Homological objects of min-pure exact sequences

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Abstract

In a recent paper, Mao has studied min-pure injective modules to investigate the existence of min-injective covers. A min-pure injective module is one that is injective relative only to min-pure exact sequences. In this paper, we study the notion of min-pure projective modules which is the projective objects of min-pure exact sequences. Various ring characterizations and examples of both classes of modules are obtained. Along this way, we give conditions which guarantee that each min-pure projective module is either injective or projective. Also, the rings whose injective objects are min-pure projective are considered. The commutative rings over which all injective modules are min-pure projective are exactly quasi-Frobenius. Finally, we are interested in the rings all of its modules are min-pure projective. We obtain that a ring $R$ is two-sided Köthe if all right $R$-modules are min-pure projective. Also, a commutative ring over which all modules are min-pure projective is quasi-Frobenius serial. As consequence, over a commutative indecomposable ring with $J(R)^2 = 0$, it is proven that all $R$-modules are min-pure projective if and only if $R$ is either a field or a quasi-Frobenius ring of composition length 2.

Mathematics Subject Classification (2020). 16E10, 16E30, 16D40, 13C10, 16D50, 13C11

Keywords. (min-)purity, Köthe rings, universally mininjective rings, quasi-Frobenius rings

1. Introduction

Throughout, $R$ will stand for associative ring with identity, and $R$-modules will be unitary modules unless otherwise specified. $A_R$ ($R_A$) stands for any module $A$ considered as a right (left) $R$-module.

As a generalization of injectivity, the concept of min-injectivity is introduced by Harada (see [22]). $R_A$ is called min-injective provided that $\text{Ext}^1_R(R/S, A) = 0$ for any minimal left ideal $S$. $A_R$ is called min-flat provided that $\text{Tor}^1_R(A, R/S) = 0$ for any minimal left ideal $S$ (see [29]). By the natural equivalence $\text{Ext}^1_R(R/S, A^+) \cong \text{Tor}^1_R(A, R/S)^+$ for any minimal left ideal $S$, we can conclude a right module $A$ is min-flat provided that $A^+$ is min-injective.

Min-injective rings and min-injective modules are the most important and most studied subjects of homological algebra along with module and ring theory. The main reason for this is that...
min-injective rings are naturally occurring in characterizing quasi-Frobenius (QF) rings. The importance of finite (quasi-Frobenius) rings has increased with the study of the rings of algebraic coding theory (see [21,24,38]).

In [28], Mao introduced the concept of min-purity and min-pure injectivity, to give further homologic characterizations of min-injective modules and to investigate the existence of min-injective covers. In the literature, purity has a considerable impact on module-ring theory, and several crucial generalizations of this notion are given since it was firstly introduced (see, [1,5,6,8,10,31,35,36]). In accordance with the terminology of Mao [28], a sequence $0 \to D \to E \to F \to 0$ of right $R$-modules is called min-pure exact if $\text{Hom}(R/aR,E) \to \text{Hom}(R/aR,F) \to 0$ is epic for any $a \in R$ such that $Ra$ is simple. $A_R$ is called min-pure injective provided that $A$ has injective property relative to all min-pure exact sequences. So far, min-purity, min-injectivity, min-pure injectivity and their homological objects are studied by many authors (see [22,28,29,33]).

Motivated by min-pure injective modules, in this article, we first introduce the homological objects which are flat and projective relative to the min-pure exact sequences. We shall call $A_R$ is min-pure projective provided that $A_R$ is projective relative to min-pure exact sequences. Also, $R_A$ is called min-pure flat if $R_A$ is flat relative to min-pure exact sequences. Naturally, flat left modules are min-pure flat, and projective right modules are min-pure projective, but not conversely (see Example 2.2(2)).

In section 2, we give some preliminary properties of min-pure projective and min-pure flat modules. After giving various equivalent conditions of min-purity, absolutely min-purity of modules are described via min-purity. Moreover, it is shown that $R$ is left universally mininjective if and only if all min-pure projective (resp. flat) right $R$-modules are projective (resp. flat). Also, we show that $R$ is left FS if and only if injective dimensions of min-pure injective right $R$-modules $\leq 1$ if and only if flat dimension of min-pure flat left $R$-modules $\leq 1$. Finally, injective dimensions of min-pure projective right $R$-modules $\leq 1$ equivalent to that for any $a \in R$ such that $Ra$ is simple, $aR$ is projective.

In section 3, we consider the covering and enveloping properties of min-pure injective and min-pure projective modules. We show that all min-pure injective modules have an injective cover, and if $R$ is left min-coherent, then all min-pure projective right $R$-modules have a projective preenvelope. Also, we get that all right modules have a min-pure projective precovers and min-pure injective envevelope.

In section 4, we focused on the rings whose all injective modules are min-pure projective. Along the way, being $R$ is quasi-Frobenius equivalent to that $R$ is right CF and every injective right $R$-module is min-pure projective. For a commutative ring $R$ we prove that all injective $R$-modules are min-pure projective if and only if $R$ is quasi-Frobenius. Moreover, it is shown that $R$ is semisimple if and only if every min-pure projective (resp. injective) right $R$-module is injective (resp. projective). Finally, we focused on the rings whose all $R$-modules are min-pure projective (resp. injective). For this purpose, we prove that $R$ is a two-sided Köthe ring provided that every right $R$-module is min-pure projective (resp. injective). Consequently, for a commutative indecomposable ring with $J(R)^2 = 0$, it is shown that $R$ is either a field or a quasi-Frobenius ring of composition length 2 if and only if all $R$-modules are min-pure projective.

For future research, we close the paper by giving some questions that are partially answered inside the paper.

2. Min-pure projective and min-pure flat modules

A sequence $0 \to D \to E \to F \to 0$ of right $R$-modules is called min-pure exact if for any $a \in R$ such that $Ra$ is simple, $0 \to D \otimes (R/Ra) \to E \otimes (R/Ra) \to F \otimes (R/Ra) \to 0$ is exact. Moreover, $A_R$ is called min-pure injective provided that $A_R$ is injective relative to every min-pure sequence (see [28]). Motivated by min-pure injective modules, we introduce the homological objects which are projective and flat with respect to the min-pure sequences.
**Definition 2.1.** (a) $A_R$ is called min-pure projective if for all min-pure sequences $0 \to D \to E \to F \to 0$ of right $R$-modules, the induced map $\beta : \text{Hom}(A, E) \to \text{Hom}(A, F)$ is an epimorphism.

