR, B)$ the isomorphism. Projectivity of aR gives that $\mathrm{Ext}^2(R/aR, B) = 0$. Thus $\mathrm{Ext}^1(R/aR, A/B) = 0$, and so A/B is absolutely min-pure by Proposition 2.8.

(3) $\Rightarrow$ (2). Let A be a min-pure projective right R-module. For any right R-module C, we always have $0 \to C \to D \to E \to 0$ with D injective, that gives the exactness of $0 = \mathrm{Ext}^1(A, D) \to \mathrm{Ext}^1(A, E) \to \mathrm{Ext}^2(A, C) \to \mathrm{Ext}^2(A, D) = 0$. By (2), E is absolutely min-pure and so $\mathrm{Ext}^2(A, C) \cong \mathrm{Ext}^1(A, E) = 0$ by Proposition 2.8. This means that projective dimension of A is ≤ 1.

(2) $\Rightarrow$ (1). Since R/aR is min-pure projective for any $a \in R$ such that Ra is simple, projective dimension of R/aR is ≤ 1. In this case aR is projective.

(1) $\Leftrightarrow$ (4). If R is commutative, it is easy. □

3. Some (pre)envelopes and (pre)covers

Let $\mathfrak{Y}$ be a class of right modules.

For a module X_R, a module $Y \in \mathfrak{Y}$ is called a $\mathfrak{Y}$-envelope of X, if there is a homomorphism $f : X \to Y$ such that the next conditions hold:
(1) For any homomorphism $g : X \to Z$ with $Z \in \mathfrak{Y}$, there is a map $h : Y \to Z$ with $g = hf$.
(2) If an endomorphism $h : Y \to Y$ is such that $f = hf$, then f must be an automorphism.

If only (1) holds, we call $f : X \to Y$ a $\mathfrak{Y}$-preenvelope. Dually, it can be defined a $\mathfrak{Y}$-cover and $\mathfrak{Y}$-precover. In general $\mathfrak{Y}$-envelopes and $\mathfrak{Y}$-covers not always exsist, but they are unique (up to isomorphism) if they exist (see [15]).

Lemma 3.1. *Let R be a ring. Then:*

(1) *Extensions, pure submodules, pure quotients, direct sums and direct summands of absolutely min-pure right R-modules are absolutely min-pure.*
(2) *Finite direct sums, direct summands and direct products of min-pure injective right R-modules are min-pure injective.*
(3) *Direct sums and direct summands of min-pure projective right R-modules are min-pure projective.*
(4) *Direct sums, pure quotients and pure submodules of min-pure flat left R-modules are min-pure flat.*

Proof. (1). Using the properties of the Ext functor, closedness of absolutely min-purity under extensions is obvious by Proposition 2.8. Also, using the properties of the tensor functor, closedness under direct sums and direct summands is easy. Also closedness of absolutely min-pure modules under pure submodules is by Proposition 2.8. Now let C a pure submodule of an absolutely min-pure right module D. Then the exact sequence $0 \to (D/C)^+ \to D^+ \to C^+ \to 0$ splits. So, the isomorphism

$$Tor_1(R/aR, D^+) \cong Tor_1(R/aR, C^+) \oplus Tor_1(R/aR, (D/C)^+)$$

induces the isomorphism

$$Ext^1(R/aR, D)^+ \cong Ext^1(R/aR, C)^+ \oplus Ext^1(R/aR, D/C)^+$$

for any $a \in R$ such that Ra is simple. Since D and C absolutely min-pure, for any $a \in R$ such that Ra is simple, $Ext^1(R/aR, D) = 0$ and $Ext^1(R/aR, C) = 0$ by Proposition 2.8, and so $Ext^1(R/aR, D/C) = 0$. Thus D/C is absolutely min-pure by Proposition 2.8, again.

(2) and (3). By using a standard technique as in the proofs of (pure-)injectivity and (pure-)projectivity.

