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Analysis of Inverse Euler-Bernoulli Equation with periodic boundary conditions

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Abstract. In this study, which aims to solve the inverse problem of a linear Euler-Bernoulli equation, the boundary condition has been periodically defined and integral overdetermination conditions. The conditions of the data used in the generalized Fourier method used to solve the problem have regularity and consistency.

1. Introduction

$T(t,x)$ is the displacement at time t and at position x , $o(x)$ is the bending stiffness, and $k(x) > 0$ is the linear mass on the Euler-Bernoulli problem . The behavior of an unloaded thin beam moving transversely can be described using the fourth-order partial differential equation:

$$k(x)(\partial^2 T)/(\partial t^2) + o(x)(\partial^2 T)/(\partial x^4) = 0, t > 0, 0 < x < L. \tag{1}$$

[1] studied isospectral properties and inhomogeneous variants of this equation. [2] used the Lie symmetry approach. [3] tried to solve it with Cartan's equivalence method. [4] obtained exact equivalence transformations by dealing with this problem initially [3] with some ambiguous functions. [5] investigated the transverse vibrations of a beam moving with time using the symmetry method and obtained approximate solutions for the problem.

If elastic modulus, area of inertia, mass per unit length, transverse displacement position x at time t and applied load are described as $E, l, \alpha, T(x,t)$ and f respectively, the PDE which is fourth-order can be given as below [6];

$$(EIT_{xx})_{xx} + \alpha T_{tt} = f(x,t), t > 0, 0 < x < L. \tag{2}$$

$$T_{xxxx} + T_{tt} = f(x,t), t > 0, 0 < x < L. \tag{3}$$

where E, I, α as constants [7].

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Vibration, buckling and dynamic behavior, which are frequently encountered in many fields from engineering to medicine, can be defined in a much broader way with the Euler-Bernoulli equations [8–16]. Many studies have been conducted on linear and quasi-linear equations and their applications in different fields [8–13].

Since periodic boundary conditions are encountered in many events, especially heat transfer, it has many application areas [14–16]. The existence and uniqueness of the solution of the problem are proved in section 2 using the Fourier and iteration methods. The stability of the method used to solve the problem is shown in section 3. Finally, a numerical procedure for solving the problem is presented in section 4.

Let $T > 0$ be fixed number and denote by $\Omega := \{0 < x < \pi, 0 < t < T\}$.

In case it is desired to obtain the function pairs $\{q(t), T(x,t)\}$ that will provide the equation given by Equation 4:

$$\frac{\partial^2 T}{\partial t^2} + \frac{\partial^4 T}{\partial x^4} = q(t) f(x,t),\ (x,t) \in \Omega \tag{4}$$

$$\begin{aligned} T(0,t) &= T(\pi,t) \\ T_x(0,t) &= T_x(\pi,t) \\ T_{xx}(0,t) &= T_{xx}(\pi,t) \\ T_{xxx}(0,t) &= T_{xxx}(\pi,t),\ t \in [0,T] \end{aligned} \tag{5}$$

$$T(x,0) = \varphi(x), T_t(x,0) = \psi(x),\ x \in [0,\pi] \tag{6}$$

$$H(t) = \int_0^\pi x T(x,t) dx,\ t \in [0,T] \tag{7}$$

The known functions $f(x,t)$,$\varphi(x)$, $\psi(x)$ and $H(t)$ expressed in equations (4)-/7) are known functions and are always continuous and have positive values. gets. The functions $u(x,t)$ and $r(t)$ are unknown. In the heat dissipation in a thin rod, studies have been made to obtain the total amount of heat dissipated [**?**].

