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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 3, Pages 721–736 (2023)
DOI:10.31801/cfsuasmas.1186785
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: October 10, 2022; Accepted: March 21, 2023

EXISTENCE OF SOLUTIONS FOR IMPULSIVE BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

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Abstract. The paper deals with the existence of solutions for a general class of second-order nonlinear impulsive boundary value problems defined on an infinite interval. The main innovative aspects of the study are that the results are obtained under relatively mild conditions and the use of principal and nonprincipal solutions that were obtained in a very recent study. Additional results about the existence of bounded solutions are also provided, and theoretical results are supported by an illustrative example.

1. Introduction

Differential equations with impulses are very convenient mathematical tools for perfectly modeling real-world phenomena with sudden changes in their states. Since it is more realistic to have abrupt changes or jumps in the state than to show constant behavior, they frequently occur in natural sciences. In addition, the efficiency and richness of the relevant theory have contributed to many researchers paying attention to impulsive differential equations in recent years. We refer the reader to the famous books [6, 16] that involve extensive knowledge about qualitative theory and some applications of impulsive differential equations. On the other hand, boundary value problems (BVPs) on unbounded domains naturally appear in fluid mechanics problems such as the unstable gas flow through a porous medium, in plasma physics, and to model many other phenomena, see [1]. In particular, some applications of impulsive BVPs can be found in the papers [9, 17, 18] that have recently been published. There are many results in the literature regarding the existence of solutions to impulsive BVPs, e.g. [2,3,10–12]. Below, we mention some recent results about impulsive BVPs on unbounded domains.

2020 *Mathematics Subject Classification.* 34B37, 34B40.
Keywords. Second-order, nonlinear, impulsive, boundary value problem, principal/nonprincipal solution.

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In [10], for an impulsive BVP with integral boundary conditions of the form

$$\begin{cases} \frac{1}{a(t)}(a(t)x'(t))' + f(t,x(t),x'(t)) = 0, \quad t \neq \tau_k, \\ \Delta x|_{t=\tau_k} = I_k(x(\tau_k)), \quad k = 1,2,\dots, \\ \Delta x'|_{t=\tau_k} = J_k(x(\tau_k)), \quad k = 1,2,\dots, \\ a_1 \lim_{t\to-\infty} x(t) - b_1 \lim_{t\to-\infty} a(t)x'(t) = \int\limits_{-\infty}^{\infty} g(x(s))\varphi(s)\,\mathrm{d}s, \\ a_2 \lim_{t\to\infty} x(t) + b_2 \lim_{t\to\infty} a(t)x'(t) = \int\limits_{-\infty}^{\infty} h(x(s))\varphi(s)\,\mathrm{d}s, \end{cases}$$

the existence of solutions is shown under the following hypotheses:

(i) $a_1b_2 + a_2b_1 + a_1a_2\int_{-\infty}^{\infty}\frac{1}{a(s)}\mathrm{d}s > 0$,

(ii) $f \in C(\mathbb{R}\times[0,\infty)\times\mathbb{R},[0,\infty))$ such that $f(t,y,z) \le u_1(t)u_2(y,z)$ where $u_1 \in L(\mathbb{R},(0,\infty))$ and $u_2 \in C([0,\infty)\times\mathbb{R},[0,\infty))$,

(iii) $g, h \in C(\mathbb{R},[0,\infty))$ are nondecreasing, and $g(x)$, $h(x)$ are bounded provided that x is defined on a bounded set,

(iv) I_k and J_k are bounded functions, and

$$\left[a_2 + b_2\int_{\tau_k}^{\infty}\frac{1}{a(s)}\mathrm{d}s\right]J_k(x(\tau_k)) - \frac{a_2}{a(\tau_k)}I_k(x(\tau_k)) > 0,$$

(v) $\varphi \in C(\mathbb{R},[0,\infty))$ and $\int\limits_{-\infty}^{\infty}\varphi(s)\,\mathrm{d}s < \infty$,

(vi) $a \in C(\mathbb{R},(0,\infty))$ and $\int\limits_{-\infty}^{\infty}\frac{1}{a(s)}\mathrm{d}s < \infty$.

In [12], the second order impulsive BVP

$$\begin{cases} x''(t) = -f(t,x(t),x'(t)), \quad t \neq \tau_k, \\ x(\tau_k+) = a_k x(\tau_k), \quad k = 1,2,\dots, \\ a_0x(0) - b_0x'(0) = \alpha, \\ a_1x(1) - b_1x'(1) = \beta \end{cases} \tag{1}$$

is studied, and the existence of solutions is shown via the upper and lower solutions method.

In [2], the existence of solutions was shown for the impulsive BVP

$$\begin{cases} (a(t)y')' + b(t)y = f(t,y), & t \neq \tau_k, \\ \Delta y' + b_k y = g_k(y), & t = \tau_k, \\ y(t_0) = y_0, \\ y(t) = c_1v(t) + c_2u(t) + o(v^{\mu}(t)u(t)), & t\to\infty,\ \mu\in(0,1), \end{cases} \tag{2}$$

where u and v are the principal and nonprincipal solutions of the corresponding homogeneous equation. Observe in (1) that impulse effects occur only on the solutions while (2) has continuous solutions as the impulse effects occur only on the derivatives of the solutions. The method of the paper [2] is different from the other studies in the literature as it relies on principal and nonprincipal solutions. A similar approach was applied in [3] and [7], where a particular case of the impulsive BVP (2) was considered in [3], while [7] dealt with a BVP without impulse effects.