(b) $R_A$ is called min-pure flat if for all min-pure sequences $0 \to D \to E \to F \to 0$ of right $R$-modules, the induced map $\alpha : D \otimes A \to E \otimes A$ is a monomorphism.

**Example 2.2.** (1) For any $a \in R$ such that $Ra$ is simple, $R/aR$ is min-pure projective and $R/Ra$ is min-pure flat.

(2) Any projective right module is min-pure projective and any flat left module is min-pure flat. However, in general case, the converses need not be true. Consider the ring $R := \mathbb{Z}/p^2\mathbb{Z}$ for some prime integer $p$. $R/pR$ is a min-pure flat and min-pure projective $R$-module since $pR$ is simple ideal. Whereas the module $R/pR$ is not flat, otherwise $R/pR$ would be projective by [29, Corollary 3.3]. But the epimorphism $R \to R/pR \to 0$ does not split.

By the following theorem, further equivalent conditions of min-pure flatness are given.

**Theorem 2.3.** Let $\mathcal{F} = \{ R/Ra \ | \text{for any } a \in R \text{ such that } Ra \text{ is simple} \}$. The following are equivalent for $R_A$:

1. $A$ is min-pure flat;
2. $A^+ = \text{min-pure injective};$
3. $A \cong E/D$ where $E$ is in $\text{Add}(\mathcal{F} \cup \{ R\})$ and $D$ is pure in $E$;
4. $A$ can be written as a direct limit of finite direct sums of modules from $\mathcal{F} \cup \{ R\}$.

Also, when $R$ is commutative, above statements are equivalent to:

5. $\text{Hom}(A, D)$ is min-pure injective, for any injective $R$-module $D$;
6. $A \otimes C$ is min-pure flat, for any flat $R$-module $C$.

**Proof.** (1) $\Rightarrow$ (2). Let $R_A$ be min-pure flat and $0 \to D \to E \to F \to 0$ a min-pure sequence of right $R$-modules. So, $0 \to D \otimes A \to E \otimes A$ is monic, whence $(E \otimes A)^+ \to (D \otimes A)^+ \to 0$ is epic. This implies that $\text{Hom}(E, A^+) \to \text{Hom}(D, A^+) \to 0$ is also epic and so $A^+$ is min-pure injective.

(2) $\Rightarrow$ (3). Assume that $A^+$ is min-pure injective. By [30, Proposition 1.2], there exist an $\mathcal{F}$-pure sequence $0 \to D \to E \to A \to 0$ where $E$ is in $\text{Add}(\mathcal{F} \cup \{ R\})$. Also, by the isomorphism used in (2) $\Leftrightarrow$ (7) from the Lemma 2.4, the sequence $0 \to A^+ \to E^+ \to D^+ \to 0$ would be min-pure. Since $A^+$ is min-pure injective, $0 \to A^+ \to E^+ \to D^+ \to 0$ splits and so the sequence $0 \to D \to E \to A \to 0$ is pure.

(3) $\Rightarrow$ (4). Easily follows by [37, Theorem 34.2].

(4) $\Rightarrow$ (1). Let $0 \to D \to E \to F \to 0$ be a min-pure sequence of right $R$-modules and $\{ F_\alpha \}_{\alpha \in \Lambda}$ is a finite family of right $R$-modules such that for each $\alpha \in \Lambda$, $A = \lim F_\alpha$, where $F_\alpha$’s is a finite direct sums of modules from $\mathcal{F} \cup \{ R\}$. Since $F_\alpha$ is min-pure flat for each $\alpha \in \Lambda$, $0 \to D \otimes F_\alpha \to E \otimes F_\alpha \to F \otimes F_\alpha \to 0$ is exact. So by [37, Theorem 24.11], the sequence $0 \to D \otimes \lim F_\alpha \to E \otimes \lim F_\alpha \to F \otimes \lim F_\alpha \to 0$ is exact. Therefore, $A$ is min-pure flat.

(1) $\Rightarrow$ (5). Let $D$ be an injective $R$-module. If we consider the splitting map $0 \to D \to \prod R^+$, we would have the map $0 \to \text{Hom}(A, D) \to \text{Hom}(A, \prod R^+)$ which is also splits. Being $A^+$ is min-pure injective by (1) together with the isomorphisms $\prod A^+ \cong \text{Hom}(A, \prod R^+)$ implies that $\prod A^+$ is min-pure injective. This gives the min-pure injectivity of $\text{Hom}(A, D)$.

(5) $\Rightarrow$ (6). Assume that $C$ is any flat $R$-module. Since $(A \otimes C)^+$ is isomorphic to $\text{Hom}(A, C^+)$, it is min-pure injective by (5) and by the injectivity of $C^+$. This gives the min-pure flatness of $A \otimes C$.

(6) $\Rightarrow$ (1) straightforward by putting $C = R$. \qed

Now, we are ready to give further characterizations of min-purity.

**Lemma 2.4.** Let $0 \to D \to E \to F \to 0$ be a sequence of right $R$-modules. The following are equivalent:

...
Corollary 2.6. The next statements are equal for a ring $R$:

1. $0 \to D \to E \to F \to 0$ is min-pure;
2. $\text{Hom}(R/aR, E) \to \text{Hom}(R/aR, F) \to 0$ is epic for any $a \in R$ such that $Ra$ is simple;
3. $\text{Hom}(A, E) \to \text{Hom}(A, F) \to 0$ is epic for any min-pure projective $R$-module $A$;
4. $\text{Hom}(E, A) \to \text{Hom}(D, A) \to 0$ is epic for any min-pure injective $R$-module $A$;
5. $\text{Hom}(R/Ra, E^+) \to \text{Hom}(R/Ra, D^+) \to 0$ is epic for $a \in R$ such that $Ra$ is simple;
6. $0 \to D \otimes B \to E \otimes B$ is monic for any min-pure flat $R$-module $RB$;
7. $0 \to R/aR \otimes F^+ \to R/aR \otimes E^+$ is monic for any $a \in R$ such that $Ra$ is simple.