Definition 1.1. *$\{q(t), T(x,t)\}$ is called the inverse problem .*

Definition 1.2. *$v(x,t) \in C(\overline{\Omega})$ is a test function and satisfies these conditions;*

$$v(x,T) = v_t(x,T) = 0, v(0,t) = v(\pi,t), v_x(0,t) = v_x(\pi,t), v_{xx}(0,t) = v_{xx}(\pi,t), v_{xxx}(0,t,) = v_{xxx}(\pi,t), t\epsilon\,[0,T]\,.$$

Definition 1.3. *$u(x,t) \in C(\overline{\Omega})$ can be called as generalized equation. Following equation can be obtained with the generalized equation.:*

$$\int_0^T \int_0^\pi \left(\left\{ \frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} \right\} u - r(t) f v \right) dxdt - \int_0^\pi v(x,0)\psi(x)dx + \int_0^\pi v_t(x,0)\varphi(x)dx = 0.$$

Nomenclature
$\varphi(x), \psi(x)$ Initial condition
$q(t)$ Unknown coefficient
$H(t)$ Energy
$T(x,t)$ Temperature distribution
$f(x,t)$ Source function
$T_0(t), T_{ck}(t), T_{sk}(t)$ Fourier coefficients
M_1, M_2, M_3 constants
$F(t)$ Continous function
$K(t,\tau)$ Kernel function
$\Omega := \{0 < x < \pi,\ 0 < t < T\}$ Domain of x, t

2. Solution of this problem

Let us look for solution of (1)-(4) in the form:

$$T(x,t) = \frac{u_0(t)}{2} + \sum_{k=1}^{\infty} \left(T_{ck}(t)\cos 2kx + T_{sk}(t)\sin 2kx\right)$$

The Fourier coefficients in Equation 8 can be obtained by applying the standard procedure of the Fourier method:

$$\begin{aligned} T(x,t) &= \frac{1}{2}\left[\varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)q(\tau)f(\xi,\tau)d\xi d\tau\right] \\ &+ \sum_{k=1}^{\infty}\left[\varphi_{ck}\cos(2k)^2 t + \frac{\psi_{ck}}{\pi(2k)^2}\sin(2k)^2 t\right]\cos 2kx \\ &+ \sum_{k=1}^{\infty}\left[\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi f(\xi,\tau,T)q(\tau)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau\right]\cos 2kx \\ &+ \sum_{k=1}^{\infty}\left[\varphi_{sk}\cos(2k)^2 t + \frac{\psi_{sk}}{\pi(2k)^2}\sin(2k)^2 t\right]\sin 2kx \\ &+ \sum_{k=1}^{\infty}\left[\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi f(\xi,\tau,T)q(\tau)\sin(2k)^2(t-\tau)\sin 2k\xi d\xi d\tau\right]\sin 2kx \end{aligned} \tag{8}$$

Definition 2.1. *The pair* $\{q(t), T(x,t)\} \in C\left(\overline{\Omega}\right)$ *is called the classical solution of the problems (1)-(4) .*

Theorem 2.2. *Suppose that the following conditions hold:*

(A1) $H(t) \in C^2[0,T]$,
(A2) $\varphi(x) \epsilon C^3[0,\pi], \psi(x)\epsilon C^1[0,\pi]$,
$\varphi(0) = \varphi(\pi), \varphi'(0) = \varphi'(\pi), \varphi''(0) = \varphi''(\pi), \psi(0) = \psi(\pi), \psi'(0) = \psi'(\pi)$,
(A3) $f(x,t)\epsilon C\left(\overline{\Omega}\right), f(0,t) = f(\pi,t), f_x(0,t) = f_x(\pi,t)$,
(A4) $\int_0^\pi xf(x,t)dx \neq 0, \forall x\epsilon[0,\pi]$
then the solution of system (1)-(4) has unique solutions.