Motivated by the studies above, we consider the second-order nonlinear differential equation under impulse effects

$$\begin{cases} (a(t)y')' + b(t)y = f(t, y), & t \neq \tau_k, \\ \Delta y + a_k y = f_k(y), & t = \tau_k, \\ \Delta(a(t)y') + b_k y + c_k y' = g_k(y), & t = \tau_k, \end{cases} \tag{3}$$

satisfying the boundary conditions

$$y(a) = 0, \quad y(t) = O(v(t)),\ t \to \infty \tag{4}$$

where $a \geq t_0$, $a(t), b(t) \in \text{PLC}([t_0, \infty), \mathbb{R})$ with $a(t) > 0$, $f \in \text{PLC}([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ are sequences of real numbers, $f_k, g_k \in \text{PLC}(\mathbb{R}, \mathbb{R})$ for each $k \in \mathbb{N}$, $\{\tau_k\}$ is the sequence of impulses satisfying $\tau_{k+1} > \tau_k$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} |\tau_k| = \infty$, and Δ is the impulse operator defined by $\Delta y(\tau_k) = y(\tau_k^+) - y(\tau_k^-)$ with $y(\tau_k^{\pm}) = \lim_{t\to\tau_k^{\pm}} y(t)$. Note that $\text{PLC}[t_0, \infty)$ is the set of functions y such that $y(t)$ is continuous on $(\tau_k, \tau_{k+1}]$, $y(\tau_k-) = y(\tau_k)$ and $y(\tau_k+)$ exists for each $k = 1, 2, \ldots$. For brevity, we use the notations $\underline{n}(t) := \inf\{k : \tau_k \geq t\}$ and $\overline{n}(t) := \sup\{k : \tau_k < t\}$.

We aim to prove the existence of solutions of the second-order nonlinear impulsive BVP (3)-(4) with discontinuous solutions under some mild conditions that depend on the principal and nonprincipal solutions of the homogeneous equation

$$\begin{cases} (a(t)y')' + b(t)y = 0, & t \neq \tau_k, \\ \Delta y + a_k y = 0, & t = \tau_k, \\ \Delta(a(t)y') + b_k y + c_k y' = 0, & t = \tau_k \end{cases} \tag{5}$$

associated with equation (3).

In the present work, the impulses affect both the solutions and their derivatives, and the impulse conditions occurring in the third line of (3) are the so-called mixed type conditions because they include both the solution and its derivative. Hence, the equation under consideration is quite general. On the other hand, the conditions determined on the functions that are on the right-hand side of the nonhomogeneous equation (3) are weaker than the conditions in previous studies. Our conditions do not directly require the functions to be bounded or monotonic. Another novelty is the use of principal and nonprincipal solutions of the corresponding homogeneous equation (5).

2. Preliminaries

In this section, we state some auxiliary lemmas that will be utilized in the rest of the paper.

The existence and some properties of principal and nonprincipal solutions for impulsive differential equations with continuous solutions

$$\begin{cases} (a(t)y')' + b(t)y = 0, & t \neq \tau_k, \\ \Delta a(t)y' + b_k y = 0, & t = \tau_k \end{cases} \tag{6}$$

was proved in [13], where it was shown that equation (6) has two linearly independent solutions u_0 and v_0 satisfying

$$\lim_{t\to\infty} \frac{u_0(t)}{v_0(t)} = 0, \quad \int_a^\infty \frac{\mathrm{d}t}{a(t)u_0^2(t)} = \infty, \quad \int_a^\infty \frac{\mathrm{d}t}{a(t)v_0^2(t)} < \infty, \quad \frac{u_0'(t)}{u_0(t)} < \frac{v_0'(t)}{v_0(t)},\ t \geq a$$

provided that (6) has a positive solution, and a is sufficiently large. Such functions u_0 and v_0 are said to be principal and nonprincipal solutions of (6), respectively.

The counterpart of the above lemma for differential equations having impulse effects not only on the derivative of the solution but also on the solution was given very recently in [4] and improved in [5] for the more general impulsive differential equations of the form (5). The statement of the related lemma is given below for completeness.

Lemma 1. ([5]) Let $(1 - a_k)(1 - c_k/a(\tau_k)) > 0$, $k \in \mathbb{N}$ and suppose equation (5) has a positive solution. Then, there exist two linearly independent solutions u and v of (5) satisfying the following conditions:

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0, \tag{7}$$

$$\int_a^\infty \frac{\mu(t,a)}{a(t)u^2(t)}\mathrm{d}t = \infty, \quad \int_a^\infty \frac{\mu(t,a)}{a(t)v^2(t)}\mathrm{d}t < \infty,$$

$$\frac{u'(t)}{u(t)} < \frac{v'(t)}{v(t)}, \quad t \geq a,$$

where a is arbitrarily large, and

$$\mu(t,a) = \prod_{k=\underline{n}(a)}^{\overline{n}(t)} (1 - a_k)(1 - c_k/a(\tau_k)).$$

Namely, u is the principal, and v is a nonprincipal solution.