Also, if $R$ is commutative or two sided mininjective, then the above are equivalent to:
8. $\text{Hom}(R/aR, E) \to \text{Hom}(R/aR, F) \to 0$ is epic for any $a \in R$ such that $aR$ is simple.

Proof. (1) $\iff$ (2) follows by [28, Lemma 2.1].

(1) $\iff$ (3) $\implies$ (4) and (1) $\implies$ (6) are obvious.

(4) $\implies$ (1). Let $a \in R$ such that $Ra$ is simple. Min-pure flatness of $R/Ra$ implies the min-pure injectivity of $(R/Ra)^+$ by Theorem 2.3. Thus by (4), the induced sequence $0 \to \text{Hom}(F, (R/Ra)^+) \to \text{Hom}(E, (R/Ra)^+) \to \text{Hom}(D, (R/Ra)^+) \to 0$ can be obtained, and that gives the sequence $0 \to (F \otimes R/Ra)^+ \to (E \otimes R/Ra)^+ \to (D \otimes R/Ra)^+ \to 0$. So (1) follows by the exactness of $0 \to D \otimes R/Ra \to E \otimes R/Ra \to F \otimes R/Ra \to 0$.

(1) $\iff$ (5). Let $a \in R$ such that $Ra$ is simple. Then the right exactness of $0 \to D \otimes (R/Ra) \to E \otimes (R/Ra) \to F \otimes (R/Ra) \to 0$ is equivalent to the left exactness of $0 \to (F \otimes (R/Ra))^+ \to (E \otimes (R/Ra))^+ \to (D \otimes (R/Ra))^+ \to 0$, equivalently $0 \to \text{Hom}(R/Ra, F^+) \to \text{Hom}(R/Ra, E^+) \to \text{Hom}(R/Ra, D^+) \to 0$ is exact. Now, (1) $\iff$ (5) is obvious.

(6) $\implies$ (1) is obvious since every $R/S$ is min-pure flat for any simple left ideal $S$.

(2) $\iff$ (7). Let $a \in R$ such that $Ra$ is simple. Take into consideration the next diagram:

\[
\begin{array}{cccccc}
0 & \to & R/aR \otimes F^+ & \to & R/aR \otimes E^+ & \to & R/aR \otimes D^+ & \to & 0 \\
\mu^+ & & \delta^+ & & \lambda^+ & & & \\
0 & \to & \text{Hom}(R/aR, F^+) & \to & \text{Hom}(R/aR, E^+) & \to & \text{Hom}(R/aR, D^+) & \to & 0
\end{array}
\]

By [12, Lemma 2], $\mu$, $\delta$ and $\lambda$ are isomorphisms. Thus exactness of the first row is equivalent to the exactness of the second row, and equivalently the map $\text{Hom}(R/aR, E) \to \text{Hom}(R/aR, F)$ $\to 0$ is epic.

(2) $\iff$ (8). If $R$ is commutative, it is easy.

Let $R$ be left-right mininjective and $a \in R$. Then being $aR$ is a minimal right ideal equivalent to that $Ra$ is a minimal left ideal by [33, Theorem 1.14]. So in either cases (2) $\iff$ (8) follows.

Remark 2.5. (1). Obviously purity implies the min-purity, but not conversely. Indeed, by [13, Example 3.1(ii)], there is an $R$-algebra $S$ over a local Artinian ring $R$, such that the inclusion homomorphism $R \to S$ is cyclically pure, and so is min-pure. But $R \to S$ is not pure.

(2). By (1), every min-pure injectivity (resp. min-pure projectivity) of modules implies pure-injectivity (resp. pure-projectivity), but not conversely. Every Artinian $R$-module is well known as pure-injective. Hence the Artinian ring $R$ in [13, Example 3.1(ii)] is pure-injective. But it is not min-pure injective, otherwise the inclusion map $R \to S$ above splits.

(3). By (2) and the following corollary, we ensure that the existence of pure-projective module which is not min-pure projective.

A ring $R$ is a valuation ring (commutative but not necessarily a domain) provided that all ideals of $R$ are totally ordered by inclusion.

Corollary 2.6. The next statements are equal for a ring $R$:

1. All left modules are min-pure flat;
2. All pure-projective right modules are min-pure projective;
3. All pure-injective right modules are min-pure injective;
4. All min-pure exact sequences of right modules are pure.

Moreover, if $R$ is commutative, $R_p$ is a valuation ring for every prime ideal $p$. 
Proposition 2.9. The ring is exactly an arithmetic ring, i.e., the rings with a distributive lattice of ideals. Thus by Theorem 2.3, we conclude that is min-pure flat.

Since cyclically pure exact sequences are min-pure, the last statement follows by [13, Theorem 2.7]. □

Let \( \mathscr{C} \) denotes the set \( \mathscr{C} = \{ R/aR \mid \text{for any } a \in R \text{ such that } Ra \text{ is simple} \} \). Note that min-pure= \( \mathscr{C} \)-pure= \( \mathscr{C} \cup \{ R_R \} \)-pure. The following due to Warfield Jr. (see [36, Proposition 1, p. 700]).

Lemma 2.7. ([30, Proposition 1.2]) For a module \( A_R \) we have:

1. There exists a min-pure exact sequence \( 0 \to D \to E \to A \to 0 \) where \( E \) is a direct sum of copies of modules in \( \mathscr{C} \cup \{ R_R \} \).
2. The class of all min-pure projective right modules is exactly \( \text{Add}(\mathscr{C} \cup \{ R_R \}) \).

We will call \( A_R \) is absolutely min-pure (similar to absolutely purity) provided that \( A \) is min-pure in every extension of it.

Proposition 2.8. The next statements are equal for an \( R \)-module \( A_R \):

1. \( A_R \) is absolutely min-pure;
2. All exact sequences starting with \( A \) are min-pure;
3. \( \text{Ext}^1(D, A) = 0 \) for any min-pure projective \( R \)-module \( D_R \);
4. \( \text{Ext}^1(R/aR, A) = 0 \) for any \( a \in R \) such that \( Ra \) is simple;
5. There exists a min-pure sequence \( 0 \to A \to E \to C \to 0 \) with \( E \) injective;
6. For all min-pure injective \( R \)-modules \( D_R \), all homomorphisms from \( A \) to \( D \) factors through an injective \( R \)-module.

