http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 4, Pages 1019–1033 (2023)
DOI:10.31801/cfsuasmas.1186649
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: October 12, 2022; Accepted: April 6, 2023

DEMONSTRATION OF THE STRENGTH OF STRONG CONVEXITY VIA JENSEN'S GAP

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Abstract. This paper demonstrates through a numerical experiment that utilization of strongly convex functions strengthens the bound presented for the Jensen gap in [4]. Consequently the improved result enables to present improvements in the bounds obtained for the Hölder and Hermite-Hadamard gaps and proposes such improvements in the results obtained for various entropies and divergences in information theory.

1. Introduction

Being a part of analysis, the field of mathematical inequalities in the sense of convexity has seen exponential growth in numerous domains of science, art, and technology [1, 3, 5, 7–9, 14–16, 22–24, 26, 27, 29–31, 36, 37, 40, 41, 43, 45]. Among these inequalities, the Jensen inequality is the most important inequality. Many other well-known used inequalities such as Young's, Hölder's, the arithmetic-geometric, the Hermite-Hadamard, and Minkowski's inequality etc can be obtained from this inequality by manipulating suitable substitutions. Furthermore, this inequality is comprehensively used in distinct areas of science and technology for example statistics [33], qualitative theory of differential and integral equations [32], engineering [17], economics [34], finance [10], information theory and coding [6, 25] etc. In addition, there are countless papers dealing with counterparts, refinements, generalizations, improvements and converse results of Jensen's inequality, (see, for instance [11–13, 19, 20, 39]). In fact, this inequality generalizes the classical notion of convexity and states that [28]:

Theorem 1. *If $\varphi : [\sigma_1, \sigma_2] \to \mathbb{R}$ is a convex function and $\vartheta_i \in [\sigma_1, \sigma_2]$, $k_i \geq 0$ for each $i \in \{1, 2, \dots, n\}$ with $\sum_{i=1}^{n} k_i := K_n > 0$, then for $\frac{1}{K_n}\sum_{i=1}^{n} k_i\vartheta_i := \bar{\vartheta}$, the*

2020 *Mathematics Subject Classification.* 26A51, 26D15, 68P30.

Keywords. Jensen inequality, strongly convex function, Hölder inequality, Hermite-Hadamard inequality, Taylor Formula.

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following inequality holds

$$\varphi\left(\bar{\vartheta}\right) \leq \frac{1}{K_n}\sum_{i=1}^{n} k_i\varphi\left(\vartheta_i\right).$$

In reference [2], the integral form of Theorem 1 can be seen, also here it is:

Theorem 2. *Assume that* $[\sigma_1, \sigma_2] \subset \mathbb{R}$ *and* $\xi_2, \xi_1 : [\rho_1, \rho_2] \to \mathbb{R}$ *are two functions with the condition that* $\xi_2(t) \in [\sigma_1, \sigma_2]$, $\forall\ t \in [\rho_1, \rho_2]$. *Further, assume that the function* $\varphi : [\sigma_1, \sigma_2] \to \mathbb{R}$ *is convex and* ξ_1, $\xi_2\xi_1$, $(\varphi \circ \xi_2).\xi_1$ *are integrable on* $[\rho_1, \rho_2]$. *Furthermore, suppose that* $\xi_1(t) \geq 0$ *for all* $t \in [\rho_1, \rho_2]$ *and* $\int_{\rho_1}^{\rho_2} \xi_1(t)dt := D > 0$, $\frac{1}{D}\int_{\rho_1}^{\rho_2} \xi_2(t)\xi_1(t)dt := \bar{\xi}$, *then*

$$\varphi\left(\bar{\xi}\right) \leq \frac{1}{D}\int_{\rho_1}^{\rho_2} (\varphi \circ \xi_2)(t)\xi_1(t)dt.$$

Following is the definition of a strongly convex function while the next theorem gives a criteria for checking the strong convexity of twice differentiable functions [44]:

Definition 1. *Let* $\varphi : [\sigma_1, \sigma_2] \to \mathbb{R}$ *be a function, then with modulus* $\lambda > 0$, *it is strongly convex, if the following inequality holds*

$$\varphi(\gamma\vartheta_1 + (1-\gamma)\vartheta_2) \leq \gamma\varphi(\vartheta_1) + (1-\gamma)\varphi(\vartheta_2) - \lambda\gamma(1-\gamma)(\vartheta_1 - \vartheta_2)^2,$$

for all $\vartheta_1, \vartheta_2 \in [\sigma_1, \sigma_2]$ *and* $\gamma \in [0, 1]$.

It is significant that every strongly convex function is convex but the converse is not true generally.

Theorem 3. *If the function* φ *is twice differentiable then it is strongly convex with modulus* $\lambda > 0$, *if and only if* $\varphi''(\vartheta_1) \geq 2\lambda$ *for all* $\vartheta_1 \in [\sigma_1, \sigma_2]$.