Remark 1. *If $u > 0$ is a principal solution of* (5)*, then, a nonprincipal solution is of the form*

$$v(t) = u(t) \int_{t_0}^{t} \frac{\mu(s,a)}{a(s)u^2(s)}\mathrm{d}s. \tag{8}$$

Conversely, if $v > 0$ is a nonprincipal solution of (5)*, then, the principal solution is of the form*

$$u(t) = v(t) \int_t^\infty \frac{1}{\mu(\infty, s)a(s)v^2(s)} \mathrm{d}s.$$

In addition, we provide below some definitions and compactness criteria that will be needed in the future.

Definition 1. ([15]) Let $1 \leq p < \infty$ and Y be an arbitrary measure space. We define $L^p(Y)$ to be the space of functions f such that

$$||f||_p = \left(\int_Y |f|^p \mathrm{d}\mu \right)^{1/p} < \infty.$$

Definition 2. ([15]) Let $1 \leq p < \infty$. We define $\ell^p(Y)$ to be the space of sequences y_k such that

$$\sum_{k=1}^{\infty} |y_k|^p < \infty.$$

Theorem 1. ([14]) Let $Y \in \mathbb{R}^n$. A set $S \subset L^p(Y)$, $1 \leq p < \infty$ is compact if

(i) there exists some $a > 0$ such that $\|y\|_{L^p(Y)} \leq a$ for all $y \in S$,
(ii) $\|(\varphi_h y) - y\|_{L^p(Y)} \to 0$ as $h \to 0$, where $(\varphi_h y)(x) := y(x_1 + h, x_2 + h \ldots x_n + h)$, $x \in Y$.

Theorem 2. ([8]) Let $Y \in \mathbb{R}^n$. A set $S \subset l^p(Y)$, $1 \leq p < \infty$ is totally bounded if, and only if

(i) S is pointwise bounded,
(ii) for every $\epsilon > 0$ and $y \in S$, there is some $n \in \mathbb{N}$ such that $\sum_{k>n} |y_k|^p < \epsilon^p$.

3. Main Results

We define the Banach space

$$X = \left\{ y \in \mathrm{PLC}([a, \infty), \mathbb{R}) : \frac{|y(t)|}{v(t)} \text{ is bounded} \right\}$$

endowed with the norm

$$\|y\| = \sup_{t \in [a, \infty)} \frac{|y(t)|}{v(t)}$$

and, introduce the operator

$$(\mathcal{T}y)(t) = -\,u(t)\Bigg\{\int\limits_a^t \frac{1}{a(s)u^2(s)}\Bigg(\int\limits_s^\infty \frac{u(r)}{\mu(r,s)} f(r,y(r))\mathrm{d}r + \sum_{k=\underline{n}(s)}^{\infty} \frac{h_k}{\mu(k,s)}\Bigg)\mathrm{d}s - \sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)}\Bigg\}, \tag{9}$$

where $h_k = (1-a_k)u(\tau_k)g_k(y(\tau_k)) - \big[(a(\tau_k)-c_k)u'(\tau_k) - b_k u(\tau_k)\big] f_k(y(\tau_k))$ and $y \in X$.

We aim to show that $\mathcal{T}y$ has at least one fixed point by applying Schauder fixed point theorem.

For this purpose, we define the set $E := \big\{y \in X : |y(t)| \le v(t)\big\}$ which is convex, closed, and bounded, and assume the following hypotheses hold:

(H1) There exist some functions $q_j \in \mathrm{C}(\mathbb{R}_+,\mathbb{R}_+)$, $j = 1,2,3$, $p_i \in \mathrm{C}([t_0,\infty),\mathbb{R}_+)$, $i = 1,2$ and real sequences $\{\alpha_k\}, \{\beta_k\}$ such that

$$|f(t,y)| \le p_1(t)q_1\left(\frac{|y|}{v(t)}\right) + p_2(t), \quad t \ge a, \tag{10}$$

$$\big|f_k(y)\big| \le \alpha_k q_2\left(\frac{|y|}{v(\tau_k)}\right), \quad \big|g_k(y)\big| \le \beta_k q_3\left(\frac{|y|}{v(\tau_k)}\right), \quad \tau_k \ge a.$$

(H2) $\displaystyle\int_a^\infty \frac{u(s)}{\mu(s,a)}(p_1(s)+p_2(s))\mathrm{d}s + \sum_{k=\underline{n}(a)}^{\infty} \frac{H_k}{\mu(k,a)} = O(1), \quad t \to \infty,$

where $H_k = (1-a_k)u(\tau_k)\beta_k + |(a(\tau_k)-c_k)u'(\tau_k) - b_k u(\tau_k)|\alpha_k$

(H3) $\displaystyle\sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{\alpha_k}{(1-a_k)u(\tau_k)} = O(1), \quad t \to \infty.$

Lemma 2. *The operator $\mathcal{T}$ given in* (9) *maps E onto E.*

Proof. First, we prove that $\mathcal{T}y \in \mathrm{PLC}[a,\infty)$.

