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Some Soft Algebraic Structures

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Abstract - Soft set theory was initiated by Molodtsov [9]. Then, Maji et al [8] defined some operations on soft sets. Aktaş and Çağman [2] defined soft groups and examined some related properties. Feng et al. [5] defined soft semirings. Acar et al. [1] defined soft rings and studied some of their properties In this study, we have defined soft fields and introduced some of their properties. Moreover, we have defined some kinds of ideals of soft rings and soft quotient rings.

Keywords - *Soft sets, Soft rings, Soft fields, Soft ideals, Soft quotient rings.*

1. Introduction

Facing up to uncertainties is a serious problem in many areas such as economics, engineering, environmental science, medical science and social sciences. Classical methods may not be successful to manage with such kind of problems. Because, classical methods have structural restrictions and difficulties.

In order to deal with these kinds of restrictions and difficulties, Molodtsov [9] introduced a completely new approach, which is called soft set theory, for modeling uncertainty. Then, Maji et al. [8] introduced several operations on soft sets. Aktaş and Çağman [2] defined soft groups and obtained the main properties of these groups.

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Moreover, they compared soft sets with fuzzy sets and rough sets. In addition, Jun et al. [7] defined soft ideals on BCK/BCI-algebras. Feng et al. [5] defined soft semirings, soft ideals on soft semirings and idealistic soft semirings. Qiu-Mei Sun et al. [10] defined the concept of soft modules and studied their basic properties. Finally, Acar et al. [1] defined soft rings and introduced some of their elementary properties. In this paper, we introduce soft fields and study their basic properties. Moreover, we introduce some other soft algebraic structures.

The rest of the paper is organized as follows. In Section 2, we give some necessary definitions about soft sets. We have defined soft integral domains and soft fields in Section 3 and studied some of their properties. We have presented some results on soft ideals, soft quotient rings and idealistic soft rings in Section 4. Finally, we give a conclusion in Section 5.

2. Preliminaries

Through this paper, U denotes an initial universe set, $\mathcal{P}(U)$ denotes the power set of U, E is a set of parameters and A is a subset of E. In addition, R denotes an arbitrary ring unless it is specified, D is any integral domain and Γ and Γ' are arbitrary fields. When R is a commutative ring with unity, for any element $a \in R$, $\langle a \rangle = \{ar \mid r \in R\}$ denotes the principal ideal of R generated by a.

If K is a subring (resp. subfield) of a ring (resp. field) L we shall write $K \leq L$. If I is an ideal of a ring R we write $I \lhd R$, the quotient ring (or factor ring) of R modulo I is denoted by R/I.

Definition 2.1. *[9] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \longrightarrow \mathcal{P}(U)$.*

In other words, a soft set (F, A) over U is a parametrized family of subsets of U, and for each $x \in A$, $F(x)$ may be considered as the x-elements of (F, A) or x-approximate elements of (F, A).

Definition 2.2. *[8] Let (F, A) and (G, B) be two soft sets over a common universe U. Then (G, B) is called a soft subset of (F, A) if it satisfies the following conditions:*
(1) $B \subset A$.
(2) For all $x \in B$, $F(x)$ and $G(x)$ are identical approximations.

Definition 2.3. *[3] Let (F, A) and (G, B) be soft sets over a common universe U, where $C = A \cap B \neq \emptyset$. The restricted intersection of (F, A) and (G, B) is defined as the soft set (H, C) satisfying $H(x) = F(x) \cap G(x)$ for all $x \in C$. This is denoted by $(F, A) \Cap (G, B) = (H, C)$.*

Similarly, the restricted intersection of a nonempty family of suitable soft sets is defined as follows.