In this manuscript, we make use of the well-known Taylor formula and the concept of strong convexity to improve an existing bound for the Jensen gap. Many results may be found in the literature regarding Jensen's inequality for strongly convex functions (see for instance [21, 35, 38, 42]).
Following is the Taylor Formula [4]:

Theorem 4. *If* $\vartheta_2 \in [\sigma_1, \sigma_2] \subseteq \mathbb{R}$ *and* $\varphi : [\sigma_1, \sigma_2] \to \mathbb{R}$ *is a function, then for a point* $\mu \in [\sigma_1, \sigma_2]$, *the well-known Taylor's formula is given by*

$$\varphi(\vartheta_2) = \sum_{i=0}^{n-1} \frac{\varphi^{(i)}(\mu)}{i!}(\vartheta_2 - \mu)^i + \frac{1}{(n-1)!}\int_{\mu}^{\vartheta_2} \varphi^{(n)}(t)(\vartheta_2 - t)^{n-1}dt, \tag{1}$$

provided that φ^{n-1} *is absolutely continuous for natural number* n.
Setting $n = 2$ *in Equation (1), we get*

$$\varphi(\vartheta_2) = \varphi(\mu) + \varphi^{'}(\mu)(\vartheta_2 - \mu) + \int_{\mu}^{\vartheta_2} \varphi^{''}(t)(\vartheta_2 - t)dt. \tag{2}$$

2. Main Results

The following theorem actually gives the improvement in an existing bound for the classical Jensen gap through the concept of strong convexity:

Theorem 5. *Let $|\varphi''|$ be a strongly convex function with modulus λ for twice differentiable functions φ defined on $[\sigma_1, \sigma_2]$. Also, let $\mu, \vartheta_i \in [\sigma_1, \sigma_2]$, $k_i \geq 0$ for $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n k_i := K_n > 0$ and $\frac{1}{K_n}\sum_{i=1}^n k_i\vartheta_i := \overline{\vartheta}$, then*

$$\begin{aligned}&\left|\frac{1}{K_n}\sum_{i=1}^n k_i\varphi(\vartheta_i) - \varphi(\overline{\vartheta})\right| \\ &\quad \leq \frac{1}{K_n}\sum_{i=1}^n k_i(\vartheta_i-\mu)^2\left[\frac{|\varphi''(\mu)|}{3} + \frac{|\varphi''(\vartheta_i)|}{6} + \frac{\mu\lambda}{12}(\vartheta_i-\mu) - \frac{\vartheta_i\lambda}{12}(\vartheta_i-\mu)\right] \\ &\quad +(\overline{\vartheta}-\mu)^2\left[\frac{|\varphi''(\mu)|}{3} + \frac{|\varphi''(\overline{\vartheta})|}{6} + \frac{\mu\lambda}{12}(\overline{\vartheta}-\mu) - \frac{\overline{\vartheta}\lambda}{12}(\overline{\vartheta}-\mu)\right]. \end{aligned} \tag{3}$$

Proof. Using (2) in $\frac{1}{K_n}\sum_{i=1}^n k_i\varphi(\vartheta_i)$ and $\varphi(\overline{\vartheta})$, then some calculations lead towards the following identity

$$\frac{1}{K_n}\sum_{i=1}^n k_i\varphi(\vartheta_i) - \varphi(\overline{\vartheta}) = \frac{1}{K_n}\sum_{i=1}^n k_i\int_\mu^{\vartheta_i}(\vartheta_i - t)\varphi''(t)dt - \int_\mu^{\overline{\vartheta}}(\overline{\vartheta}-t)\varphi''(t)dt. \tag{4}$$

Inequality (5) can be acquired by taking absolute value of both sides of (4) and then applying triangle inequality

$$\begin{aligned}&\left|\frac{1}{K_n}\sum_{i=1}^n k_i\varphi(\vartheta_i) - \varphi(\overline{\vartheta})\right| \\ &\qquad = \left|\frac{1}{K_n}\sum_{i=1}^n k_i\int_\mu^{\vartheta_i}(\vartheta_i - t)\varphi''(t)dt - \int_\mu^{\overline{\vartheta}}(\overline{\vartheta}-t)\varphi''(t)dt\right| \\ &\qquad \leq \frac{1}{K_n}\sum_{i=1}^n k_i\int_\mu^{\vartheta_i}(\vartheta_i - t)\left|\varphi''(t)\right|dt + \int_\mu^{\overline{\vartheta}}(\overline{\vartheta}-t)\left|\varphi''(t)\right|dt. \end{aligned} \tag{5}$$

Change of the variable $t = \theta\mu + (1-\theta)\vartheta_i$ for $\theta \in [0,1]$ will give the following result

$$\int_\mu^{\vartheta_i}(\vartheta_i - t)|\varphi''(t)|dt = (\vartheta_i-\mu)\int_0^1(\vartheta_i\theta - \mu\theta)|\varphi''(\theta\mu + (1-\theta)\vartheta_i)|d\theta. \tag{6}$$

Using the strong convexity of $|\varphi''|$ in (6), the following result acquires

$$\begin{aligned}&\int_\mu^{\vartheta_i}(\vartheta_i - t)|\varphi''(t)|dt \\ &\quad \leq (\vartheta_i-\mu)\int_0^1(\vartheta_i\theta-\mu\theta)\times\Big(\theta|\varphi''(\mu)| + (1-\theta)|\varphi''(\vartheta_i)| - \mu\theta(1-\theta)(\mu-\vartheta_i)^2\Big)\,d\theta\end{aligned}$$