Let $y \in X$ and $t_1 \in [a,\infty)$ with $t < t_1$, and $t_1 \ne \tau_l$, $l = 1,2,\dots$. Then

$$\begin{aligned}|(\mathcal{T}y)(t) - (\mathcal{T}y)(t_1)| \le & |u(t)-u(t_1)|\Bigg\{\int\limits_a^t \frac{1}{a(s)u^2(s)}\Bigg(\int\limits_s^\infty \frac{u(r)}{\mu(r,s)}|f(r,y(r))|\mathrm{d}r \\ & + \sum_{k=\underline{n}(s)}^{\infty} \frac{|h_k|}{\mu(k,s)}\Bigg)\mathrm{d}s + \sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{|f_k(y(\tau_k))|}{(1-a_k)u(\tau_k)}\Bigg\} \\ & + u(t_1)\Bigg\{\int\limits_t^{t_1} \frac{1}{a(s)u^2(s)}\Bigg(\int\limits_s^\infty \frac{u(r)}{\mu(r,s)}|f(r,y(r))|\mathrm{d}r\end{aligned}$$

$$+ \sum_{k=\underline{n}(s)}^{\infty} \frac{|h_k|}{\mu(k,s)}\Bigg)\mathrm{d}s + \sum_{k=\underline{n}(t)}^{\overline{n}(t_1)} \frac{|f_k(y(\tau_k))|}{(1-a_k)u(\tau_k)}\Bigg\}.$$

Since $|y(t)|/v(t)$ is bounded, $\exists M > 0$ such that

$$\frac{|y(t)|}{v(t)} \le M.$$

So, from continuity of q_j, there can be found positive constants c_j such that $\max_{0\le t\le M} q_j(t) = c_j$, $j = 1,2,3$. Hence, in view of (H1), we have the following estimates:

$$|f(r,y(r))| \le p_1(r)q_1\left(\frac{|y(r)|}{v(r)}\right) + p_2(r) \le c_1p_1(r) + p_2(r) \le c\big[p_1(r)+p_2(r)\big], \quad (11)$$

$$\begin{aligned}|h_k| &\le (1-a_k)u(\tau_k)\beta_k q_3\Big(\frac{|y(\tau_k)|}{v(\tau_k)}\Big) + |(a(\tau_k)-c_k)u'(\tau_k) - b_ku(\tau_k)|\alpha_k q_2\Big(\frac{|y(\tau_k)|}{v(\tau_k)}\Big)\\ &\le c_3(1-a_k)u(\tau_k)\beta_k + c_2|(a(\tau_k)-c_k)u'(\tau_k) - b_ku(\tau_k)|\alpha_k \;\le\; cH_k,\end{aligned} \quad (12)$$

$$|f_k(y(\tau_k))| \le \frac{\alpha_k}{(1-a_k)u(\tau_k)}q_2\left(\frac{|y(\tau_k)|}{v(\tau_k)}\right) \le c_2\frac{\alpha_k}{(1-a_k)u(\tau_k)} \le c\frac{\alpha_k}{(1-a_k)u(\tau_k)}, \quad (13)$$

where $c = \max\{1, c_1, c_2, c_3\}$.

Using the above estimates and the expansion $1/\mu(s,\nu) = \mu(s,a)/\mu(\nu,a)$, we can proceed as follows:

$$\begin{aligned}|(\mathcal{T}y)(t) - (\mathcal{T}y)(t_1)| \le{}& c|u(t)-u(t_1)|\Bigg\{\int_a^t \Bigg(\frac{\mu(s,a)}{a(s)u^2(s)}\int_a^\infty \frac{u(r)}{\mu(r,a)}\,(p_1(r)+p_2(r))\,\mathrm{d}r\\ &+ \sum_{k=\underline{n}(a)}^{\infty}\frac{H_k}{\mu(k,a)}\Bigg)\mathrm{d}s + \sum_{k=\underline{n}(a)}^{\overline{n}(t)}\frac{\alpha_k}{(1-a_k)u(\tau_k)}\Bigg\}\\ &+ cu(t_1)\Bigg\{\int_t^{t_1}\frac{\mu(s,a)}{a(s)u^2(s)}\Bigg(\int_a^\infty \frac{u(r)}{\mu(r,a)}\,(p_1(r)+p_2(r))\,\mathrm{d}r\\ &+ \sum_{k=\underline{n}(a)}^{\infty}\frac{H_k}{\mu(k,a)}\Bigg)\mathrm{d}s + \sum_{k=\underline{n}(t)}^{\overline{n}(t_1)}\frac{\alpha_k}{(1-a_k)u(\tau_k)}\Bigg\}.\end{aligned}$$

It follows from (H2) that $(\mathcal{T}y)(t) \to (\mathcal{T}y)(t_1)$ as $t \to t_1-$.

In a similar way, one can show that $\lim_{t\to t_1+}(\mathcal{T}y)(t) = (\mathcal{T}y)(t_1)$ for $t_1 \ne \tau_l$, $l = 1,2,\dots$, and $\lim_{t\to\tau_l+}(\mathcal{T}y)(t)$ exist for all $l = 1,2,\dots$ Hence, $\mathcal{T}y(t)$ is piecewise left continuous on $[a,\infty)$.