(4). For a pure exact sequence $0 \to X \to Y \to Z \to 0$ of left R-modules with Y min-pure flat, we get the splitting of $0 \to Z^+ \to Y^+ \to X^+ \to 0$. Since Y^+ is min-pure injective by Theorem 2.3, X^+ and Z^+ are min-pure injective by (2), whence X and Z are min-pure flat by Theorem 2.3. Moreover, direct sums of min-pure flat left R-modules are min-pure flat can be easily seen by using the tensor product properties. □

Proposition 3.2. *Let R be a ring. Then:*

(1) *All min-pure injective right R-modules have an injective cover.*
(2) *If R is left min-coherent (all minimal left ideals are finitely presented), then all min-pure projective right R-modules have a projective preenvelope.*

Proof. (1). Lemma 3.1(1) and [23, Theorem 2.5] yield that any min-pure injective R-module A_R has an absolutely min-pure cover $\beta : B \to A$. Absolutely min-purity of B gives a min-pure sequence $0 \to B \xrightarrow{i} E \to C \to 0$ with E injective by Proposition 2.8, whence there exists $\alpha : E \to A$ such that $\alpha i = \beta$. Being β an absolutely min-pure cover gives the existence of $\lambda : E \to B$ such that $\beta\lambda = \alpha$. So $\beta(\lambda i) = (\beta\lambda)i = \alpha i = \beta$, whence λi is an isomorphism. This means that E has a summand which is isomorphic to B. This makes B injective and g an injective cover of A.

(2). If A_R is min-pure projective, then A_R has a min-flat preenvelope $\beta : A \to B$ by [29, Theorem 4.6]. By [28, Proposition 2.2], there exist $\alpha : A \to D$ and $\lambda : D \to B$ with D projective such that $\beta = \lambda\alpha$. It follows that α is a projective preenvelope of A. □

Proposition 3.3. *All left R-modules can be embedded as a min-pure submodule of a min-pure injective module.*

Proof. Let $\mathscr{F} = \{R/aR \mid \text{for any } a \in R \text{ such that } aR \text{ is simple}\}$ and A a left R-module. Then by [30, Proposition 1.2], there exist an $\mathscr{F}$-pure sequence $0 \to C \to D \to A^+ \to 0$ where D is a direct sum of copies of modules in $\mathscr{F} \cup \{R_R\}$. By the isomorphism used in (2) $\Leftrightarrow$ (7) from the Lemma 2.4, the sequence $0 \to A^{++} \to D^+ \to C^+ \to 0$ is min-pure. Since A is pure in A^{++} by [18, Corollary 1.30], A is min-pure in D^+. Moreover, since any $R/aR \in \mathscr{F}$, any $(R/aR)^+$ is min-pure injective by Theorem 2.3, D^+ is min-pure injective by Lemma 3.1(2). □

Next, we consider the existence of a min-pure injective envelope and a min-pure projective (pre-)cover.

Proposition 3.4. *Let R be a ring. Then:*

(1) *All right R-modules have a min-pure injective envelope.*
(2) *All right R-modules have a min-pure projective precover. Moreover, if min-pure projective right R-modules is closed under pure quotients, all right R-module have a min-pure projective cover.*
(3) *All left R-modules have a min-pure flat cover.*

Proof. (1). By Proposition 3.3, all right R-modules have a min-pure injective preenvelope. Let a pair $(\mathfrak{E}, \mathfrak{A})$, with $\mathfrak{E}$ is a class of min-pure monomorphism between right R-modules and $\mathfrak{A}$ is a class of min-pure injective right R-modules. Then the pair $(\mathfrak{E}, \mathfrak{A})$ is an injective structure on the category of right R-modules determined by the class $\mathscr{F} = \{R/aR \mid \text{for any } a \in R \text{ such that } aR \text{ is simple}\}$ by Lemma 2.4 and [15, Definitions 6.6.2 and 6.6.3]. Thus, (1) follows by [15, Theorem 6.6.4].

(2). Min-pure projective modules are precovering by Lemma 2.7. If min-pure projective right R-modules are closed under pure quotients, every right R-module has a min-pure projective cover by [23, Theorem 2.5].