$$= (\vartheta_i - \mu)\int_0^1 (\vartheta_i\theta - \mu\theta)\Big[\theta|\varphi^{''}(\mu)| + |\varphi''(\vartheta_i)| - \theta|\varphi^{''}(\vartheta_i)|$$
$$-\lambda\theta(\mu-\vartheta_i)^2 + \lambda\theta^2(\mu-\vartheta_i)^2\Big]d\theta$$
$$= (\vartheta_i - \mu)\int_0^1 \Big(\vartheta_i\theta^2|\varphi^{''}(\mu)| - \mu\theta^2|\varphi^{''}(\mu)| + \vartheta_i\theta|\varphi''(\vartheta_i)| - \mu\theta|\varphi^{''}(\vartheta_i)|$$
$$-\vartheta_i\theta^2|\varphi''(\vartheta_i)| + \mu\theta^2|\varphi''(\vartheta_i)| - \vartheta_i\lambda\theta^2(\mu-\vartheta_i)^2 + \mu\lambda\theta^2(\mu-\vartheta_i)^2$$
$$+\vartheta_i\lambda\theta^3(\mu-\vartheta_i)^2 - \mu\lambda\theta^3(\mu-\vartheta_i)^2\Big)d\theta.$$
$$= (\vartheta_i - \mu)\left[(\vartheta_i-\mu)\left(\frac{|\varphi^{''}(\mu)|}{3} + \frac{|\varphi^{''}(\vartheta_i)|}{6}\right) + \frac{\mu\lambda}{12}(\vartheta_i-\mu)^2 - \frac{\vartheta_i\lambda}{12}(\vartheta_i-\mu)^2\right]$$
$$= (\vartheta_i-\mu)^2\left[\frac{|\varphi^{''}(\mu)|}{3} + \frac{|\varphi^{''}(\vartheta_i)|}{6} + \frac{\mu\lambda}{12}(\vartheta_i-\mu) - \frac{\vartheta_i\lambda}{12}(\vartheta_i-\mu)\right]. \tag{7}$$

Substitution of ϑ_i by $\overline{\vartheta}$ in (7) gives the following inequality

$$\int_\mu^{\overline{\vartheta}}(\overline{\vartheta}-t)|\varphi^{''}(t)|dt \le (\overline{\vartheta}-\mu)^2\left[\frac{|\varphi^{''}(\mu)|}{3} + \frac{|\varphi^{''}(\overline{\vartheta})|}{6} + \frac{\mu\lambda}{12}(\overline{\vartheta}-\mu) - \frac{\overline{\vartheta}\lambda}{12}(\overline{\vartheta}-\mu)\right]. \tag{8}$$

From (5), (7) and (8) we get (3). □

Example 1. *Let* $\varphi(t) = t^4, t \in [0,1]$*, then* $\varphi^{''}(t) = 12t^2 > 0,\ |\varphi^{''}|^{''}(t) = 24 \ge 2(12)$ *for all* $t \in [0,1]$*. Which show that* φ *is convex and with modulus* $\lambda = 12$ *the function* $|\varphi^{''}|$ *is strongly convex on* $[0,1]$*. Now, let* $k_1, k_2, k_3 = 0.2, 0.3, 0.5$ *and* $\upsilon_1, \upsilon_2, \upsilon_3 = 0.5, 0.25, 0.2$ *respectively, then applying these values in (3) , we get*

$$0 < \sum_{i=1}^{3} k_i\varphi(\vartheta_i) - \varphi\left(\sum_{i=1}^{3} k_i\vartheta_i\right) \le 6\mu^4 - 2.2\mu^3 + 0.0001\mu^2 + 0.0202 = T(\mu) \tag{9}$$

Here at $\mu = 0.275$, $T(\mu)$ will reach towards its minimum value, which is 0.0086 and hence from (9) we have

$$0 < \sum_{i=1}^{3} k_i\varphi(\vartheta_i) - \varphi\left(\sum_{i=1}^{3} k_i\vartheta_i\right) \le 0.0086. \tag{10}$$

For $|\varphi^{''}|$ as a convex function and for the above values of k_1, k_2, k_3 and $\upsilon_1, \upsilon_2, \upsilon_3$, the following result has been obtained in [4].

$$0 < \sum_{i=1}^{3} k_i\varphi(\vartheta_i) - \varphi\left(\sum_{i=1}^{3} k_i\vartheta_i\right) \le 0.0092. \tag{11}$$

It is easy to understand that inequality (10) gives better result than the results obtained in inequality (11) for the Jensen gap. Thus through this gap it is understandable that strongly convex functions actually strengthens the results.

Proposition 1. *Let $(a_1, \dots, a_n)$, and $(b_1, \dots, b_n)$ be two positive $n-$ tuples and $[\sigma_1, \sigma_2]$ be a positive interval. Then*

***1**. for $q > 1, p \in (1,2) \cup (3,4)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ with $\mu, \frac{\sum_{i=1}^n a_i b_i}{\sum_{i=1}^n b_i^q}, a_i b_i^{-\frac{q}{p}} \in [\sigma_1, \sigma_2]$ for $i = 1, \dots, n$, the following inequality holds*

$$\begin{aligned}
&\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} - \sum_{i=1}^n a_i b_i \\
&\quad \le \frac{p(p-1)}{24} \frac{1}{\sum_{i=1}^n b_i^q} \sum_{i=1}^n b_i^q \left(a_i b_i^{-\frac{q}{p}} - \mu\right)^2 \\
&\quad\times \left[8\mu^{p-2} + 4a_i^{p-2} b_i^{\frac{q}{p}-1} - (p-2)(p-3)\sigma_2^{p-4} \left(a_i b_i^{-\frac{q}{p}} - \mu\right)^2\right] \\
&\quad + \left(\frac{\sum_{i=1}^n a_i b_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q} - \mu\right)^2 \\
&\quad\times \left[8\mu^{p-2} + 4\left(\frac{\sum_{i=1}^n a_i b_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q}\right)^{p-2} - (p-2)(p-3)\sigma_2^{p-4} \left(\frac{\sum_{i=1}^n a_i b_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q} - \mu\right)^2\right]
\end{aligned} \tag{12}$$