Also, if \( R \) is commutative, then the above conditions are equivalent to:
7. \( A \) is min-injective.

Proof. (1) \( \iff \) (2) is easy by definition.

(2) \( \Rightarrow \) (5). Let \( f : A \to B \) be a homomorphism for any min-pure injective \( R \)-module \( B_R \).

(5) \( \Rightarrow \) (6). Let \( g : A \to D \) be any homomorphism with \( D \) min-pure injective and \( \xi : 0 \to A \to K \to L \to 0 \) be an exact sequence. So, there are a map \( h : A \to E \) with \( E \) injective and a map \( f : E \to D \) such that \( fh = g \) by (6). By injectivity of \( E \), there is a map \( \alpha : K \to E \) such that \( \alpha \xi = h \). So \( g = f \alpha \xi \), whence \( \xi \) is min-pure by Lemma 2.4.

(1) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (4) follows from Lemma 2.7.

(4) \( \Rightarrow \) (6). We always have a sequence \( \varepsilon : 0 \to A \to K \to L \to 0 \) with \( K \) injective. Since by (4), \( \text{Ext}^1(R/aR, A) = 0 \) for any \( a \in R \) such that \( Ra \) is simple, \( \text{Hom}(R/aR, K) \to \text{Hom}(R/aR, L) \) \( \to 0 \) is epic. Thus \( \varepsilon \) is min-pure by Lemma 2.4, and so every homomorphism \( A \to B \) with \( B \) min-pure injective factors through \( E \).

(2) \( \iff \) (7) follows by [28, Proposition 2.3]. □

Recall by Puninski et al. [34] that, \( R \) is an \( RD \)-ring provided that purity and \( RD \)-purity coincides (this property is right-left symmetric). A serial ring and a regular ring are always \( RD \) (see [11, Theorem 1.4] and [34, Remark 2.7]). By Puninski et al. [34, Proposition 4.5], a commutative \( RD \) ring is exactly an arithmetic ring, i.e., the rings with a distributive lattice of ideals.

Proposition 2.9. The next statements hold for a ring \( R \):

1. If all min-pure sequences are pure, then \( R \) is an \( RD \)-ring.
2. If \( R \) is commutative and all min-pure sequences are pure, then \( R \) is arithmetic and all min-injective \( R \)-modules are Absolutely pure.
If $R$ is commutative Noetherian ring such that all min-pure sequences are pure, then $R$ is quasi-Frobenius arithmetic.

**Proof.** (1). If we assume that every min-pure exact sequence is pure, then every $RD$-exact sequence is pure, whence $R$ is an $RD$-ring.

(2). By (1) and [34, Proposition 4.5], $R$ is arithmetic. Also, if $A$ is min-injective, then $0 \to A \to E(A) \to E(A)/A \to 0$ is min-pure by Proposition 2.8. So, it is pure exact and this implies that $A$ is Absolutely pure.

(3). Being arithmetic comes from (2). Again by (2) and Noetherianity of $R$, all min-injective $R$-modules are injective, whence $R$ is Artinian by [2, Theorem 1] and the fact that simple injectives are min-injective. Thus $R_R$ is pure-injective, whence $R_R$ is min-pure injective by hypothesis. Hence by Theorem 4.6, $R$ is quasi-Frobenius. □

Relationship between min-pure injective (resp. min-pure projective, min-pure flat) modules and injective (resp. projective, flat) modules is given below.

**Corollary 2.10.** The next conditions are true for any ring $R$:

1. Any min-pure injective absolutely min-pure right $R$-module is injective.
2. Any min-flat min-pure projective right $R$-module is projective.
3. Any min-pure injective min-injective $R$-module is injective.
4. Any min-flat min-flat $R$-module is flat.

**Proof.** (1). For any min-pure injective absolutely min-pure right $R$-module $A$, By Proposition 2.8, there is a min-pure sequence $0 \to A \to E \to B \to 0$ with $E$ injective. Splitting of this sequence gives us the injectivity of $A$.

(2). For any min-pure projective min-flat right $R$-module $A$, we always have $0 \to K \to F \to A \to 0$ where $F$ projective. Since $A$ is min-flat, this exact sequence is min-pure. Splitting of this sequence gives us the projectivity of $A$.

(3) follows by Proposition 2.8.

(4). For any min-flat min-pure flat module $A$, $A^+$ is min-pure injective and min-injective by Theorem 2.3. This gives the injectivity of $A^+$ by (2), whence is flatness of $A$. □

Recall by [33] that, $R$ is left universally mininjective ring if all left $R$-modules are min-injective, equivalently $R$ is left min-injective and left PS. Now, we obtain new equivalent conditions of left universally mininjective rings via min-purity.

**Proposition 2.11.** The next statements are equal for a ring $R$:

1. $R$ is left universally mininjective;
2. Every exact sequences of right $R$-modules is min-pure;
3. Every right $R$-module is absolutely min-pure;
4. Every min-pure injective right $R$-module is injective;
5. Every min-pure injective right $R$-module is absolutely min-pure;
6. Every min-pure flat left module is flat;
7. Every min-pure projective right $R$-module is projective.

**Proof.** (1) $\iff$ (2) follows by [28, Theorem 4.3] and (3) $\iff$ (2) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (4). Hypothesis implies that any min-pure injective right $R$-module is a direct summand of an injective module, and so (2) follows.

(4) $\Rightarrow$ (6). For any min-pure flat left module $A$, $A^+$ is min-pure injective, whence is injective by (2). Therefore $A$ would be flat.

(6) $\Rightarrow$ (1). Let $M$ be a min-pure flat left $R$-module. We always have a sequence $\varepsilon : 0 \to D \to E \to A \to 0$ where $E$ is projective. Flatness of $M$, gives the monic map $0 \to D \otimes M \to E \otimes M$, and so $\varepsilon$ is min-pure by Lemma 2.4. Thus, any right $R$-module $A$ is min-flat by [28, Proposition 2.2], whence $R$ is left universally mininjective by [28, Theorem 4.3].
Some (pre)envelopes and (pre)covers

(1) ⇒ (7). Since $R$ is left universally mininjective, for any $a \in R$ such that $Ra$ is simple, $R/aR$ is min-flat by [28, Theorem 4.3], whence is projective by [29, Corollary 3.3]. If $\mathcal{C} = \{R/aR \mid \text{for any } a \in R \text{ such that } Ra \text{ is simple}\}$, any min-pure projective module contained in $\text{Add}(\mathcal{C} \cup \{R_i\})$ by Lemma 2.7(2). Since any $R/aR \in \mathcal{C}$ is projective, (7) follows.