(3) follows by Lemma 3.1(4) and [23, Theorem 2.5]. □

4. Rings whose injective modules are min-pure projective

Next we characterize min-pure injctive and min-pure projective modules via min-purity.

Proposition 4.1. *For a module A_R, the next statements are equal:*

1. *A is min-pure injective;*
2. *All min-pure sequences $0 \to A \to M \to N \to 0$ are split;*
3. *A is injective relative to all min-pure sequences $0 \to M \to N \to L \to 0$ with N min-pure projective;*
4. *A is a direct summand of every min-pure extension of it.*

Proof. (1) ⇒ (2) is obvious and (1) ⇔ (3) follows by [30, Theorem 1.6].

(2) ⇒ (1). By Proposition 3.3, there is a min-pure exact sequence $0 \to A \to M \to N \to 0$ with M min-pure injective. So A is min-pure injective by (2).

(1) ⇒ (4). Suppose A is a min-pure submodule of a module B. Since A is min-pure injective then the identity map of A extends to a map $B \to A$ meaning that A is a direct summand of B.

(4) ⇒ (1) is clear by Lemma 3.1(2). □

Proposition 4.2. *For a module A_R, the next statements are equal:*

1. *A is min-pure projective;*
2. *All min-pure exact sequences $0 \to M \to N \to A \to 0$ are split;*
3. *A is projective with respect to all min-pure sequences $0 \to M \to N \to L \to 0$ with N min-pure injective.*

Proof. (1) ⇒ (2) is clear and (1) ⇔ (3) follows by [30, Theorem 1.6].

(2) ⇒ (1). By Lemma 2.7, there is a min-pure exact sequence $0 \to M \to N \to A \to 0$ with N min-pure projective. So, A is min-pure projective by (2). □

Recall that R is called a semisimple ring provided that all right (or left) R-modules are projective (resp. injective). A ring R is said to be quasi-Frobenius if R is left (or right) artinian and left (or right) self-injective. By a well-known result of Faith and Walker [16], R is quasi-Frobenius if and only if the class of injective modules and the class of projective modues are the same.

Theorem 4.3. *The next statements are equal for a ring R:*

1. *R is semisimple;*
2. *All min-pure injective right R-modules are projective;*
3. *All min-pure projective right R-modules are injective.*

Proof. (1) ⇒ (3) and (1) ⇒ (2) are easy.

(2) ⇒ (1). Our hypothesis implies that all injective right R-modules are projective, whence R is quasi-Frobenius. For each right R-module A, by Proposition 3.3, there is a min-pure extension B of A such that B is min-pure injective. Since B is projective by (2), B is injective. This means that A is absolutely min-pure by Proposition 2.8. Thus R is left universally mininjective by Proposition 2.11, whence R is left PS. Being R is left Kasch gives that all simple left R-modules are projective, i.e. R is semisimple.

(3) ⇒ (1) By our hypothesis again, R is quasi-Frobenius. Let A be a min-pure projective right R-module. By hypothesis, A is injective, and so is projective. Thus, R is left universally mininjective by Theorem 2.11, whence R is left PS. By the same reason of (2) ⇒ (1), R is semisimple. □

Proposition 4.4. *Let R be a right Artinian ring and $\mathscr{C} = \{R/aR \mid$ such that Ra is simple for any $a \in R\}$. Then a right R-module A is min-pure projective if and only if $A \cong P \oplus L$ where P is projective and $L \in Add(\mathscr{C})$.*

Proof. The sufficiency follows directly. For the necessity, let A_R be min-pure projective R-module. Then $A \oplus B = (\oplus_{i\in I} R_i) \oplus (\oplus_{\lambda\in\Lambda} A_\lambda)$ where $R_i \cong R$, A_λ is in $\mathscr{C}$ for all $i \in I$ and $\lambda \in \Lambda$ for some index sets I and Λ, and B a right R-module by Lemma 2.7. Artinianity of R implies that composition lengths of each R_i and A_λ are finite, and each R_i and A_λ can be written as a finite direct sum of indecomposable cyclic modules. So, each indecomposable components of R_i and A_λ has local endomorphism ring by [18, Lemma 2.21]. Thus each A_λ have the exchange property, this means that there exist some submodules A_1, A_1', B_1, B_1' such that $A \oplus B = A_1 \oplus B_1(\oplus_{\lambda\in\Lambda} A_\lambda)$ and $A_1 \oplus A_1' = A$ and $B_1 \oplus B_1' = B$. Thus, $A_1 \oplus B_1 \cong \oplus_{i\in I} R_i$ and $A_1' \oplus B_1' \cong \oplus_{\lambda\in\Lambda} A_\lambda$. So A_1 is projective and A_1' is in $\mathrm{Add}(\mathscr{C})$. □