***2**. If the statement of part **1** satisfied, then the following inequality holds but this time keeping the condition that $p > 4$*

$$\begin{aligned}
&\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} - \sum_{i=1}^n a_i b_i \\
&\quad \le \frac{p(p-1)}{24} \frac{1}{\sum_{i=1}^n b_i^q} \sum_{i=1}^n b_i^q \left(a_i b_i^{-\frac{q}{p}} - \mu\right)^2 \\
&\quad\times \left[8\mu^{p-2} + 4a_i^{p-2} b_i^{\frac{q}{p}-1} - (p-2)(p-3)\sigma_1^{p-4} (a_i b_i^{-\frac{q}{p}} - \mu)^2\right] \\
&\quad + \left(\frac{\sum_{i=1}^n a_i b_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q} - \mu\right)^2 \\
&\quad\times \left[8\mu^{p-2} + 4\left(\frac{\sum_{i=1}^n a_i b_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q}\right)^{p-2} - (p-2)(p-3)\sigma_1^{p-4} \left(\frac{\sum_{i=1}^n a_i b_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q} - \mu\right)^2\right]
\end{aligned} \tag{13}$$

Proof. **1**. Let $\varphi(t) = t^p, t \in [\sigma_1, \sigma_2]$ then $\varphi''(t) = p(p-1)t^{p-2} > 0$, $|\varphi''|''(t) = p(p-1)(p-2)(p-3)t^{p-4}$, which show that the function φ is convex, and for the given value of p, the function $|\varphi''|''(t)$ is decreasing while $|\varphi''|''(t) \ge 2\left(\frac{p(p-1)(p-2)(p-3)\sigma_2^{p-4}}{2}\right)$ for all $t \in [\sigma_1, \sigma_2]$. Therefore the function $|\varphi''|$ is strongly convex with $\lambda =$

$\frac{p(p-1)(p-2)(p-3)\sigma_2^{p-4}}{2}$, so using (3) for $\varphi(t)=t^p, k_i=b_i^q$ and $\vartheta_i=a_ib_i^{-\frac{q}{p}}$, we derive

$$\left(\left(\sum_{i=1}^n a_i^p\right)\left(\sum_{i=1}^n b_i^q\right)^{p-1}-\left(\sum_{i=1}^n a_ib_i\right)^p\right)^{\frac{1}{p}}$$
$$\leq \frac{p(p-1)}{24}\frac{1}{\sum_{i=1}^n b_i^q}\sum_{i=1}^n b_i^q\left(a_ib_i^{-\frac{q}{p}}-\mu\right)^2$$
$$\times\left[8\mu^{p-2}+4a_l^{p-2}b_i^{\frac{q}{p}-1}-(p-2)(p-3)\sigma_2^{p-4}\left(a_lb_i^{-\frac{q}{p}}-\mu\right)^2\right]$$
$$+\left(\frac{\sum_{i=1}^n a_ib_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q}-\mu\right)^2$$
$$\times\left[8\mu^{p-2}+4\left(\frac{\sum_{i=1}^n a_ib_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q}\right)^{p-2}-(p-2)(p-3)\sigma_2^{p-4}\left(\frac{\sum_{i=1}^n a_ib_i^{-\frac{q}{p}}}{\sum_{i=1}^n b_i^q}-\mu\right)^2\right] \quad (14)$$

By applying the inequality $\alpha^s-\beta^s\leq(\alpha-\beta)^s, \beta\in[0,\alpha], s\in[0,1]$ for $\alpha=\left(\sum_{i=1}^n a_i^p\right)\left(\sum_{i=1}^n b_i^q\right)^{p-1}, \beta=\left(\sum_{i=1}^n a_ib_i\right)^p$ and $s=\frac{1}{p}$ we obtain

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}-\left(\sum_{i=1}^n a_ib_i\right)\leq\left(\left(\sum_{i=1}^n a_i^p\right)\left(\sum_{i=1}^n b_i^q\right)^{p-1}-\left(\sum_{i=1}^n a_ib_i\right)^p\right)^{\frac{1}{p}} \quad (15)$$

From (14) and (15), we get (12)

2. We get $\lambda=\frac{p(p-1)(p-2)(p-3)\sigma_1^{p-4}}{2}$, as by applying the same proposed value of p, the function $|\varphi''|''$ become an increasing function. Now by applying the same method of part **1** the inequality (13) can be obtained. □

Here in the following theorem, we present a generalized version of Theorem 5.