Definition 2.4. *Let* $(F_i, A_i)_{i\in I}$ *be a nonempty family of soft sets over a common universe* U*, where* $C = \bigcap_{i\in I} A_i \neq \emptyset$*. The restricted intersection of these soft sets is defined as the soft set* (H, C) *such that* $H(x) = \bigcap_{i\in I} F_i(x)$ *for all* $x \in C$*. This is denoted by* $\Cap_{i\in I}(F_i, A_i) = (H, C)$*.*

Definition 2.5. *[3]) Let* (F, A) *and* (G, B) *be soft sets over a common universe* U *such that* $C = A \cap B \neq \emptyset$*. The restricted union of* (F, A) *and* (G, B) *is defined as the soft set* (H, C) *satisfying* $H(x) = F(x) \cup G(x)$ *for all* $x \in C$*. This is denoted by* $(F, A) \cup_{\mathcal{R}} (G, B) = (H, C)$*.*

Definition 2.6. *[8] Let* (F, A) *and* (G, B) *be two soft sets over a common universe* U*. The union of* (F, A) *and* (G, B) *is defined as the soft set* (H, C) *satisfying the following conditions:*
(1) $C = A \cup B$*.*
(2) For all $x \in C$*,*

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B, \\ G(x) & \text{if } x \in B - A, \\ F(x) \cup G(x) & \text{if } x \in A \cap B. \end{cases}$$

This is denoted by $(F, A) \,\tilde{\cup}\, (G, B) = (H, C)$*.*

Definition 2.7. *[5] Let* $(F_i, A_i)_{i\in I}$ *be a nonempty family of soft sets over a common universe* U*. The union of these soft sets is defined as the soft set* (H, C) *satisfying the following conditions:*
(1) $C = \bigcup_{i\in I} A_i$*.*
(2) For all $x \in C$*,* $H(x) = \bigcup_{i\in I(x)} F_i(x)$ *where* $I(x) = \{i \in I \mid x \in A_i\}$*.*
This is denoted by $\widetilde{\bigcup}_{i\in I}(F_i, A_i) = (H, C)$*.*

Definition 2.8. *[8] If* (F, A) *and* (G, B) *are two soft sets over a common universe* U*, then* (F, A) *AND* (G, B) *denoted by* $(F, A) \wedge (G, B)$ *is defined as* $(F, A) \wedge (G, B) = (H, C)$*, where* $C = A \times B$ *and* $H(x, y) = F(x) \cap G(y)$*, for all* $(x, y) \in C$*.*

Definition 2.9. *[8] If* (F, A) *and* (G, B) *are two soft sets over a common universe* U*, then* (F, A) *OR* (G, B) *denoted by* $(F, A) \vee (G, B)$ *is defined as* $(F, A) \vee (G, B) = (H, C)$*, where* $C = A \times B$ *and* $H(x, y) = F(x) \cup G(y)$*, for all* $(x, y) \in C$*.*

We define a nonempty soft set as follows.

Definition 2.10. *Let* (F, A) *be a soft set over* U*. If* $F(x) \neq \emptyset$ *for all* $x \in A$*, then* (F, A) *is called a nonempty soft set.*

Definition 2.11. *[4] Let (F, A) and (G, B) be nonempty soft sets over U. The Cartesian product of (F, A) and (G, B) is defined as the soft set $(F, A) \times (G, B) = (H, A \times B)$ over $U \times U$, where $H : A \times B \longrightarrow \mathcal{P}(U \times U)$ is the mapping given by $H(x, y) = F(x) \times G(y)$ for all $(x, y) \in A \times B$.*

The Cartesian product of three or more nonempty soft sets can be defined similarly.

Definition 2.12. *[5] Let (F, A) be a soft set. The set $Supp(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called the support of the soft set (F, A). A soft set is said to be non-null if its support is not equal to the empty set.*

In this study, all considered soft sets are supposed to be non-null.

Definition 2.13. *[1] Let (F, A) be a soft set over R. Then (F, A) is called a soft ring over R if $F(x)$ is a subring of R for all $x \in A$.*

Definition 2.14. *[1] Let (F, A) and (G, B) be soft rings over R. Then (G, B) is called a soft subring of (F, A), which is denoted by $(G, B)\widetilde{\leq}(F, A)$, if it satisfies the following conditions:*

(1) $B \subset A$.