A ring R is *right CF* if all cyclic right R-modules embedded in a free module. In general, a right CF ring need not be a quasi-Frobenius ring even if it is two-sided Artinian (see [7]). Now, we attempt to understand when the right CF rings would be quasi-Frobenius by min-purity.

Theorem 4.5. *The next statements are equal for a ring R:*

(1) *R is right CF and all injective right R-modules are min-pure projective;*
(2) *R is a quasi-Frobenius ring.*

Proof. (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2). Let A_R be an R-module with its injective hull $E(A)$. Since $E(A)$ is min-pure projective, by Lemma 2.7, $E(A)$ is contained in a direct sum of fininitely generated modules, and so A can be embedded in a direct sum of fininitely generated modules, whence R is right artinian by [17, Theoram 3.1]. Artinianity of R implies that all injective modules E can be seen as a direct sum of indecomposable cyclic modules by Proposition 4.4, and by (2), each cyclic indecomposable summands of E can be embedded in a free right R-module. By this we say that E can be embedded in a free module, whence R is quasi-Frobenius. □

By the next result, commutative quasi-Frobenius rings are determined in terms of min-pure injective and min-pure projective modules.

Theorem 4.6. *The next statements are equal for a commutative ring R:*

(1) *R is a quasi-Frobenius ring;*
(2) *All injective R-modules are min-pure projective;*
(3) *R is Artinian and $E(R)$ is min-pure projective;*
(4) *R is an Artinian ring and all projective R-modules are min-pure injective;*
(5) *R is an Artinian and min-pure injective ring.*

Proof. (1) $\Rightarrow$ (4) $\Rightarrow$ (5) and (1) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (3). Let A be any R-module. Since A embeds in a min-pure projective R-module $E(A)$, by Lemma 2.7, $E(A)$ is a direct summand of a direct sum of fininitely generated modules, whence R is artinian by [17, Theoram 3.1].

(3) $\Rightarrow$ (1). Since $E(R)$ is min-pure projective, by Proposition 4.4, $E(R)$ is a direct sum of finitely many cyclic indecomposable modules. Thus, by similar arguments used in [5, Theorem 4.12] from (6) $\Rightarrow$ (1), we conclude that R is a quasi-Frobenius.

(5) $\Rightarrow$ (1). Without loss of the generality, we may assume that R is a local ring with maximal ideal J. Let E be the injective hull of the field R/J. Since R is a commutative min-pure injective ring and $R \cong Hom_R(E, E)$, E is min-pure flat by Theorem 2.3(5), and so by Theorem 2.3, there exists a pure exact sequence $\xi : 0 \to A \to B \to E \to 0$ where B is in $\mathrm{Add}(\mathscr{F} \cup \{R_R\})$. But it is known that E is finitely presented. It follows that ξ splits

and so E is min-pure projective. Thus by Proposition 4.4, E can be written as a direct sum of cyclic indecomposable modules. Moreover, E is indecomposable by [20, Lemma 5.14], whence is finitely presented cyclic. Also, [26, Theorem 3.64] implies that E is faithful, and so $E \cong R$. Thus R is quasi-Frobenius. □

The rings whose all right R-modules are direct sum of cyclic modules are called *right Köthe ring*. By a *Köthe ring* we mean that both right and left Köthe ring. Köthe in [25] proved that an Artinian principal ideal ring is a Köthe ring and then Cohen and Kaplansky in [9] showed that a commutative ring R is a Köthe ring if and only if R is an Artinian principal ideal ring. Recently, in [4, Theorem 3.1], it is shown that every normal (i.e., all the idempotents are central) right Köthe ring is an Artinian principal left ideal ring.