Proof. The assumptions $\varphi(0)=\varphi(\pi)$, $\varphi^{'}(0)=\varphi^{'}(\pi), \psi(0)=\psi(\pi)$, $f(0,t)=f(\pi,t)$, are verify for the representation (7) of the solution $T(x,t)$. Further, under $\varphi(x)\epsilon C^3[0,\pi]$, $\psi(x)\epsilon C[0,\pi]$, $f(x,t)\,\epsilon C\left(\overline{\Omega}\right)$, the series (7) converge uniformly in $\overline{\Omega}$ since their majorizing sums are absolutely convergent. Under the conditions, since the majorizing sum of the t-partial derivative series are convergent, $T_t(x,t), T_{tt}(x,t)$ i is continuous in $\overline{\Omega}$. because the majorizing sum of t-partial derivative series is absolutely convergent under the conditions $\varphi(0)=\varphi(\pi)$, $\varphi^{'}(0)=\varphi^{'}(\pi), \varphi^{''}(0)=\varphi^{''}(\pi), \psi(0)=\psi(\pi), \psi^{'}(0)=\psi^{'}(\pi)$, $f(0,t)=f(\pi,t), f_x(0,t)=f_x(\pi,t)$ in $\overline{\Omega}$.

From the (5) and under the condition $(A1)$ to obtain:

$$H^{''}(t)=\int_0^{\pi} xT_{tt}(x,t)dx \tag{9}$$

The formulas (5)-(6) yield the following equation:

$$q(t)=\frac{H^{''}(t)-\pi\sum_{k=1}^{\infty}(2k)^3\left\{\varphi_{sk}\cos(2k)^2t+\frac{\psi_{sk}}{(2k)^2}\sin(2k)^2t+\frac{1}{(2k)^2}\int_0^t f_{sk}(\tau)q(\tau)\sin(2k)^2(t-\tau)d\tau\right\}}{\int_0^{\pi} xf(x,t)dx}$$

From The second kind Volterra integral equation:

$$q(t)=F(t)+\int_0^t K(t,\tau)q(\tau)d\tau, t\epsilon[0,T] \tag{10}$$

$$F(t)=\frac{H^{''}(t)-\pi\sum_{k=1}^{\infty}(2k)^3\,\varphi_{sk}\cos(2k)^2t-\pi\sum_{k=1}^{\infty}(2k)\,\psi_{sk}\sin(2k)^2t}{\int_0^{\pi} xf(x,t)dx}, \tag{11}$$

$$K(t,\tau)=\frac{-\pi\sum_{k=1}^{\infty}(2k)\int_0^t f_{sk}(\tau)q(\tau)\sin(2k)^2(t-\tau)d\tau}{\int_0^{\pi} xf(x,t)dx}. \tag{12}$$

Let $F(t)$ and the kernel $K(t,\tau)$ are continuous functions:

$$F(t)=\frac{H^{''}(t)-\pi\sum_{k=1}^{\infty}(2k)^3\left(\int_0^{\pi}\varphi(\xi)\sin 2k\xi d\xi\right)\cos(2k)^2t-\pi\sum_{k=1}^{\infty}(2k)\left(\int_0^{\pi}\psi(\xi)\sin 2k\xi d\xi\right)\sin(2k)^2t}{\int_0^{\pi} xf(x,t)dx},$$

we applying partial integration method for convergence ,

$\varphi_{sk}=\frac{2}{\pi}\int_0^{\pi}\varphi(\xi)\sin 2k\xi d\xi=-\frac{1}{2k}\varphi_{ck}^{'}=\frac{1}{(2k)^2}\varphi_{sk}^{''}=\frac{-1}{(2k)^3}\varphi_{ck}^{'''}$

$\psi_{sk}=\frac{2}{\pi}\int_0^{\pi}\psi(\xi)\sin 2k\xi d\xi=-\frac{1}{2k}\psi_{ck}^{'}$

$$F(t)=\frac{H^{''}(t)+\pi\sum_{k=1}^{\infty}(2k)^3\,\frac{1}{(2k)^3}\varphi_{ck}^{'''}\cos(2k)^2t+\pi\sum_{k=1}^{\infty}(2k)\,\frac{1}{2k}\psi_{ck}^{'}\sin(2k)^2t}{\int_0^{\pi} xf(x,t)dx},$$