Now, from (11), (12) and (13) one has

$$|(\mathcal{T}y)(t)| \leq cu(t)\Big\{\int_a^t \frac{\mu(s,a)}{a(s)u^2(s)}\Big(\int_a^\infty \frac{u(r)}{\mu(r,a)}\big(p_1(r)+p_2(r)\big)\mathrm{d}r + \sum_{k=\underline{n}(a)}^{\infty}\frac{H_k}{\mu(k,a)}\Big)\mathrm{d}s + \sum_{k=\underline{n}(a)}^{\overline{n}(t)}\frac{\alpha_k}{(1-a_k)u(\tau_k)}\Big\}.$$

In view of (H2) and (H3) we may write

$$\int_a^\infty \frac{u(r)}{\mu(r,a)}\big(p_1(r)+p_2(r)\big)\mathrm{d}r + \sum_{k=\underline{n}(a)}^{\infty}\frac{H_k}{\mu(k,a)} \leq \frac{1}{2c} \tag{14}$$

and

$$\sum_{k=\underline{n}(a)}^{\overline{n}(t)}\frac{\alpha_k}{(1-a_k)u(\tau_k)} \leq \frac{1}{2c}$$

for some sufficiently large a. Then, from the relation (8) we have

$$|(\mathcal{T}y)(t)| \leq \frac{u(t)}{2}\Big\{\int_a^t \frac{\mu(s,a)}{a(s)u^2(s)}\mathrm{d}s + \sum_{k=\underline{n}(a)}^{\overline{n}(t)}\frac{\alpha_k}{(1-a_k)u(\tau_k)}\Big\} = \frac{v(t)}{2} + \frac{u(t)}{2}.$$

Using (7) we conclude that $|(\mathcal{T}y)(t)| \leq v(t)$. Hence, $\mathcal{T}y \in E$. □

Lemma 3. *$\mathcal{T}$ is a continuous operator.*

Proof. Take a sequence $\{y_n\} \in E$ such that $\lim_{n\to\infty} y_n = y \in E$. Using (11), (12) and (13) we can write

$$\begin{aligned}|(\mathcal{T}y_n)(t) - (\mathcal{T}y)(t)| \leq& u(t)\Big\{\int_a^t \frac{\mu(s,a)}{a(s)u^2(s)}\Big(\int_s^\infty \frac{u(r)}{\mu(r,a)}|f(r,y_n(r)) - f(r,y(r))|\mathrm{d}r \\ &+ \sum_{k=\underline{n}(s)}^{\infty}\frac{1}{\mu(k,a)}\Big[(1-a_k)u(\tau_k)|g_k(y_n(\tau_k)) - g_k(y(\tau_k))| \\ &+ |(a(\tau_k)-c_k)u'(\tau_k) - b_k u(\tau_k)|f_k(y_n(\tau_k)) - f_k(y(\tau_k))|\Big]\Big)\mathrm{d}s \\ &+ \sum_{k=\underline{n}(a)}^{\overline{n}(t)}\frac{1}{(1-a_k)u(\tau_k)}|f_k(y_n(\tau_k)) - f_k(y(\tau_k))|\Big\} \\ \leq& 2cu(t)\Big\{\int_a^t \frac{\mu(s,a)}{a(s)u^2(s)}\Big(\int_s^\infty \frac{u(r)}{\mu(r,a)}\left(p_1(r)+p_2(r)\right)\mathrm{d}r\end{aligned}$$

$$+ \sum_{k=\underline{n}(s)}^{\infty} \frac{H_k}{\mu(k,a)}\Big)\mathrm{d}s + \sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{\alpha_k}{(1-a_k)u(\tau_k)}\Big\}.$$

From (H2) and (H3), it can be seen that the above expression is finite for all $t \in [a,\infty)$. Thus, applying Lebesgue dominated convergence theorem and Weierstrass-M test, we obtain

$$\lim_{n\to\infty} \|\mathcal{T}y_n - \mathcal{T}y\| \to 0.$$

Hence, $\mathcal{T}$ is a continuous operator.

□

Lemma 4. *$\mathcal{T}$ is a relatively compact operator.*

Proof. Pick an arbitrary sequence $\{y_n\} \in E$. We wish to prove that there exists a subsequence $\{y_{n_i}\} \in E$ such that $\mathcal{T}y_{n_i}$ is convergent in E. If we define

$$f_n(r) := \frac{u(r)}{\mu(r,s)} f(r, y_n(r)), \qquad g_n(\tau_k) := \frac{f_k(y_n(\tau_k))}{(1-a_k)u(\tau_k)},$$

and

$$h_n(\tau_k) := \frac{1}{\mu(k,s)}\big[(1-a_k)u(\tau_k)g_k(y_n(\tau_k)) + [(a(\tau_k)-c_k)u'(\tau_k) - b_k u(\tau_k)]f_k(y_n(\tau_k))\big]$$

then, $\mathcal{T}$ can be decomposed as $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$, where

$$(\mathcal{T}_1 y_n)(t) = u(t) \int_a^t \frac{\mu(s,a)}{a(s)u^2(s)} \int_s^\infty f_n(r)\mathrm{d}r\mathrm{d}s,$$

$$(\mathcal{T}_2 y_n)(t) = u(t) \int_a^t \frac{\mu(s,a)}{a(s)u^2(s)} \sum_{k=\underline{n}(s)}^{\infty} h_n(\tau_k)\mathrm{d}s, \qquad (\mathcal{T}_3 y_n)(t) = u(t) \sum_{k=\underline{n}(a)}^{\overline{n}(t)} g_n(\tau_k).$$