(7) ⇒ (1). Since by (7), $R/aR$ is projective for any minimal left ideal $Ra$, (1) follows by [29, Theorem 5.10].

The rings all of whose minimal left ideals are projective is called left PS [32]. Nonsingular rings, Semiprime rings and V-rings are left PS. A ring $R$ is left FS [27], if every simple left ideal of $R$ is flat.

**Proposition 2.12.** The next statements are equal for a ring $R$:

1. $R$ is left FS;
2. $\text{Id}(A) \leq 1$ for any min-pure injective module $AR$;
3. $\text{Fid}(A) \leq 1$ for any min-pure flat module $RA$.

**Proof.** (1) ⇒ (2). By [28, Theorem 4.1], for any right $R$-module $F$, we have $0 \to D \to E \to F \to 0$ with $E$ projective and $D$ min-flat. This gives by [28, Proposition 2.2], for any min-pure injective right $R$-module $A$, $\text{Ext}^2(F, A) \cong \text{Ext}^2(D, A) = 0$. That is, $\text{Id}(A) \leq 1$.

(2) ⇒ (3). For any min-pure flat $R$-module $RA$, $A^+$ is min-pure injective by Theorem 2.3. By (2), for any $R$-module $DR$, we have $\text{Tor}_2(D, A^+) \cong \text{Ext}^2(D, A^+) = 0$. So, $\text{Tor}_2(D, A) = 0$, and hence $\text{fd}(A) \leq 1$.

(3) ⇒ (1). Since $R/S$ is min-pure flat for any minimal left ideal $S$, flat dimension of $R/S$ is $\leq 1$. In this case $S$ is flat and so $R$ is left FS.

Next we discuss the conditions related to min-pure projective modules which exactly characterizes left PS rings as follows.

**Proposition 2.13.** The next statements are equal for a ring $R$:

1. $aR$ is projective for any $a \in R$ such that $Ra$ is simple;
2. $\text{Pd}(A) \leq 1$ for any min-pure projective module $AR$;
3. Absolutely min-pure left $R$-modules is closed under homomorphic images.

Also, when $R$ is commutative, above conditions are equal to:

4. $R$ is PS.

**Proof.** (1) ⇒ (3). Let $B$ be a submodule of an absolutely min-pure right $R$-module $A$. We shall show that $A/B$ is absolutely min-pure. For any $a \in R$ such that $Ra$ is simple, consider the induced exact sequence

$$\text{Ext}^1(R/aR, A) \to \text{Ext}^1(R/aR, A/B) \to \text{Ext}^2(R/aR, B)$$

By Proposition 2.8, $\text{Ext}^1(R/aR, A) = 0$. Consider $\text{Ext}^2(R/aR, B) \cong \text{Ext}^1(aR, B)$ the isomorphism. Projectivity of $aR$ gives that $\text{Ext}^2(R/aR, B) = 0$. Thus $\text{Ext}^1(R/aR, A/B) = 0$, and so $A/B$ is absolutely min-pure by Proposition 2.8.

(3) ⇒ (2). Let $A$ be a min-pure projective right $R$-module. For any right $R$-module $C$, we always have $0 \to C \to D \to E \to 0$ with $D$ injective, that gives the exactness of $0 = \text{Ext}^1(A, D) \to \text{Ext}^1(A, E) \to \text{Ext}^2(A, C) \to \text{Ext}^2(A, D) = 0$. By (2), $E$ is absolutely min-pure and so $\text{Ext}^2(A, C) \cong \text{Ext}^1(A, E) = 0$ by Proposition 2.8. This means that projective dimension of $A$ is $\leq 1$.

(2) ⇒ (1). Since $R/aR$ is min-pure projective for any $a \in R$ such that $Ra$ is simple, projective dimension of $R/aR$ is $\leq 1$. In this case $aR$ is projective.

(1) ⇔ (4). If $R$ is commutative, it is easy.

3. Some (pre)envelopes and (pre)covers

Let $\mathcal{B}$ be a class of right modules.
For a module $X_R$, a module $Y \in \mathcal{P}$ is called a $\mathcal{P}$-envelope of $X$, if there is a homomorphism $f : X \to Y$ such that the next conditions hold:

1. For any homomorphism $g : X \to Z$ with $Z \in \mathcal{P}$, there is a map $h : Y \to Z$ with $g = hf$.
2. If an endomorphism $h : Y \to Y$ is such that $f = hf$, then $f$ must be an automorphism.

If only (1) holds, we call $f : X \to Y$ a $\mathcal{P}$-precover. Dually, it can be defined a $\mathcal{P}$-cover and $\mathcal{P}$-preenvelope. In general $\mathcal{P}$-envelopes and $\mathcal{P}$-covers not always exist, but they are unique (up to isomorphism) if they exist (see [15]).

**Lemma 3.1.** Let $R$ be a ring. Then:

1. Extensions, pure submodules, pure quotients, direct sums and direct summands of absolutely min-pure right $R$-modules are absolutely min-pure.
2. Finite direct sums, direct summands and direct products of min-pure injective right $R$-modules are min-pure projective.
3. Direct sums and direct summands of min-pure projective right $R$-modules are min-pure injective.
4. Direct sums, pure quotients and pure submodules of min-pure flat left $R$-modules are min-pure flat.