(2) $G(x)$ is a subring of $F(x)$, for all $x \in Supp(G, B)$.

Definition 2.15. *[1] Let (F, A) be a soft ring over R. A non-null soft set (γ, I) over R is called soft ideal of (F, A), which will be denoted by $(\gamma, I)\widetilde{\lhd}(F, A)$, if it satisfies the following conditions:*
(1) $I \subset A$.
(2) $\gamma(x)$ is an ideal of $F(x)$ for all $x \in Supp(\gamma, I)$.

Definition 2.16. *[1] Let (F, A) be a soft set over R. Then (F, A) is called an idealistic soft ring over R if $F(x)$ is an ideal of R for all $x \in Supp(F, A)$.*

Definition 2.17. *[1] An idealistic soft ring (F, A) over R is said to be trivial if $F(x) = \{0_R\}$ for all $x \in A$. An idealistic soft ring (F, A) over R is said to be whole if $F(x) = R$ for all $x \in A$.*

Let (F, A) be a soft set over R and $f : R \longrightarrow R'$ be a mapping of rings. As mentioned in [1], we can define a soft set $(f(F), A)$ over R', where $f(F) : A \longrightarrow \mathcal{P}(R')$ is defined as $f(F)(x) = f(F(x))$ for all $x \in A$, see [1]. Observe that $Supp(f(F), A) = Supp(F, A)$.

Definition 2.18. *[1] Let (F, A) and (G, B) be soft rings over the rings R and R' respectively. Let $f : R \longrightarrow R'$ and $g : A \longrightarrow B$ be two mappings. The pair (f, g) is called a soft ring homomorphism if the following conditions are satisfied:*
(1) f is a ring epimorphism,

(2) g is surjective,
(3) $f(F(x)) = G(g(x))$ for all $x \in A$.
If we have a soft ring homomorphism between (F, A) and (G, B), (F, A) is said to be soft homomorphic to (G, B), which is denoted by $(F, A) \sim (G, B)$. In addition, if f is a ring isomorphism and g is bijective, then (f, g) is called a soft ring isomorphism. In this case, we say that (F, A) is soft isomorphic to (G, B), which is denoted by $(F, A) \simeq (G, B)$.

3. Soft Integral Domains and Soft Fields

We define a commutative soft ring and a soft ring with identity as follows.

Definition 3.1. *Let (F, A) be a soft ring over R. If $F(x)$ is a commutative ring for all $x \in A$, then (F, A) is called a commutative soft ring. Moreover, If $F(x)$ is a ring with identity for all $x \in A$, then (F, A) is called a soft ring with identity.*

Example 3.2. Let $A = R = \mathbb{Z}$, $2 < n \in \mathbb{Z}^+$, $B = \{m \in \mathbb{Z}^+ \ : \ 1 \leq m \leq n\}$ and $S = M_n(\mathbb{Z})$ the ring of all $n \times n$ matrices over $\mathbb{Z}$ under the usual matrix addition and matrix multiplication. Consider the functions $F : A \longrightarrow \mathcal{P}(R)$ and $G : B \longrightarrow \mathcal{P}(S)$ defined by $F(x) = x\mathbb{Z}$ and $G(y) = D_y(\mathbb{Z})$ which is the subring of S consisting of all $n \times n$ diagonal matrices whose jth rows and jth columns are zero for all $y + 1 \leq j \leq n$ if $1 \leq y \leq (n-1)$ and $G(n) = D_n(\mathbb{Z})$ the subring of S consisting of all $n \times n$ diagonal matrices. Then (F, A) is a commutative soft ring and (G, B) is a commutative soft ring with identity.

It is known that very ring R can be embedded in a ring S with identity, see e. g. [6, Theorem 1.10]. Actually, if R is a ring then $S = R \times \mathbb{Z}$ is a ring with identity $(0, 1)$ and characteristic zero under the following operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 + y_2x_1 + y_1x_2, y_1y_2)$$

and the map $f : R \longrightarrow S$ given by $f(r) = (r, 0)$ is a ring monomorphism.