$$F(t) = \frac{H^{''}(t) + \pi \sum_{k=1}^{\infty} \varphi_{ck}^{'''} \cos(2k)^2 t + \pi \sum_{k=1}^{\infty} \psi_{ck}^{'} \sin(2k)^2 t}{\int_0^{\pi} x f(x,t)dx},$$

$$|F(t)| \leq \frac{\left|H^{''}(t)\right| + \pi \sum_{k=1}^{\infty} \left|\varphi_{ck}^{'''}\right| + \pi \sum_{k=1}^{\infty} \left|\psi_{ck}^{'}\right|}{\left|\int_0^{\pi} x f(x,t)dx\right|}$$

$$|F(t)| \leq \frac{2\left(\left|H^{''}(t)\right| + \pi \sum_{k=1}^{\infty} \left|\varphi_{ck}^{'''}\right| + \pi \sum_{k=1}^{\infty} \left|\psi_{ck}^{'}\right|\right)}{M\pi^2}$$

Taking maximum both of sides

$$\|F(t)\| \leq \frac{2\left(\left\|H^{''}(t)\right\| + \pi \sum_{k=1}^{\infty} \left\|\varphi_{ck}^{'''}\right\| + \pi \sum_{k=1}^{\infty} \left\|\psi_{ck}^{'}\right\|\right)}{M\pi^2}.$$

$$K(t,\tau) = \frac{-\pi \sum_{k=1}^{\infty} (2k) \int_0^{t} f_{sk}(\tau) q(\tau) \sin(2k)^2 (t-\tau) d\tau}{\int_0^{\pi} x f(x,t)dx}$$

$f_{sk} = \frac{2}{\pi} \int_0^{\pi} f(\xi,\tau) \sin 2k\xi d\xi = -\frac{1}{2k} \left(f_{ck}\right)_x$

$$K(t,\tau) = \frac{-\pi \sum_{k=1}^{\infty} (2k) \int_0^{t} \frac{2}{\pi} \int_0^{\pi} f(\xi,\tau) q(\tau) \sin(2k)^2 (t-\tau) \sin 2k\xi \, d\xi d\tau}{\int_0^{\pi} x f(x,t)dx}$$

$$K(t,\tau) = \frac{\pi \sum_{k=1}^{\infty} \int_0^{t} \frac{(2k)}{(2k)} \left(f_{ck}\right)_x \sin(2k)^2 (t-\tau) d\tau}{\int_0^{\pi} x f(x,t)dx}$$

$$|K(t,\tau)| \leq \frac{\pi \sum_{k=1}^{\infty} \left|\left(f_{ck}\right)_x\right| \left|\int_0^{t} \sin(2k)^2 (t-\tau) d\tau\right|}{\left|\int_0^{\pi} x f(x,t)dx\right|}$$

Taking maximum both of sides

$$\|K(t,\tau)\| \leq \frac{2 \sum_{k=1}^{\infty} \left\|\left(f_{ck}\right)_x\right\| \|T\|}{M\pi}$$

Under the assumption (A1)-(A2) and according to Weierstrass M test the function $F(t)$ and the kernel $K(t,\tau)$ are continuous functions The unique solution of the inverse problem (1)-(4) according to Volterra Theorem. □

3. Stability of Problem

The result in the theorem given below is valid for solving problems from equality (1) to (4).

Theorem 3.1. $\Phi = \{\varphi, \psi,\ H,\ f\}$ *satisfy the assumptions (A1)-(A4) of theorem 1 then the solution* (u, r) *of the problem (1)-(4) depends continuously upon the data* f, φ, ψ, H.

Proof. Suppose that there exist positive constants M_i, $i = 1, 2, 3$.