**Proof.** (1). Using the properties of the Ext functor, closedness of absolutely min-purity under extensions is obvious by Proposition 2.8. Also, using the properties of the tensor functor, closedness under direct sums and direct summands is easy. Also closedness of absolutely min-pure modules under pure submodules is by Proposition 2.8. Now let $C$ a pure submodule of an absolutely min-pure right module $D$. Then the exact sequence $0 \to (D/C)^+ \to D^+ \to C^+ \to 0$ splits. So, the isomorphism

$$\text{Tor}_1(R/aR, D^+) \cong \text{Tor}_1(R/aR, C^+) \oplus \text{Tor}_1(R/aR, (D/C)^+)$$

induces the isomorphism

$$\text{Ext}^1(R/aR, D^+) \cong \text{Ext}^1(R/aR, C^+) \oplus \text{Ext}^1(R/aR, D/C)^+$$

for any $a \in R$ such that $Ra$ is simple. Since $D$ and $C$ absolutely min-pure, for any $a \in R$ such that $Ra$ is simple, $\text{Ext}^1(R/aR, D) = 0$ and $\text{Ext}^1(R/aR, C) = 0$ by Proposition 2.8, and so $\text{Ext}^1(R/aR, D/C) = 0$. Thus $D/C$ is absolutely min-pure by Proposition 2.8, again.

(2) and (3). By using a standard technique as in the proofs of (pure-)injectivity and (pure-)projectivity.

(4). For a pure exact sequence $0 \to X \to Y \to Z \to 0$ of left $R$-modules with $Y$ min-pure flat, we get the splitting of $0 \to Z^+ \to Y^+ \to X^+ \to 0$. Since $Y^+$ is min-pure injective by Theorem 2.3, $X^+$ and $Z^+$ are min-pure injective by (2), whence $X$ and $Z$ are min-pure flat by Theorem 2.3. Moreover, direct sums of min-pure flat left $R$-modules are min-pure flat can be easily seen by using the tensor product properties. 

**Proposition 3.2.** Let $R$ be a ring. Then:

1. All min-pure injective right $R$-modules have an injective cover.
2. If $R$ is left min-coherent (all minimal left ideals are finitely presented), then all min-pure projective right $R$-modules have a projective preenvelope.

**Proof.** (1). Lemma 3.1(1) and [23, Theorem 2.5] yield that any min-pure injective $R$-module $A_R$ has an absolutely min-pure cover $\beta : B \to A$. Absolutely min-purity of $B$ gives a min-pure sequence $0 \to B \xrightarrow{i} E \to C \to 0$ with $E$ injective by Proposition 2.8, whence there exists $\alpha : E \to A$ such that $\alpha i = \beta$. Being $\beta$ an absolutely min-pure cover gives the existence of $\lambda : E \to B$ such that $\beta \lambda = \alpha$. So $\beta(\lambda i) = (\beta \lambda)i = \alpha i = \beta$, whence $\lambda i$ is an isomorphism. This means that $E$ has a summand which is isomorphic to $B$. This makes $B$ injective and $g$ an injective cover of $A$. 
(2). If \( A_R \) is min-pure projective, then \( A_R \) has a min-flat preenvelope \( \beta : A \to B \) by [29, Theorem 4.6]. By [28, Proposition 2.2], there exist \( \alpha : A \to D \) and \( \lambda : D \to B \) with \( D \) projective such that \( \beta = \lambda \alpha \). It follows that \( \alpha \) is a projective preenvelope of \( A \).

\[ \square \]

**Proposition 3.3.** All left \( R \)-modules can be embedded as a min-pure submodule of a min-pure injective module.

**Proof.** Let \( \mathcal{F} = \{ R/aR \mid \text{for any } a \in R \text{ such that } aR \text{ is simple} \} \) and \( A \) a left \( R \)-module. Then by [30, Proposition 1.2], there exist an \( \mathcal{F} \)-pure sequence \( 0 \to C \to D \to A^+ \to 0 \) where \( D \) is a direct sum of copies of modules in \( \mathcal{F} \cup \{ R_R \} \). By the isomorphism used in (2) \( \Leftrightarrow \) (7) from the Lemma 2.4, the sequence \( 0 \to A^{++} \to D^+ \to \tilde{C}^+ \to 0 \) is min-pure. Since \( A \) is pure in \( A^{++} \) by [18, Corollary 1.30], \( A \) is min-pure in \( D^+ \). Moreover, since any \( R/aR \in \mathcal{F} \), any \( (R/aR)^+ \) is min-pure injective by Theorem 2.3, \( D^+ \) is min-pure injective by Lemma 3.1(2).

Next, we consider the existence of a min-pure injective envelope and a min-pure projective (pre-)cover.

**Proposition 3.4.** Let \( R \) be a ring. Then:

1. All right \( R \)-modules have a min-pure injective envelope.
2. All right \( R \)-modules have a min-pure projective precover. Moreover, if min-pure projective right \( R \)-modules is closed under pure quotients, all right \( R \)-module have a min-pure projective cover.
3. All left \( R \)-modules have a min-pure flat cover.

**Proof.** (1). By Proposition 3.3, all right \( R \)-modules have a min-pure injective preenvelope. Let a pair \( (\mathcal{E}, \mathfrak{A}) \), with \( \mathcal{E} \) is a class of min-pure monomorphism between right \( R \)-modules and \( \mathfrak{A} \) is a class of min-pure injective right \( R \)-modules. Then the pair \( (\mathcal{E}, \mathfrak{A}) \) is an injective structure on the category of right \( R \)-modules determined by the class \( \mathcal{F} = \{ R/aR \mid \text{for any } a \in R \text{ such that } aR \text{ is simple} \} \) by Lemma 2.4 and [15, Definitions 6.6.2 and 6.6.3]. Thus, (1) follows by [15, Theorem 6.6.4].

(2). Min-pure injective modules are precovering by Lemma 2.7. If min-pure projective right \( R \)-modules are closed under pure quotients, every right \( R \)-module has a min-pure projective cover by [23, Theorem 2.5].

(3) follows by Lemma 3.1(4) and [23, Theorem 2.5].

\[ \square \]

4. Rings whose injective modules are min-pure projective

Next we characterize min-pure injective and min-pure projective modules via min-purity.

**Proposition 4.1.** For a module \( A_R \), the next statements are equal:

1. \( A \) is min-pure injective;
2. All min-pure sequences \( 0 \to A \to M \to N \to 0 \) are split;
3. \( A \) is injective relative to all min-pure sequences \( 0 \to M \to N \to L \to 0 \) with \( N \) min-pure projective;
4. \( A \) is a direct summand of every min-pure extension of it.