Theorem 6. *Let $\varphi\in\mathbb{C}^2[\sigma_1,\sigma_2]$ such that $|\varphi''|$ is strongly convex function with modulus λ, and $\xi_1\geq 0$ integrable function such that $\xi_1:[\rho_1,\rho_2]\to\mathbb{R}$ with $\int_{\rho_1}^{\rho_2}\xi_1(t)dt=D>0$. Also, assuming the integrable function ξ_2 such that $\xi_2:[\rho_1,\rho_2]\to\mathbb{R}$ where $\xi_2(t)\in[\sigma_1,\sigma_2], \forall \quad t\in[\rho_1,\rho_2]$. Then the following inequality holds for $\overline{\xi}=\frac{1}{D}\int_{\rho_1}^{\rho_2}\xi_2(t)\xi_1(t)dt$ and $\mu\in[\sigma_1,\sigma_2]$.*

$$\left|\frac{1}{D}\int_{\rho_1}^{\rho_2}\xi_1(t)\left(\varphi\circ\xi_2\right)(t)dt-\varphi(\overline{\xi})\right|$$
$$\leq\frac{1}{D}\int_{\rho_1}^{\rho_2}\xi_1(t)\left(\xi_2(t)-\mu\right)^2$$

$$\times\left(\frac{\varphi''(\mu)}{3}+\frac{\varphi''(\xi_2(t))}{6}+\frac{\mu\lambda}{12}\left(\xi_2(t)-\mu\right)-\frac{\lambda\xi_2(t)}{12}(\xi_2(t)-\mu)\right)dt$$

$$+\left(\overline{\xi}-\mu\right)^2\left[\frac{\varphi''(\mu)}{3}+\frac{\varphi''(\overline{\xi})}{6}+\frac{\mu\lambda}{12}\left(\overline{\xi}-\mu\right)-\frac{\overline{\xi}\lambda}{12}(\overline{\xi}-\mu)\right]. \tag{16}$$

Proof. Using (2) in $\frac{1}{D}\int_{\rho_1}^{\rho_2}\xi_1(t)\left(\varphi\circ\xi_2\right)(t)dt$ and $\varphi(\overline{\xi})$, then some calculations lead towards the following identity.

$$\frac{1}{D}\int_{\rho_1}^{\rho_2}\xi_1(t)\left(\varphi\circ\xi_2\right)(t)dt-\varphi(\overline{\xi})$$

$$=\frac{1}{D}\int_{\rho_1}^{\rho_2}\left(\xi_1(t)\int_{\mu}^{\xi_2(t)}(\xi_2(t)-t)\varphi^{''}(t)dt\right)dt-\int_{\mu}^{\overline{\xi}}(\overline{\xi}-t)\varphi^{''}(t)dt. \tag{17}$$

From here, adopting the procedure of the proof of Theorem 5, we get the result. □

Remark 1. *The integral form of Proposition 1 may be shown as an application of Theorem 6.*

Corollary 1. *Let $\Phi:[\rho_1,\rho_2]\to\mathbb{R}$ be such that $|\Phi''|$ is strongly convex function with modulus λ and $\mu\in[\rho_1,\rho_2]$, then the following inequality can be obtained:*

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\Phi(t)dt-\Phi\left(\frac{\rho_1+\rho_2}{2}\right)\right|$$

$$\leq\frac{1}{6(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}(t-\mu)^2\Phi''(t)dt+\frac{1}{24}\left|\Phi^{''}\left(\frac{\rho_1+\rho_2}{2}\right)\right|(\rho_1+\rho_2-2\mu)^2$$

$$+\frac{\Phi^{''}(\mu)}{36}\left(7\rho_1^2+7\rho_2^2+10\rho_1\rho_2-24\mu\rho_1-24\mu\rho_2+24\mu^2\right)$$

$$+\frac{\mu\lambda}{48\times 96}\left(2\left(\rho_1+\rho_2\right)\left(\rho_1^2+\rho_2^2\right)-4\left(\rho_1^2+\rho_2^2+\rho_1\rho_2\right)-4\mu^3\right)+\left(\rho_1+\rho_2-2\mu\right)^3$$

$$-\frac{\lambda}{12(\rho_2-\rho_1)}\left(\frac{\rho_2^5-\rho_1^5}{5}-\frac{\mu^3\left(\rho_2^2-\rho_1^2\right)}{2}-\frac{3\mu(\rho_2^4-\rho_1^4)}{4}+\mu^2\left(\rho_2^3-\rho_1^3\right)\right)$$

$$-\frac{\lambda(\rho_1+\rho_2)}{192}(\rho_1+\rho_2-2\mu)^3. \tag{18}$$

Proof. By utilizing (16) for $\Phi=\varphi,[\sigma_1,\sigma_2]=[\rho_1,\rho_2]$ and $\xi_1(t)=1,\xi_2(t)=t$ for all $t\in[\rho_1,\rho_2]$, we get (18). □

3. Applications in Information Theory

In information theory, we study about the storage, quantification and communication of information about certain events in different aspects. In this field, various such events may be practiced through different divergences, distances or entropies for example Kullback-Leibler and Rényi-divergences, Hellinger distance, Shannon and Zipf-Mandelbrot entropies etc, which are special cases of the Csiszár divergence. In analysis, the Jensen inequality is one of the most important inequalities which produces various results for Csiszár divergence by manipulating suitable substitutions. Such practices are made in this section. Following is the Csiszár divergence functional [18]:

Definition 2 (Csiszár divergence)**.** *Let $[\sigma_1, \sigma_2] \subset \mathbb{R}$ and $T : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a function, then for $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$ and $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n_+$ such that $\frac{h_i}{z_i} \in [\sigma_1, \sigma_2]$, for $i = 1, \ldots, n$, the Csiszár divergence is defined as:*

$$C(\mathbf{h}, \mathbf{z}) = \sum_{i=1}^{n} z_i T\left(\frac{h_i}{z_i}\right).$$