Let us denote $\|\Phi\| = (\|H\|_{C^1[0,T]} + \|\varphi\|_{C^1[0,\pi]} + \|\psi\|_{C[0,\pi]} + \|f\|_{C(\overline{\Omega})})$. Let (u, r) and $(\overline{u}, \overline{r})$ be solutions of inverse problems (1)-(4) corresponding to the data $\Phi = \{\varphi, \psi,\ H,\ f\}$ and $\overline{\Phi} = \left\{\overline{\varphi}, \overline{\psi}, \overline{H}, \overline{E}, \overline{f}\right\}$ respectively.

$$F(t) - \overline{F(t)} = \frac{H^{''}(t) - \overline{H^{''}(t)} + \pi \sum\limits_{k=1}^{\infty} \left(\varphi_{ck}^{'''} - \overline{\varphi_{ck}^{'''}}\right) \cos(2k)^2 t + \pi \sum\limits_{k=1}^{\infty} \left(\psi_{ck}^{'} - \overline{\psi_{ck}^{'}}\right) \sin(2k)^2 t}{\int\limits_0^{\pi} x f(x,t) dx}$$

Equation (13) can be obtained with the maximum of both sides of this equation:

$$\left\|F - \overline{F}\right\| \leq \frac{2}{\pi^2 M} \left\{ \left\|H^{'}(t) - \overline{H^{'}(t)}\right\| + \pi \sum_{k=1}^{\infty} \left\|\varphi_{ck}^{'''} - \overline{\varphi_{ck}^{'''}}\right\| + \left\|\psi_{ck}^{'} - \overline{\psi_{ck}^{'}}\right\| \right\}. \tag{13}$$

$$K(t, \tau) = \frac{\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t (f_{ck})_x \sin(2k)^2 (t - \tau) d\tau}{\int\limits_0^{\pi} x f(x,t) dx}$$

$$K - \overline{K} = \frac{\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t (f_{ck})_x \sin(2k)^2 (t - \tau) d\tau}{\int\limits_0^{\pi} x f(x,t) dx} - \frac{\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \overline{(f_{ck})_x} \sin(2k)^2 (t - \tau) d\tau}{\int\limits_0^{\pi} x \overline{f(x,t)} dx}$$

$$K - \overline{K} = \frac{\left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t (f_{ck})_x \sin(2k)^2 (t - \tau) d\tau\right) \left(\int\limits_0^{\pi} x \overline{f(x,t)} dx\right) - \left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \overline{(f_{ck})_x} \sin(2k)^2 (t - \tau) d\tau\right) \left(\int\limits_0^{\pi} x f(x,t) dx\right)}{\left(\int\limits_0^{\pi} x f(x,t) dx\right) \left(\int\limits_0^{\pi} x \overline{f(x,t)} dx\right)}$$

$$\begin{aligned} K-\overline{K} \; = \; & \frac{\left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t (f_{ck})_x \sin(2k)^2(t-\tau)d\tau\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)}{\left(\int\limits_0^\pi xf(x,t)dx\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)} \\ & - \frac{\left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \overline{(f_{ck})_x} \sin(2k)^2(t-\tau)d\tau\right)\left(\int\limits_0^\pi xf(x,t)dx\right)}{\left(\int\limits_0^\pi xf(x,t)dx\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)} \\ & + \frac{\left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \overline{(f_{ck})_x} \sin(2k)^2(t-\tau)d\tau\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)}{\left(\int\limits_0^\pi xf(x,t)dx\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)} \\ & - \frac{\left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \overline{(f_{ck})_x} \sin(2k)^2(t-\tau)d\tau\right)\left(\int\limits_0^\pi xf\overline{(x,t)}dx\right)}{\left(\int\limits_0^\pi xf(x,t)dx\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)} \end{aligned}$$

$$\begin{aligned} K-\overline{K} \; = \; & \frac{\left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \left((f_{ck})_x - \overline{(f_{ck})_x}\right) \sin(2k)^2(t-\tau)d\tau\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)}{\left(\int\limits_0^\pi xf(x,t)dx\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)} \\ & \frac{\left(\int\limits_0^\pi x\left(\overline{f(x,t)} - f(x,t)\right)dx\right)\left(\pi \sum\limits_{k=1}^{\infty} \int\limits_0^t \overline{(f_{ck})_x} \sin(2k)^2(t-\tau)d\tau\right)}{\left(\int\limits_0^\pi xf(x,t)dx\right)\left(\int\limits_0^\pi x\overline{f(x,t)}dx\right)} \end{aligned}$$

$$\left|K-\overline{K}\right| \leq \frac{\frac{\pi^3}{2}M \sum\limits_{k=1}^{\infty} \left|(f_{ck})_x - \overline{(f_{ck})_x}\right|}{\frac{\pi^4}{2}M^2} + \frac{\frac{\pi^3}{2}M\left|f-\overline{f}\right|}{\frac{\pi^4}{2}M^2}$$