Proposition 4.7. *The next statements are equal for a ring R:*

(1) *All right R-modules are min-pure projective;*
(2) *All right R-modules are min-pure injective;*
(3) *All min-pure exact sequences $0 \to M \to N \to L \to 0$ are split;*
(4) *All right R-modules are a direct sum of a module in $Add(\mathscr{C})$ and a projective module.*

Proof. (4) ⇒ (3) ⇔ (2) ⇔ (1) are obvious.

(1) ⇒ (4). Since min-pure projectivity implies pure projectivity, R is right pure-semisimple, whence is right Artinian. Thus (4) follows by Proposition 4.4. □

Proposition 4.8. *The next statements hold for a ring R:*

(1) *If all right R-modules are min-pure projective, then R is two-sided Köthe.*
(2) *If R is normal and all right R-modules are min-pure projective, then R is an Artinian principal ideal ring.*
(3) *If R is commutative and all right R-modules are min-pure projective, then R is a quasi-Frobenius serial ring.*

Proof. (1). Our hypothesis implies that every right R-module is RD-projective and so R is a right pure-semisimple RD-ring. Thus, [34, Proposition 6.5] implies that R is two-sided Köthe.

(2) follows from (1) and [4, Theorem 3.1].

(3). By (2), R is a commutative Artinian principal ideal ring and so it is Artinian serial. In this case, R is quasi-Frobenius serial by Theorem 4.6. □

Recall by [8] that, a submodule C of a right R-module D is said to be *neat* in D provided that for any simple right R-module S, $Hom_R(S, D) \to Hom_R(S, D/C)$ is epic. Now, the following gives a particular answer to Proposition 4.7.

Corollary 4.9. *Let R be a commutative indecomposable ring with $J(R)^2 = 0$. Then R is either a field or a quasi-Frobenius ring of composition length 2 if and only if all R-modules are min-pure projective.*

Proof. There is nothing to prove for if R is a field. If R is not a field, $cl(R) = 2$, whence R is local with unique simple and maximal ideal S such that $(R/S) \cong S$. Thus any min-pure exact sequence is neat-exact, and so closed-exact by [19, Theorem 5]. On the other hand, since R is Artinian serial with $J(R)^2 = 0$, every closed exact sequence is splitting by [14, 13.5]. Thus, every min-pure exact sequence is splitting by Proposition 4.7, whence the necessity follows by Proposition 4.7. Conversely, R is quasi-Frobenius serial by Proposition4.8. Since R is indecomposable and $J(R)^2 = 0$, either R is a field or R is a quasi-Frobenius ring of $cl(R) = 2$ by [3, Proposition 3.4]. □

5. Questions

For future research, we close the paper by giving next questions that are partially answered throughout the paper.

It was shown in [31, Theorem 2.4] that right perfectness of a ring R is equivalent to the fact that each RD-flat right R-module is RD-projective. Now, we have if every min-pure flat right R-module min-pure projective, then R is right perfect.
Q1: "*Whether the converse of this fact is true or not?*"

In Proposition4.8, we know that if every right R-module is min-pure projective, then R is a right and left Köthe ring. Also, a commutative ring over which modules are min-pure projective is quasi-Frobenius serial. Finally, in Corollary4.9, it is shown that over a commutative indecomposable ring with $J(R)^2 = 0$, every R-module is min-pure projective if and only if R is either a field or a quasi-Frobenius ring of composition length 2. Now,
Q2: "*What is the class of (commutative) rings R for which every R-module is min-pure projective?*"

Obviously every pure (resp. RD) exact sequence is min-pure, but not conversely (see Remark 2.5). Now,
Q3: "*What is the class of rings R for which min-pure exact sequences are pure (resp. RD-pure)?*"

Acknowledgment. The research of the second author was in part supported by a grant from IPM (No. 1402160411). This research is partially carried out in the IPM-Isfahan Branch.

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