Theorem 7. *Let for the function $T \in C^2[\sigma_1, \sigma_2]$, such that with modulus λ, $|T''|$ is strongly convex function. Also, let $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$ and $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n_+$, such that $\mu, \frac{\sum_{i=1}^n h_i}{\sum_{i=1}^n z_i}, \frac{h_i}{z_i} \in [\sigma_1, \sigma_2] \subset \mathbb{R}$, for $i = 1, 2, \ldots, n$ then,*

$$\begin{aligned}
&\left|\frac{1}{\sum_{i=1}^n z_i} C(\mathbf{h}, \mathbf{z}) - T\left(\frac{\sum_{i=1}^n h_i}{\sum_{i=1}^n z_i}\right)\right| \\
&\quad \le \frac{1}{\sum_{i=1}^n z_i} \sum_{i=1}^{n} z_i \left(\frac{h_i}{z_i} - \mu\right)^2 \\
&\quad \times \left[\frac{|T''(\mu)|}{3} + \frac{\left|T''\left(\frac{h_i}{z_i}\right)\right|}{6} + \frac{\mu\lambda}{12}\left(\frac{h_i}{z_i} - \mu\right) - \frac{\left(\frac{h_i}{z_i}\right)\lambda}{12}\left(\frac{h_i}{z_i} - \mu\right)\right] \\
&\quad + \left(\frac{\sum_{i=1}^n h_i}{\sum_{i=1}^n z_i} - \mu\right)^2 \left[\frac{|T''(\mu)|}{3} + \frac{\left|T''\left(\frac{\sum_{i=1}^n h_i}{\sum_{i=1}^n z_i}\right)\right|}{6} + \frac{\mu\lambda}{12}\left(\frac{\sum_{i=1}^n h_i}{\sum_{i=1}^n z_i} - \mu\right)\right. \\
&\quad \left. - \frac{\left(\frac{\sum_{i=1}^n h_i}{\sum_{i=1}^n z_i}\right)\lambda}{12}\left(\frac{\sum_{i=1}^n h_i}{\sum_{i=1}^n z_i} - \mu\right)\right].
\end{aligned} \tag{19}$$

Proof. Utilizing $\varphi = T, \vartheta_i = \frac{h_i}{z_i}$ and $k_i = \frac{z_i}{\sum_{i=1}^n z_i}$ in (3) we obtain (19). □

Definition 3 (Rényi-divergence)**.** *For two positive probability distributions $\mathbf{h} = (h_1, \ldots, h_n), \mathbf{z} = (z_1, \ldots, z_n)$ and $\eta \ge 0, \eta \ne 1$, the Rényi-divergence is defined as:*

$$R(\mathbf{h}, \mathbf{z}) = \frac{1}{\eta - 1} \log\left(\sum_{i=1}^{n} h_i^{\eta} z_i^{1-\eta}\right).$$

Corollary 2. *Let* $\mathbf{h} = (h_1, \ldots, h_n), \mathbf{z} = (z_1, \ldots, z_n)$ *be two positive probability distributions and* $\eta > 1$ *such that* $\mu, \sum_{i=1}^{n} z_i \left(\frac{h_i}{z_i}\right)^{\eta}, \left(\frac{h_i}{z_i}\right)^{\eta-1} \in [\sigma_1, \sigma_2] \subseteq \mathbb{R}$ *for* $i = 1, 2 \ldots, n$ *then*

$$R(\mathbf{h}, \mathbf{z}) - \frac{1}{\eta - 1} \sum_{i=1}^{n} h_i \log \left(\frac{h_i}{z_i}\right)^{\eta-1}$$

$$\leq \frac{1}{\eta - 1} \sum_{i=1}^{n} h_i \left(\frac{h_i}{z_i} - \mu\right)^2 \left[\frac{1}{3\mu_2} + \frac{1}{6}\left(\frac{z_i}{h_i}\right)^{2(\eta-1)} - \frac{1}{4\sigma_2^4}\left(\left(\frac{h_i}{z_i}\right)^{\eta-1} - \mu\right)^2\right]$$

$$+ \left(\sum_{i=1}^{n} h_i^{\eta} z_i^{\eta-1} - \mu\right)^2 \left[\frac{1}{3\mu^2} + \frac{1}{6\left(\sum_{i=1}^{n} h_i^{\eta} v_i^{1-\eta}\right)^2} - \frac{1}{4\sigma_2^4}\left(\sum_{i=1}^{n} h_i^{\mu} z_i^{1-\eta} - \mu\right)^2\right]. \tag{20}$$

Proof. Let $\varphi(t) = -\frac{1}{\eta-1}\log(t), t \in [\sigma_1, \sigma_2]$, then $\varphi''(t) = \frac{1}{(\eta-1)t^2}$ and $|\varphi''|''(t) = \frac{6}{(\eta-1)t^4} \geq 2\left(\frac{3}{(\eta-1)\sigma_2^4}\right)$ which implies that φ is convex and with $\lambda = \frac{3}{(\eta-1)\sigma_2^4}$, $|\varphi''|$ is strongly convex. Thus we get (20) by applying (3) for $\varphi(t) = -\frac{1}{\eta-1}\log(t), k_i = h_i$ and $\vartheta_i = \left(\frac{h_i}{z_i}\right)^{\eta-1}$. □