Equation (14) can be obtained with the maximum of both sides of this equation:

$$\left\|K-\overline{K}\right\| \leq \frac{2}{\pi M}\,\|T\|\,\left\|f-\overline{f}\right\| + \frac{2}{\pi M}\,\|T\| \sum_{k=1}^{\infty} \left\|(f_{ck})_x - \overline{(f_{ck})_x}\right\| \tag{14}$$

Using same estimations, we obtain

$$\left\|q-\overline{q}\right\| \leq \left\|F-\overline{F}\right\| + |T|\,\|K\|\,\left\|r-\overline{r}\right\| + \left\|\overline{r}\right\|\,\left\|K-\overline{K}\right\|,$$

From (10)-(11) we also obtain that

$$\begin{aligned}\left\|q-\overline{q}\right\| &\leq \frac{2}{\pi^{2}M(1-|T|\,|K|)}\left\{\left\|H^{'}(t)-\overline{H^{'}(t)}\right\|+\pi\sum_{k=1}^{\infty}\left\|\varphi_{ck}^{'''}-\overline{\varphi_{ck}^{'''}}\right\|+\left\|\psi_{ck}^{'}-\overline{\psi_{ck}^{'}}\right\|\right\}\\ &\frac{2\left\|\overline{r}\right\|\|T\|}{\pi M(1-|T|\,|K|)}\left\|f-\overline{f}\right\|+\frac{2\left\|\overline{r}\right\|\|T\|^{2}}{\pi M(1-|T|\,|K|)}\sum_{k=1}^{\infty}\left\|(f_{ck})_{x}-\overline{(f_{ck})_{x}}\right\|\end{aligned}$$

We obtain the difference u and $\overline{u}$ from (5):

$$\begin{aligned}T-\overline{T} &= \frac{1}{2}\left[(\varphi_{0}-\overline{\varphi_{0}})+\left(\psi_{0}-\overline{\psi_{0}}\right)t+\int_{0}^{t}(t-\tau)q(\tau)\left(f_{0}-\overline{f_{0}}\right)d\tau\right]\\ &+\sum_{k=1}^{\infty}\left[(\varphi_{ck}-\overline{\varphi_{ck}})\cos(2k)^{2}t+\frac{\left(\psi_{ck}-\overline{\psi_{ck}}\right)}{(2k)^{2}}\sin(2k)^{2}t\right]\cos 2kx\\ &+\sum_{k=1}^{\infty}\left[\frac{1}{(2k)^{2}}\int_{0}^{t}\left(f_{ck}-\overline{f_{ck}}\right)q(\tau)\sin(2k)^{2}(t-\tau)d\tau\right]\cos 2kx \qquad (15)\\ &+\sum_{k=1}^{\infty}\left[(\varphi_{sk}-\overline{\varphi_{sk}})\cos(2k)^{2}t+\frac{\left(\psi_{sk}-\overline{\psi_{sk}}\right)}{(2k)^{2}}\sin(2k)^{2}t\right]\sin 2kx\\ &+\sum_{k=1}^{\infty}\left[\frac{1}{(2k)^{2}}\int_{0}^{t}\left(f_{sk}-\overline{f_{sk}}\right)q(\tau)\sin(2k)^{2}(t-\tau)d\tau\right]\sin 2kx\end{aligned}$$