**Proof.** (1) \( \Rightarrow \) (2) is obvious and (1) \( \Rightarrow \) (3) follows by [30, Theorem 1.6].

(2) \( \Rightarrow \) (1). By Proposition 3.3, there is a min-pure exact sequence \( 0 \to A \to M \to N \to 0 \) with \( M \) min-pure injective. So \( A \) is min-pure injective by (2).

(1) \( \Rightarrow \) (4). Suppose \( A \) is a min-pure submodule of a module \( B \). Since \( A \) is min-pure injective then the identity map of \( A \) extends to a map \( B \to A \) meaning that \( A \) is a direct summand of \( B \).

(4) \( \Rightarrow \) (1) is clear by Lemma 3.1(2).

\[ \square \]

**Proposition 4.2.** For a module \( A_R \), the next statements are equal:

1. \( A \) is min-pure projective;
2. All min-pure exact sequences \( 0 \to M \to N \to A \to 0 \) are split;
A is projective with respect to all min-pure sequences $0 \to M \to N \to L \to 0$ with $N$ min-pure injective.

**Proof.** (1) $\Rightarrow$ (2) is clear and (1) $\Leftrightarrow$ (3) follows by [30, Theorem 1.6].

(2) $\Rightarrow$ (1). By Lemma 2.7, there is a min-pure exact sequence $0 \to M \to N \to A \to 0$ with $N$ min-pure projective. So, $A$ is min-pure projective by (2). \qed

Recall that $R$ is called a semisimple ring provided that all right (or left) $R$-modules are projective (resp. injective). A ring $R$ is said to be quasi-Frobenius if $R$ is left (or right) artinian and left (or right) self-injective. By a well-known result of Faith and Walker [16], $R$ is quasi-Frobenius if and only if the class of injective modules and the class of projective modules are the same.

**Theorem 4.3.** The next statements are equal for a ring $R$:

1. $R$ is semisimple;
2. All min-pure injective right $R$-modules are projective;
3. All min-pure projective right $R$-modules are injective.

**Proof.** (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) are easy.

(2) $\Rightarrow$ (1). Our hypothesis implies that all injective right $R$-modules are projective, whence $R$ is quasi-Frobenius. For each right $R$-module $A$, by Proposition 3.3, there is a min-pure extension $B$ of $A$ such that $B$ is min-pure injective. Since $B$ is projective by (2), $B$ is injective. This means that $A$ is absolutely min-pure by Proposition 2.8. Thus $R$ is left universally mininjective by Proposition 2.11, whence $R$ is left PS. Being $R$ is left Kasch gives that all simple left $R$-modules are projective, i.e. $R$ is semisimple.

(3) $\Rightarrow$ (1) By our hypothesis again, $R$ is quasi-Frobenius. Let $A$ be a min-pure projective right $R$-module. By hypothesis, $A$ is injective, and so is projective. Thus, $R$ is left universally mininjective by Theorem 2.11, whence $R$ is left PS. By the same reason of (2) $\Rightarrow$ (1), $R$ is semisimple. \qed

**Proposition 4.4.** Let $R$ be a right Artinian ring and $\mathcal{C} = \{ R/aR \mid \text{such that } Ra \text{ is simple for any } a \in R \}$. Then a right $R$-module $A$ is min-pure projective if and only if $A \cong P \oplus L$ where $P$ is projective and $L \in \text{Add}(\mathcal{C})$.

**Proof.** The sufficiency follows directly. For the necessity, let $A_R$ be min-pure projective $R$-module. Then $A \oplus B = (\oplus_{i \in I} R_i) \oplus (\oplus_{\lambda \in \Lambda} A_\lambda)$ where $R_i \cong R, A_\lambda$ is in $\mathcal{C}$ for all $i \in I$ and $\lambda \in \Lambda$ for some index sets $I$ and $\Lambda$, and $B$ a right $R$-module by Lemma 2.7. Artinianity of $R$ implies that composition lengths of each $R_i$ and $A_\lambda$ are finite, and each $R_i$ and $A_\lambda$ can be written as a finite direct sum of indecomposable cyclic modules. So, each indecomposable components of $R_i$ and $A_\lambda$ has local endomorphism ring by [18, Lemma 2.21]. Thus each $A_\lambda$ have the exchange property, this means that there exist some submodules $A_1, A'_1, B_1, B'_1$ such that $A \oplus B = A_1 \oplus B_1$ and $A_1 \oplus A'_1 = A$ and $B_1 \oplus B'_1 = B$. Thus, $A \oplus B \cong \oplus_{i \in I} R_i$ and $A_1 \oplus A'_1 \cong \oplus_{\lambda \in \Lambda} A_\lambda$. So $A_1$ is projective and $A'_1$ is in $\text{Add}(\mathcal{C})$. \qed

A ring $R$ is right CF if all cyclic right $R$-modules embedded in a free module. In general, a right CF ring need not be a quasi-Frobenius ring even if it is two-sided Artinian (see [7]). Now, we attempt to understand when the right CF rings would be quasi-Frobenius by min-purity.

**Theorem 4.5.** The next statements are equal for a ring $R$:

1. $R$ is right CF and all injective right $R$-modules are min-pure projective;
2. $R$ is a quasi-Frobenius ring.

**Proof.** (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2). Let $A_R$ be an $R$-module with its injective hull $E(A)$. Since $E(A)$ is min-pure projective, by Lemma 2.7, $E(A)$ is contained in a direct sum of finitely generated modules, and so $A$ can be embedded in a direct sum of finitely generated modules, whence $R$ is right artinian by [17, Theorem 3.1]. Artinianity of $R$ implies that all injective modules $E$ can be seen
as a direct sum of indecomposable cyclic modules by Proposition 4.4, and by (2), each cyclic indecomposable summands of $E$ can be embedded in a free right $R$-module. By this we say that $E$ can be embedded in a free module, whence $R$ is quasi-Frobenius.

By the next result, commutative quasi-Frobenius rings are determined in terms of min-pure injective and min-pure projective modules.

**Theorem 4.6.** The next statements are equal for a commutative ring $R$:

1. $R$ is a quasi-Frobenius ring;
2. All injective $R$-modules are min-pure projective;
3. $R$ is Artinian and $E(R)$ is min-pure projective;
4. $R$ is an Artinian ring and all projective $R$-modules are min-pure injective;
5. $R$ is an Artinian and min-pure injective ring.