Definition 4 (Shannon entropy)**.** *Let* $\mathbf{z} = (z_1, \ldots, z_n)$ *be a positive probability distribution , then the information divergence or shannon entropy is define as:*

$$S(\mathbf{z}) = -\sum_{i=1}^{n} z_i \log(z_i).$$

Corollary 3. *Suppose a positive probability distribution* $\mathbf{z} = (z_1, \ldots, z_n)$ *and* $\mu, \frac{1}{z_i} \in [\sigma_1, \sigma_2] \subseteq \mathbb{R}^+$ *for* $i = 1, \ldots, n$. *then*

$$\log n - S(\mathbf{z}) \leq \sum_{i=1}^{n} z_i \left(\frac{1}{z_i} - \mu\right)^2 \left[\frac{1}{3\mu^2} + \frac{z_i^2}{6} - \frac{1}{4\sigma_2^4}\left(\mu - \frac{1}{z_i}\right)^2\right]$$

$$+(n - \mu)^2 \left[\frac{1}{3\mu^2} + \frac{1}{6n^2} - \frac{1}{4\sigma_2^4}(n - \mu)^2\right]. \tag{21}$$

Proof. Let $\varphi(t) = -\log t, t \in [\sigma_1, \sigma_2]$ then $\varphi''(t) = \frac{1}{t^2} > 0$ and $|\varphi''|''(t) = \frac{6}{t^4} \geq 2\frac{3}{\sigma_2^4}$, which presented that the function φ is convex and with $\lambda = \frac{3}{\sigma_2^4}$, $|\varphi''|$ is strongly convex . Therefore applying (19) for $\varphi(t) = -\log(t), (h_1, \ldots, h_n) = (1, \ldots, 1)$, we get (21). □

Definition 5 (Kullback-Leibler divergence)**.** *For two positive probability distributions,* $\mathbf{h} = (h_1, \ldots, h_n)$ *and* $\mathbf{z} = (z_1, \ldots, z_n)$*, the Kullback-Leibler divergence is defined as:*

$$KL(\mathbf{h}, \mathbf{z}) = \sum_{i=1}^{n} h_i \log\left(\frac{h_i}{z_i}\right).$$

Corollary 4. *Consider that* $[\sigma_1, \sigma_2] \subseteq \mathbb{R}^+$ *and* $\mathbf{h} = (h_1, \ldots, h_n), \mathbf{z} = (z_1, \ldots, z_n)$ *are two positive probability distributions, with* μ, $\frac{h_i}{z_i} \in [\sigma_1, \sigma_2]$*, then*

$$KL(\mathbf{h}, \mathbf{z}) \leq \sum_{i=1}^{n} z_i \left(\frac{h_i}{z_i} - \mu\right)^2 \left[\frac{1}{3\mu} + \frac{z_i}{6h_i} - \frac{1}{12\sigma_2^3}\left(\frac{h_i}{z_i} - \mu\right)^2\right]$$
$$+(\mu - 1)^2 \left[\frac{1}{3\mu} - \frac{1}{12\sigma_2^3}(\mu - 1)^2\right]. \tag{22}$$

Proof. Let $\varphi(t) = t \log t, t \in [\sigma_1, \sigma_2]$, then $\varphi''(t) = \frac{1}{t} > 0$, which conclude that φ is convex function. Also, $|\varphi''|''(t) = \frac{2}{t^3} \geq 2\left(\frac{1}{\sigma_2^3}\right)$, which implies, with $\lambda = \frac{1}{\sigma_2^3} > 0$ the function $|\varphi''|$ is strongly convex. Thus we get (22) by applying (19) for $\varphi(t) = t \log t$. □

Definition 6 (Bhattacharyya coefficient)**.** *If* $\mathbf{z} = (z_1, \ldots, z_n)$ *and* $\mathbf{h} = (h_1, \ldots, h_n)$ *are two positive probability distributions, then the mathematical form of Bhattacharyya coefficient is given by:*

$$B(\mathbf{h}, \mathbf{z}) = \sum_{i=1}^{n} \sqrt{h_i z_i}.$$

Corollary 5. *Consider that* $\mathbf{h} = (h_1, \ldots, h_n)$ *and* $\mathbf{z} = (z_1, \ldots, z_n)$ *are some positive probability distributions with the following conditions* μ, $\frac{h_i}{z_i} \in [\sigma_1, \sigma_2]$ *where* $[\sigma_1, \sigma_1] \subseteq \mathbb{R}^+$ *and* $i = 1, 2, \ldots, n$ *then*

$$1 - B(\mathbf{h}, \mathbf{z}) \leq \sum_{i=1}^{n} z_i \left(\frac{h_i}{z_i} - \mu\right)^2 \left[\frac{1}{12\mu^{\frac{3}{2}}} + \frac{1}{24}\left(\frac{z_i}{h_i}\right)^{\frac{3}{2}} - \frac{5}{128\sigma_2^{\frac{7}{2}}}\right]$$
$$+\big(1 - \mu\big)^2 \left[\frac{1}{24} + \frac{1}{12\mu^{\frac{3}{2}}} - \frac{5}{128\sigma_2^{\frac{7}{2}}}\big(1 - \mu\big)^2\right]. \tag{23}$$

Proof. Let $\varphi(t) = -\sqrt{t}, t \in [\sigma_1, \sigma_2]$ then $\varphi''(t) = \frac{1}{4t^{\frac{3}{2}}} > 0$ and $|\varphi''|''(t) = \frac{15}{16t^{\frac{7}{2}}} \geq 2\left(\frac{15}{32\sigma_2^{\frac{7}{2}}}\right)$. Which show that the function φ is convex and with $\lambda = \frac{15}{32\sigma_2^{\frac{7}{2}}}$, $|\varphi''|$ is strongly convex. Therefore by putting $\varphi(t) = -\sqrt{t}$, in (19), we can get (23). □