Taking maximum both of sides

$$\begin{aligned}\left\|T-\overline{T}\right\| &\leq \frac{1}{2}\left\|\varphi_{0}-\overline{\varphi_{0}}\right\|+\frac{1}{2}|T|\left\|\psi_{0}-\overline{\psi_{0}}\right\|+\frac{1}{2}|T|\left\|f_{0}-\overline{f_{0}}\right\|\\ &+\sum_{k=1}^{\infty}\left(\left\|\varphi_{ck}-\overline{\varphi_{ck}}\right\|+\left\|\varphi_{sk}-\overline{\varphi_{sk}}\right\|\right)\\ &+\sum_{k=1}^{\infty}\frac{1}{(2k)^{2}}\left(\left\|\psi_{ck}-\overline{\psi_{ck}}\right\|+\left\|\psi_{sk}-\overline{\psi_{sk}}\right\|\right)\\ &+\sum_{k=1}^{\infty}|T|\left\|f_{ck}-\overline{f_{ck}}\right\|\left\|q\right\|+\sum_{k=1}^{\infty}|T|\left\|f_{sk}-\overline{f_{sk}}\right\|\|r\|\\ &+\sum_{k=1}^{\infty}|T|\left\|\overline{f_{ck}}\right\|\left\|q-\overline{q}\right\|+\sum_{k=1}^{\infty}|T|\left\|\overline{f_{sk}}\right\|\left\|q-\overline{q}\right\|\end{aligned}$$

Applying Hölder inequality,

$$
\begin{aligned}
\left\|u-\overline{u}\right\| \quad \le \quad & \frac{1}{2}\left\|\varphi_0-\overline{\varphi_0}\right\|+\frac{1}{2}|T|\left\|\psi_0-\overline{\psi_0}\right\|+\frac{1}{2}|T|\left\|f_0-\overline{f_0}\right\| \\
& +\sum_{k=1}^{\infty}\left(\left\|\varphi_{ck}-\overline{\varphi_{ck}}\right\|+\left\|\varphi_{sk}-\overline{\varphi_{sk}}\right\|\right) \\
& +\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left[\left\|\psi_{ck}-\overline{\psi_{ck}}\right\|+\left\|\psi_{sk}-\overline{\psi_{sk}}\right\|\right]^2\right)^{\frac{1}{2}} \\
& +\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\left(|T|\left\|f_{ck}-\overline{f_{ck}}\right\|\|q\|\right)^{\frac{1}{2}} \\
& +\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\left(|T|\left\|f_{sk}-\overline{f_{sk}}\right\|\|q\|\right)^{\frac{1}{2}} \\
& +\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\left(|T|\left\|\overline{f_{ck}}\right\|\left\|q-\overline{q}\right\|\right)^{\frac{1}{2}} \\
& +\frac{1}{4}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\left(|T|\left\|\overline{f_{sk}}\right\|\left\|q-\overline{q}\right\|\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\left\|T-\overline{T}\right\| \quad \le \quad & \frac{1}{2}\left\|\varphi_0-\overline{\varphi_0}\right\|+\frac{1}{2}|T|\left\|\psi_0-\overline{\psi_0}\right\|+\frac{1}{2}|T|\left\|f_0-\overline{f_0}\right\| \\
& +\sum_{k=1}^{\infty}\left(\left\|\varphi_{ck}-\overline{\varphi_{ck}}\right\|+\left\|\varphi_{sk}-\overline{\varphi_{sk}}\right\|\right) \\
& +\frac{\pi^2}{24}\sum_{k=1}^{\infty}\left\|\psi_{ck}-\overline{\psi_{ck}}\right\|+\left\|\psi_{sk}-\overline{\psi_{sk}}\right\| \\
& +\frac{\pi^2}{24}\sum_{k=1}^{\infty}|T|\left\|f_{ck}-\overline{f_{ck}}\right\|\|q\| \\
& +\frac{\pi^2}{24}\sum_{k=1}^{\infty}|T|\left\|f_{sk}-\overline{f_{sk}}\right\|\|q\| \\
& +\frac{\pi^2}{24}\sum_{k=1}^{\infty}|T|\left\|\overline{f_{ck}}\right\|\left\|q-\overline{q}\right\| \\
& +\frac{\pi^2}{24}\sum_{k=1}^{\infty}|T|\left\|\overline{f_{sk}}\right\|\left\|q-\overline{q}\right\|
\end{aligned}
$$