Knowing this fact, we can give the following example for a commutative soft ring with identity.

Example 3.3. Let $A = \mathbb{Z}$ and $R = \mathbb{Z} \times \mathbb{Z}$ which is a commutative ring with identity $(0, 1)$ under the given operations above. Let $F : A \longrightarrow \mathcal{P}(R)$ be the function defined by $F(x) = x\mathbb{Z} \times \mathbb{Z}$. Then (F, A) is a commutative soft ring with identity.

In classical algebra, a commutative ring D with identity containing no zero-divisor is called an integral domain. Now, we define soft integral domains, soft Euclidean domains, soft principal ideal domains and soft unique factorization domains.

Definition 3.4. *Let (F, A) be a soft set over an integral domain D. Then (F, A) is called a soft integral domain (soft Euclidean domain, soft principal ideal domain, soft unique factorization domain resp.) if $F(x)$ is a sub integral domain (sub euclidean domain, sub principal ideal domain, sub unique factorization domain resp.) of D for all $x \in A$.*

Example 3.5. Let $A = \mathbb{Z} - \{0\}$ and $R = \mathbb{Q}[x]$. Consider the function $F : A \longrightarrow \mathcal{P}(R)$ defined by $F(a) = \langle a \rangle = \mathbb{Q}[x]$ for $a \in A$. Since $\mathbb{Q}[x]$ is an Euclidean domain, a principal ideal domain and a unique factorization domain; (F, A) is a soft Euclidean domain, a soft principal ideal domain and a soft unique factorization domain.

We define a soft field as follows.

Definition 3.6. *Let (F, A) be a soft set over Γ. Then (F, A) is called a soft field over Γ if $F(x)$ is a subfield of Γ for all $x \in A$.*

Example 3.7. Let $A = \mathbb{Z}^+$ and $\Gamma = \mathbb{R}$. Consider the function $F : A \longrightarrow \mathcal{P}(\Gamma)$ defined by $F(a) = \mathbb{Q}(\sqrt{a})$ for $a \in A$, where $\mathbb{Q}(\sqrt{a})$ is the smallest subfield of $\mathbb{R}$ containing $\mathbb{Q}$ and $\sqrt{a}$. Then (F, A) is a soft field over Γ.

Naturally, we can define idealistic soft fields which are similar to idealistic soft rings as follows.

Definition 3.8. *A soft field (F, A) over Γ is called an idealistic soft field if $F(x)$ is an ideal of Γ for all $x \in A$.*

Combining Definition 2.17, Definition 3.6, Definition 3.8 and using the fact that every soft field is a soft ring, we see that an idealistic soft field is a whole idealistic soft ring. Moreover, we observe that if $f : \Gamma \longrightarrow \Gamma'$ is a field epimorphism and (F, A) is an idealistic soft field over Γ, then $(f(F), A)$ is an idealistic soft field over Γ'.

Definition 3.9. *Let (F, A) and (G, B) be soft fields over Γ. Then (G, B) is called a soft subfield of (F, A) if it satisfies the following conditions:*
(1) $B \subset A$.
(2) $G(x)$ is a subfield of $F(x)$ for all $x \in Supp(G, B)$.
In this case, we write $(G, B)\widetilde{\leq}(F, A)$.

For an example of soft subfield, see Example 3.17.

The following three theorems are the results of the related definitions and the fact that the intersection of any number of subfields of a field Γ is a subfield of itself. So, we present them without detailed proofs.

Theorem 3.10. *Let (F, A) and (G, B) be soft fields over Γ. Then*
(1) $(F, A) \wedge (G, B)$ is a soft field over Γ,
(2) $(F, A) \Cap (G, B)$ is a soft field over Γ.