Definition 7 (Hellinger distance)**.** *For two positive probability distributions* $\mathbf{z} = (z_1, \dots, z_n)$*,* $\mathbf{h} = (h_1, \dots, h_2)$*, the Hellinger distance is define as:*

$$H(\mathbf{h}, \mathbf{z}) = \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{h_i} - \sqrt{z_i}\right)^2.$$

Corollary 6. *let* $\mathbf{h} = (h_1, \cdots, h_n)$ *and* $\mathbf{z} = (z_1, \cdots, z_n)$ *be two positive probability distributions, such that* μ*,* $\frac{h_i}{z_i} \in [\sigma_1, \sigma_2]$*, where* $[\sigma_1, \sigma_1] \subseteq \mathbb{R}^+$ *and* $i = 1, 2, \dots, n$ *then*

$$H(\mathbf{h}, \mathbf{z}) \le \sum_{i=1}^{n} z_i\left(\frac{h_i}{z_i} - \mu\right)^2\left[\frac{1}{12\mu^{\frac{3}{2}}} + \frac{1}{24\left(\frac{h_i}{z_i}\right)^{\frac{3}{2}}} - \frac{5}{128\sigma_2^{\frac{7}{2}}}\right]$$
$$+ (1-\mu)^2\left[\frac{1}{24} + \frac{1}{12\mu^{\frac{3}{2}}} - \frac{5}{128\sigma_2^{\frac{7}{2}}}(1-\mu)^2\right]. \tag{24}$$

Proof. Let $\varphi(t) = \frac{1}{2}(1-\sqrt{t})^2$, $t \in [\sigma_1, \sigma_2]$, then $\varphi''(t) = \frac{1}{4t^{\frac{3}{2}}} > 0$ and $|\varphi''|''(t) = \frac{15}{16t^{\frac{7}{2}}} \ge 2\left(\frac{15}{32\sigma_2^{\frac{7}{2}}}\right)$. This shows that φ is convex and $|\varphi''|$ is strongly convex with modulus $\lambda = \frac{15}{32\sigma_2^{\frac{7}{2}}}$. Hence we can obtained (24) by utilizing (19) for $\varphi(t) = \frac{1}{2}(1-\sqrt{t})^2$. □

Definition 8 (Triangular discrimination)**.** *Assume that* $\mathbf{h} = (h_1, \dots, h_n)$ *and* $\mathbf{z} = (z_1, \dots, z_n)$ *are two positive probability distributions then the mathematical formula for the Triangular discrimination is given by:*

$$T_d(\mathbf{h}, \mathbf{z}) = \sum_{i=1}^{n}\frac{(h_i - z_i)^2}{(h_i + z_i)}.$$

Corollary 7. *let* $\mathbf{z} = (z_1, \dots, z_n)$ *and* $\mathbf{h} = (h_1, \dots, h_n)$ *be two positive probability distributions. Further assume that* μ*,* $\frac{h_i}{z_i} \in [\sigma_1, \sigma_2]$ *where* $[\sigma_1, \sigma_1] \subseteq \mathbb{R}^+$ *and* $i = 1, 2, \dots, n$ *then*

$$T_d(\mathbf{h}, \mathbf{z}) \le \sum_{i=1}^{n} z_i\left(\frac{h_i}{z_i} - \mu\right)^2\left[\frac{8}{3(\mu+1)^3} + \frac{4z_i^3}{3(h_i+z_i)^3} - \frac{4}{(\sigma_2+1)^5}\left(\frac{h_i}{z_i} - \mu\right)^2\right]$$
$$+ (1-\mu)^2\left[\frac{8}{3(\mu+1)^3} - \frac{4}{(\sigma_2+1)^5}(1-\mu)^2\right]. \tag{25}$$

Proof. Let $\varphi(t) = \frac{(t-1)^2}{(t+1)}, t \in [\sigma_1, \sigma_2]$, then $\varphi''(t) = \frac{8}{(t+1)^3} > 0$ and $|\varphi''|''(t) = \frac{96}{(t+1)^5} \ge 2\left(\frac{48}{(\sigma_2+1)^5}\right)$. This presented that φ is a convex function and with $\lambda = \frac{48}{(\sigma_2+1)^5}$, $|\varphi''|$ is a strongly convex function. Therefore using (19) for such values we may deduce (25). □

4. Concluding Remarks

In fact, among the mathematical inequalities for convex functions, the Jensen inequality is the most powerful inequality whose gap can be utilized for various purposes specially in the approximation of certain parameters in optimization problems. In this regard, better estimates for its gap can be used to obtain better results. The strongly convex functions are some tools to strengthen such estimates. In this paper, it is demonstrated through a numerical experiment that replacing convex functions by strongly convex functions actually strengthens the bound presented for the Jensen gap in [4]. Similarly the improved result enabled us to present improvements in the bounds obtained for the Hölder and Hermite-Hadamard gaps and proposed such improvements in the results obtained for various entropies and divergences in information theory. The idea presented in the paper, further motivates the mathematicians to establish such results in future.

Author Contribution Statements Both the authors contributed equally and significantly in writing this article. The authors read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

Acknowledgements We are thankful to the respected reviewers for their valuable suggestions

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