As in (14), there is a constant $m_1 > 0$ such that

$$\|f_n\|_{L^1([a,\infty))} \le m_1, \quad n \ge 1.$$

Thus, the first hypothesis of Lemma 1 holds. Now, for $(\varphi_h f)(s) = f(s+h)$ from (10) and (11) we may write

$$\begin{aligned}\int_a^\infty |(\varphi_h f_n)(s) - f_n(s)|\mathrm{d}s &\le \int_{a+h}^\infty |f_n(s)|\mathrm{d}s + \int_a^\infty |f_n(s)|\mathrm{d}s \\ &\le 2\int_a^\infty |f_n(s)|\mathrm{d}s \le 2c \int_a^\infty \frac{u(s)}{\mu(s,a)}(p_1(s) + p_2(s))\mathrm{d}s.\end{aligned}$$

In view of (H2), we apply the Lebesgue dominated convergence theorem, and we obtain the second hypothesis of Lemma 1. Hence, Lemma 1 asserts that there exists a convergent subsequence $\{f_{n_i}\} \in L^1([a,\infty))$. Since f_{n_i} is continuous, we conclude that

$$\int_a^\infty |\bar{f}(r)|\mathrm{d}r = \lim_{i\to\infty} \int_a^\infty |f_{n_i}(r)|\mathrm{d}r,$$

where

$$\bar{f}(r) = \frac{u(r)}{\mu(r,a)} f(r, y(r)).$$

Then,

$$\frac{|(\mathcal{T}_1 y_{n_i})(t) - (\mathcal{T}_1 y)(t)|}{v(t)} \le \frac{u(t)}{v(t)} \int_a^t \frac{\mu(s,a)}{a(s)u^2(s)} \int_s^\infty |f_{n_i}(r) - \bar{f}(r)|\mathrm{d}r\mathrm{d}s.$$

In view of (H2), again Lebesgue dominated convergence theorem applies, and so

$$\lim_{i\to\infty} \|\mathcal{T}_1 y_{n_i} - \mathcal{T}_1 y\| = 0.$$

Next, we need to utilize Lemma 2 to show that $\mathcal{T}_2$ is a compact operator. Proceeding as in (12), we see that

$$|h_n(\tau_k)| \le c\frac{H_k}{\mu(k,a)}.$$

But (H2) and (H3) imply that each element of the sets $\{f_n\}$, $\{h_n\}$ is pointwise bounded. This means that the first hypothesis of Lemma 2 holds.

For an arbitrary $\epsilon > 0$, we may choose a sufficiently large $j \in \mathbb{N}$ so that

$$\sum_{k=j}^\infty \frac{H_k}{\mu(k,a)} < \frac{\epsilon}{c},$$

then we get

$$\sum_{k=j}^\infty |h_n(\tau_k)| < \epsilon.$$

Thus, by virtue of Lemma 2, the set $\{h_n\}$ is compact in $\ell^1([a,\infty))$ which means that there exists a convergent subsequence $\{h_{n_i}\} \in \ell^1([a,\infty))$ such that

$$\lim_{i\to\infty} \sum_{k=\underline{n}(a)}^\infty |h_{n_i}(\tau_k) - \bar{h}_k| = 0,$$

where

$$\bar{h}_k := \frac{h_k}{\mu(k,a)}.$$

Hence,

$$\frac{|(\mathcal{T}_2 y_{n_i})(t) - (\mathcal{T}_2 y)(t)|}{v(t)} \le \frac{u(t)}{v(t)} \int_a^t \frac{\mu(s,a)}{a(s)u^2(s)} \sum_{k=\underline{n}(s)}^\infty |h_{n_i}(\tau_k) - \bar{h}_k|.$$

Applying Weierstrass-M test it is seen that $\mathcal{T}_2$ has a convergent subsequence in E, i.e.,

$$\lim_{i\to\infty} \|\mathcal{T}_2 y_{n_i} - \mathcal{T}_2 y\| = 0.$$

Finally, since $\mathcal{T}_3$ is a finite sum, it is uniformly convergent. Hence,

$$\lim_{i\to\infty} \frac{|(\mathcal{T}_3 y_{n_i})(t) - (\mathcal{T}_3 y)(t)|}{v(t)} = 0.$$

Since each of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ is relatively compact in E, then so is $\mathcal{T}$. This completes the proof. □

Lemma 5. *Let y be a fixed point of the operator* (9). *Then, y is a solution of equation* (3).

Proof. Suppose y is a fixed point of the operator $\mathcal{T}$. Then,

$$y(t) = u(t)\left\{ I(t) + \sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)} \right\},$$

where

$$I(t) := -\int_a^t \frac{1}{a(s)u^2(s)} \left(\int_s^\infty \frac{u(r)}{\mu(r,s)} f(r,y(r))\mathrm{d}r + \sum_{k=\underline{n}(s)}^{\infty} \frac{h_k}{\mu(k,s)} \right) \mathrm{d}s.$$

For $t \neq \tau_l$, $l = 1, 2, \ldots$, we have

$$y'(t) = u'(t)\left\{ I(t) + \sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)} \right\} - \frac{J(t)}{a(t)u(t)}$$

where

$$J(t) = \int_t^\infty \frac{u(r)}{\mu(r,t)} f(r,y(r))\mathrm{d}r + \sum_{k=\underline{n}(t)}^{\infty} \frac{h_k}{\mu(k,t)}.$$

Thus,

$$\begin{aligned}(a(t)y'(t))' + b(t)y(t) =& [(a(t)u'(t))' + b(t)u(t)]\left\{ I(t) + \sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)} \right\} \\ &+ 2a(t)u'(t)I'(t) + u(t)(a(t)I'(t))'.\end{aligned}$$