$$
\left\|T-\overline{T}\right\| \le M_1\left\|\varphi-\overline{\varphi}\right\|+M_2\left\|\psi-\overline{\psi}\right\|+M_3\left\|f-\overline{f}\right\|+M_4\left\|H''-\overline{H''}\right\|
$$

where

$$M_1 = \max\left\{\frac{1}{2}, 1, \frac{|T|\,\pi}{6(1-|T|\,|K|)}\right\},$$
$$M_2 = \max\left\{\frac{\pi^2}{24}, \frac{|T|\,\pi}{6(1-|T|\,|K|)}\right\},$$
$$M_3 = \max\left\{\frac{|T|}{2}, |T|\left\|q\right\|, \frac{|T|\,\pi\left\|\overline{q}\right\|}{6(1-|T|\,|K|)}, \frac{|T|^3\,\pi\left\|\overline{q}\right\|}{6(1-|T|\,|K|)}\right\},$$
$$M_4 = \max\left\{\frac{|T|}{6(1-|T|\,|K|)}\right\}$$

we also obtain that

$$\left\|T-\overline{T}\right\| \le M_5\left\|\Phi-\overline{\Phi}\right\|,$$

where

$$M_5 = \max\{M_1, M_2, M_3, M_4\}.$$

For $\Phi \to \overline{\Phi}$ then $u \to \overline{u}$. □

4. Numerical Method

We use finite-difference approximation for discretizing problem (1)-(3):

$$\frac{1}{\tau^2}\left(T_i^{j+1}-2T_i^j+T_i^{j-1}\right)+\frac{1}{h^4}\left(T_{i+2}^j-4T_{i+1}^j+6T_i^j-4T_{i-1}^j+T_{i-2}^j\right)=q^j\widetilde{f}_i^j$$

$$T_i^0=\phi_i, \frac{1}{\tau}\left(T_i^1-T_i^0\right)=\psi_i \quad (16)$$

$$T_0^j=T_{N_x+1}^j, \quad (17)$$

$$T_1^j=T_{N_x+2}^j, \quad (18)$$

$$T_{-1}^j=T_{N_x}^j, \quad (19)$$

$$T_2^j-T_{-2}^j=T_{N_x+3}^j-T_{N_x-1}^j, \quad (20)$$

The domain $[0,\pi]\times[0,T]$ is divided into an $N_x \times N_t$ mesh with the spatial step size $h=\pi/N_x$ in x direction and the time step size $\tau = T/N_t$, respectively.

x_i, t_j are defined by
$x_i = ih$; $i = 0;1;2;...;N_x$;
$t_j = j\tau$; $j = 0;1;2;...;N_t$;
$T_i^j = T(x_i,t_j)$, $\widetilde{f}_i^j = \widetilde{f}(x_i,t_j)$, $q^j = q(t_j)$.

Let us integrate the equation (1) respect to x from 0 to π , we obtain

$$q(t)=\frac{H''(t)}{\int_0^\pi \widetilde{f}(x,t)dx}. \quad (21)$$

The finite difference approximation of (18) is

$$q^j = \frac{\left(\left(H^{j+1} - 2H^j + H^{j-1}\right)/\tau^2\right)}{\left(\int_0^\pi \widetilde{f}_i^j dx\right)}.$$

where $H^j = H(t_j)$, $q^j = q(t_j)$, $j = 0, 1, ..., N_t$. We mention that the integral is numerically calculated using Simpson's rule of integration.The system of equations (13)-(17) is solved and u_i^j, q^j

is determined.The condition for stopping the iteration depends on the value difference between the two iterations. Iteration should be stopped when this difference is equal to the tolerance predicted previously.

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