**Proof.** (1) $\Rightarrow$ (4) $\Rightarrow$ (5) and (1) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (3). Let $A$ be any $R$-module. Since $A$ embeds in a min-pure projective $R$-module $E(A)$, by Lemma 2.7, $E(A)$ is a direct summand of a direct sum of finitely generated modules, whence $R$ is artinian by [17, Theorem 3.1].

(3) $\Rightarrow$ (1). Since $E(R)$ is min-pure projective, by Proposition 4.4, $E(R)$ is a direct sum of finitely many cyclic indecomposable modules. Thus, by similar arguments used in [5, Theorem 4.12] from (6) $\Rightarrow$ (1), we conclude that $R$ is a quasi-Frobenius.

(5) $\Rightarrow$ (1). Without loss of the generality, we may assume that $R$ is a local ring with maximal ideal $J$. Let $E$ be the injective hull of the field $R/J$. Since $R$ is a commutative min-pure injective ring and $R \cong \text{Hom}_R(E, E)$, $E$ is min-pure flat by Theorem 2.3(5), and so by Theorem 2.3, there exists a pure exact sequence $\xi : 0 \to A \to B \to E \to 0$ where $B$ is in Add($\mathcal{F} \cup \{ R_R \}$). But it is known that $E$ is finitely presented. It follows that $\xi$ splits and so $E$ is min-pure projective. Thus by Proposition 4.4, $E$ can be written as a direct sum of cyclic indecomposable modules. Moreover, $E$ is indecomposable by [20, Lemma 5.14], whence is finitely presented cyclic. Also, [26, Theorem 3.64] implies that $E$ is faithful, and so $E \cong R$. Thus $R$ is quasi-Frobenius.

The rings whose all right $R$-modules are direct sum of cyclic modules are called right Köthe ring. By a Köthe ring we mean that both right and left Köthe ring. Köthe in [25] proved that an Artinian principal ideal ring is a Köthe ring and then Cohen and Kaplansky in [9] showed that a commutative ring $R$ is a Köthe ring if and only if $R$ is an Artinian principal ideal ring. Recently, in [4, Theorem 3.1], it is shown that every normal (i.e., all the idempotents are central) right Köthe ring is an Artinian principal left ideal ring.

**Proposition 4.7.** The next statements are equal for a ring $R$:

1. All right $R$-modules are min-pure projective;
2. All right $R$-modules are min-pure injective;
3. All min-pure exact sequences $0 \to M \to N \to L \to 0$ are split;
4. All right $R$-modules are a direct sum of a module in Add($\mathcal{C}$) and a projective module.

**Proof.** (4) $\Rightarrow$ (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1) are obvious.

(1) $\Rightarrow$ (4). Since min-pure projectivity implies pure projectivity, $R$ is right pure-semisimple, whence is right Artinian. Thus (4) follows by Proposition 4.4.

**Proposition 4.8.** The next statements hold for a ring $R$:

1. If all right $R$-modules are min-pure projective, then $R$ is two-sided Köthe.
2. If $R$ is normal and all right $R$-modules are min-pure projective, then $R$ is an Artinian principal ideal ring.
3. If $R$ is commutative and all right $R$-modules are min-pure projective, then $R$ is a quasi-Frobenius serial ring.

**Proof.** (1). Our hypothesis implies that every right $R$-module is $RD$-projective and so $R$ is a right pure-semisimple $RD$-ring. Thus, [34, Proposition 6.5] implies that $R$ is two-sided Köthe.
(2) follows from (1) and [4, Theorem 3.1].

(3). By (2), R is a commutative Artinian principal ideal ring and so it is Artinian serial. In this case, R is quasi-Frobenius serial by Theorem 4.6. □

Recall by [8] that, a submodule C of a right R-module D is said to be neat in D provided that for any simple right R-module S, HomR(S, D) → HomR(S, D/C) is epic. Now, the following gives a particular answer to Proposition 4.7.

Corollary 4.9. Let R be a commutative indecomposable ring with J(R)^2 = 0. Then R is either a field or a quasi-Frobenius ring of composition length 2 if and only if all R-modules are min-pure projective.

Proof. There is nothing to prove if R is a field. If R is not a field, cl(R) = 2, whence R is local with unique simple and maximal ideal S such that (R/S) ≅ S. Thus any min-pure exact sequence is neat-exact, and so closed-exact by [19, Theorem 5]. On the other hand, since R is Artinian serial with J(R)^2 = 0, every closed exact sequence is splitting by [14, 13.5]. Thus, every min-pure exact sequence is splitting by Proposition 4.7, whence the necessity follows by Proposition 4.7. Conversely, R is quasi-Frobenius serial by Proposition 4.8. Since R is indecomposable and J(R)^2 = 0, either R is a field or R is a quasi-Frobenius ring of cl(R) = 2 by [3, Proposition 3.4]. □

5. Questions

For future research, we close the paper by giving next questions that are partially answered throughout the paper.

It was shown in [31, Theorem 2.4] that right perfectness of a ring R is equivalent to the fact that each RD-flat right R-module is RD-projective. Now, we have if every min-pure flat right R-module min-pure projective, then R is right perfect.

Q1: “Whether the converse of this fact is true or not?”

In Proposition 4.8, we know that if every right R-module is min-pure projective, then R is a right and left Köthe ring. Also, a commutative ring over which modules are min-pure projective is quasi-Frobenius serial. Finally, in Corollary 4.9, it is shown that over a commutative indecomposable ring with J(R)^2 = 0, every R-module is min-pure projective if and only if R is either a field or a quasi-Frobenius ring of composition length 2. Now,

Q2: “What is the class of (commutative) rings R for which every R-module is min-pure projective?”

Obviously every pure (resp. RD) exact sequence is min-pure, but not conversely (see Remark 2.5). Now,

Q3: “What is the class of rings R for which min-pure exact sequences are pure (resp. RD-pure)?”

There is no conflict of interest.

Acknowledgment. The research of the second author was in part supported by a grant from IPM (No. 1402160411). This research is partially carried out in the IPM-Isfahan Branch.

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