It is easy to see that

$$2a(t)u'(t)I'(t) = -\frac{2u'(t)}{u^2(t)} J(t)$$

and

$$u(t)(a(t)I'(t))' = \frac{2u'(t)}{u^2(t)} J(t) + \frac{1}{\mu(t,t)} f(t,y(t)).$$

From $\mu(t,t)=1$, we conclude that

$$(a(t)y'(t))' + b(t)y(t) = f(t, y(t)). \tag{15}$$

Now, we need to show that impulsive conditions hold. Let $t=\tau_l$. Clearly $I(t)$ is a continuous function, namely $I(\tau_l+)=I(\tau_l)$. Thus, we have

$$\begin{aligned}\Delta y|_{t=\tau_l} =& u(\tau_l+)\left\{I(\tau_l) + \sum_{k=\underline{n}(a)}^{l-1} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)} + \frac{f_l(y(\tau_l))}{(1-a_l)u(\tau_l)}\right\} - \\ & u(\tau_l)\left\{I(\tau_l) + \sum_{k=\underline{n}(a)}^{l-1} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)}\right\} \\ =& \Delta u|_{t=\tau_l}\left\{I(\tau_l) + \sum_{k=\underline{n}(a)}^{l-1} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)}\right\} + \frac{u(\tau_l+)f_l(y(\tau_l))}{(1-a_l)u(\tau_l)}.\end{aligned}$$

From $u(\tau_l+)=(1-a_l)u(\tau_l)$ it follows that

$$\Delta y|_{t=\tau_l} + a_l y(\tau_l) = f_l(y(\tau_l)). \tag{16}$$

Finally, using

$$\frac{1}{\mu(k,l+1)} = \prod_{j=l+1}^{k} (1-a_j)^{-1}(1-c_j/a(\tau_j))^{-1} = (1-a_l)(1-c_l/a(\tau_l))\frac{1}{\mu(k,l)}$$

we can write $J(\tau_l+) = (1-a_l)(1-c_l/a(\tau_l))J(\tau_l) - h_l$, and hence

$$a(\tau_l+)u(\tau_l+)I'(\tau_l+) = -\frac{1}{u(\tau_l+)}J(\tau_l+) = ((a(\tau_l)-c_l))u(\tau_l)I'(\tau_l) + \frac{h_l}{(1-a_l)u(\tau_l)}.$$

Then, we have

$$\begin{aligned}a(\tau_l+)y'(\tau_l+) =& a(\tau_l+)u'(\tau_l+)\left\{I(\tau_l) + \sum_{k=\underline{n}(a)}^{l} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)}\right\} + a(\tau_l+)u(\tau_l+)I'(\tau_l+) \\ =& \big[(a(\tau_l)-c_l)u'(\tau_l) - b_l u(\tau_l)\big]\left\{I(\tau_l) + \sum_{k=\underline{n}(a)}^{l-1} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)} + \frac{f_l(y(\tau_l))}{(1-a_l)u(\tau_l)}\right\} \\ & + a(\tau_l)u(\tau_l)I'(\tau_l) + \frac{h_l}{(1-a_l)u(\tau_l)}\end{aligned}$$

which implies that

$$\Delta(ay') + b_l y(\tau_l) + c_l y'(\tau_l) = g_l(y(\tau_l)). \tag{17}$$

Hence, from (15), (16), and (17) we conclude that $y(t)$ is a solution of (3). □

Theorem 3. *The impulsive differential equation* (3) *with the boundary conditions* (4) *has at least one solution, provided that the hypotheses (H1)-(H3) hold, where u and v are principal and nonprincipal solutions of the homogeneous equation* (5).

Proof. From Lemma 2, Lemma 3 and Lemma 4 it is seen that the Schauder fixed point theorem's all hypotheses hold. Thus, the operator $\mathcal{T}$ given in (9) has a fixed point, say y. In view of Lemma 5, the fixed point y is a solution of the equation (3).

On the other hand, by using the hypotheses (H1)-(H3) it is not hard to see that $I(a) = 0$ which implies

$$y(a) = u(a)\Big\{I(a) + \sum_{k=\underline{n}(a)}^{\overline{n}(a)} \frac{f_k(y(\tau_k))}{(1-a_k)u(\tau_k)}\Big\} = 0.$$

Proceeding as in Lemma 3, we obtain $|y(t)| \le v(t)$ from which we can write

$$\lim_{t\to\infty} \frac{|y(t)|}{v(t)} \le 1,$$

which means that $y(t) = O(v(t))$ as $t \to \infty$. Thus, the boundary conditions in (4) hold. This completes the proof. □

4. Examples

This section is devoted to illustrative examples that demonstrate the efficiency of the above result.

Example 1. *Consider the impulsive BVP*

$$\begin{cases} (t^2y')' - 2y = \ln\Big(1 + \dfrac{y^2}{t^2(y^2+1)}\Big), & t \ne \tau_k, \\ \Delta y - \dfrac{y}{k} = \sin\Big(\dfrac{y}{k^4(k+1)^2}\Big), & t = \tau_k, \\ \Delta(t^2y') - ky' = \arctan(y/k^3), & t = \tau_k, \\ y(1) = 0, \quad y(t) = O(v(t)),\ t \to \infty. \end{cases} \tag{18}$$

Observe that $a(t) = t^2$, $b(t) = -2$, $a_k = -1/k$, $b_k = 0$, $c_k = -k$, and so $(1-a_k)(1-c_k/a(\tau_k)) = (1+1/k)^2$. Furthermore,

$$f(t,y) = \ln\Big(1 + \frac{y^2}{t^2(y^2+1)}\Big) \le \frac{1}{t^2}\frac{y^2}{y^2+1} \le \frac{1}{t^2},$$

$$f_k(y) = \sin\Big(\frac{y}{k^4(k+1)^2}\Big) \le \frac{1}{(k^2+k)^2}\frac{|y|}{k^2}$$

and

$$g_k(y) = \arctan\Big(\frac{y}{k^3}\Big) \le \frac{1}{k}\frac{|y|}{k^2}.$$

By direct computations, it can be shown that $u(t) = kt^{-2}$ is the principal, and $v(t) = kt$, $t \in (k-1,k]$ is a nonprincipal solution of the associated homogeneous *impulsive equation*

$$\begin{cases} (t^2y')' - 2y = 0, & t \neq \tau_k, \\ \Delta y - \dfrac{y}{k} = 0, & t = \tau_k, \\ \Delta(t^2y') - ky' = 0, & t = \tau_k. \end{cases}$$

Thus, one may choose $p_1(t) = 1/t^2$, $p_2(t) = 0$, $q_1(y) = 1$, $q_2(y) = q_3(y) = y$, $\alpha_k = 1/(k^2+k)^2$ *and* $\beta_k = 1/k$ *so that (H1) is satisfied.*

Now, we need to check for the validity of the hypotheses (H2) and (H3). Let $a = 2$. *Observe that* $\overline{n}(s) = i$ *if* $s \in (i-1, i]$, *and*

$$\mu(s,a) = \mu(i,2) = \prod_{k=2}^{i} \frac{(k+1)^2}{k^2} = \frac{(i+1)^2}{4}, \quad H_k = \frac{k+1}{k^3} + \frac{2}{k^3(k+1)}.$$

So, we have

$$\begin{aligned} \int_a^\infty \frac{u(s)}{\mu(s,a)}(p_1(s)+p_2(s))\mathrm{d}s + \sum_{k=\underline{n}(a)}^{\infty} \frac{H_k}{\mu(k,a)} &= \sum_{i=3}^{\infty} \int_{i-1}^{i} \frac{4}{(i+1)^2} \frac{i}{s^4} \mathrm{ds} \\ &\quad + \sum_{k=2}^{\infty} \frac{4}{(k+1)^2} \Big(\frac{k+1}{k^3} + \frac{2}{k^3(k+1)} \Big) \\ &= \sum_{i=3}^{\infty} \frac{4i(3i^2-3i+1)}{3(i-1)^3 i^3 (i+1)^2} \\ &\quad + \sum_{k=2}^{\infty} \Big(\frac{4}{k^3(k+1)} + \frac{8}{k^3(k+1)^3} \Big) \end{aligned} \tag{19}$$

and

$$\sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{\alpha_k}{(1-a_k)u(\tau_k)} = \sum_{k=2}^{\overline{n}(t)} \frac{1}{(k+1)^3} \tag{20}$$

which are both finite.

Thus, all the hypotheses of Theorem 3 hold, and hence there exists a solution $y(t)$ *of the impulsive BVP* (18).

Remark 2. *If the right-hand sides of the hypotheses (H2) and (H3) are replaced with* $O(1/v(t))$, *where* v *is a nonprincipal solution of the homogeneous impulsive equation* (5), *then the impulsive BVP* (3) *satisfies the boundary condition*

$$y(t) = O(1), \quad t \to \infty,$$

i.e., the solution turns out to be bounded.

Indeed, in Example 1, it can be seen from (19) *and* (20) *that there exist some positive constants C_1 and C_2 such that*

$$\int_a^t \frac{u(s)}{\mu(s,a)}(p_1(s)+p_2(s))\mathrm{d}s + \sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{H_k}{\mu(k,a)} \le \frac{C_1}{t^3} + o(t^3) = o(v(t)), \quad t \to \infty$$

and

$$\sum_{k=\underline{n}(a)}^{\overline{n}(t)} \frac{\alpha_k}{(1-a_k)u(\tau_k)} \le \frac{C_2}{k^2} = O\Big(\frac{1}{v(t)}\Big), \quad t \to \infty$$

since $v(t) = kt$, $t \in (k-1, k]$. Hence, the impulsive BVP (18) *has at least one bounded solution.*

5. Conclusion

In this paper, the existence of solutions for impulsive BVPs on an infinite interval was obtained under some weak conditions. As the impulses act on both the solution and its derivative, i.e., the solutions have discontinuities, and both the differential equation and the impulses are nonlinear, it turns out that the impulsive BVP (3) is in a quite general form. The main innovation in the study is to use the principal and nonprincipal solutions of the associated impulsive homogeneous equation. Also, slightly modifying the hypotheses of the main theorem, it was shown that the considered impulsive BVP has a bounded solution.

Declaration of Competing Interests The author declares that there is no competing interest regarding the publication of this paper.

Acknowledgements The author would like to thank anonymous referees for carefully reading the paper and for their helpful suggestions.

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