Proof. Similar to the proof of [1, Theorem 3.3]. □

Example 3.11. Let $A = \mathbb{Z}^+$, $B = \{p \in \mathbb{Z} \mid p \text{ is prime}\}$ and $\Gamma = \mathbb{R}$. Consider the functions $F : A \longrightarrow \mathcal{P}(\Gamma)$ and $G : B \longrightarrow \mathcal{P}(\Gamma)$ given by $F(a) = \mathbb{Q}(\sqrt{a})$ and $G(b) = \mathbb{Q}(\sqrt{b})$. For $a \in A$ and $b \in B$, as being intersection of two subfields of $\mathbb{R}$, $\mathbb{Q}(\sqrt{a}) \cap \mathbb{Q}(\sqrt{b})$ is a subfield of $\mathbb{R}$. Therefore, $(F, A) \wedge (G, B)$ and $(F, A) \Cap (G, B)$ are soft fields over $\Gamma = \mathbb{R}$.

Theorem 3.12. *Let (F, A) and (G, B) be soft fields over Γ. Then $(F, A) \Cap (G, B)$ is a soft subfield of both (F, A) and (G, B).*

Proof. Similar to the proof of [1, Theorem 3.6]. □

Theorem 3.13. *Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft fields over Γ. Then we have the followings:*
(1) $\bigwedge_{i \in I} (F_i, A_i)$ is a soft field over Γ.
(2) $\Cap_{i \in I} (F_i, A_i)$ is a soft field over Γ.
(3) If $\{A_i \mid i \in I\}$ are pairwise disjoint, then $\widetilde{\bigcup_{i \in I}} (F_i, A_i)$ is a soft field over Γ.

Proof. Similar to the proof of [1, Theorem 3.8]. □

Definition 3.14. *Let Γ be the field of fractions of the integral domain D, (F, A) be a soft field over Γ and (G, B) be a soft integral domain over D, where $B \subset A$. Then (F, A) is called the soft field of fractions of (G, B) if $F(x)$ is the field of fractions of $G(x)$ for all $x \in B$.*

Example 3.15. Let $A = \mathbb{Z}^+$, $\emptyset \neq B \subset A$ and $\Gamma = \mathbb{R}(x)$ be the field of fractions of the polynomial ring $D = \mathbb{R}[x]$, which is an integral domain, with indeterminate x. Consider the functions $F : A \longrightarrow \mathcal{P}(\Gamma)$ and $G : B \longrightarrow \mathcal{P}(D)$ defined by $F(a) = \mathbb{Q}(\sqrt{a})(x)$ and $G(b) = \mathbb{Q}(\sqrt{b})[x]$. Then (F, A) is a soft field over Γ and (G, B) is a soft integral domain over D. Since $F(b) = \mathbb{Q}(\sqrt{b})(x)$ is the field of fractions of $G(b) = \mathbb{Q}(\sqrt{b})[x]$ for all $b \in B$, (F, A) is the soft field of fractions of (G, B).

Definition 3.16. *Let (F, A) and (G, B) be soft fields over Γ such that $(G, B) \widetilde{\leq} (F, A)$. Then (F, A) is called a soft field extension of the soft field (G, B).*

Example 3.17. Let $A = \mathbb{Z} - \{0\}$, $B = \mathbb{Z}^+$ and $\Gamma = \mathbb{R}$. Consider the functions $F : A \longrightarrow \mathcal{P}(\Gamma)$ and $G : B \longrightarrow \mathcal{P}(\Gamma)$ defined by $F(a) = \langle a \rangle = \mathbb{R}$ and $G(b) = \mathbb{Q}(\sqrt{b})$. Then (F, A) and (G, B) are soft fields over Γ and $(G, B) \widetilde{\leq} (F, A)$ since $G(b) = \mathbb{Q}(\sqrt{b}) \leq F(b) = \mathbb{R}$ for all $b \in B$.

Example 3.18. Let $A = \mathbb{Z}^+$, $B = \{p \in \mathbb{Z} \mid p \text{ is prime}\}$ and $\Gamma = \mathbb{R}$. Consider the functions $F : A \longrightarrow \mathcal{P}(\Gamma)$ and $G : B \longrightarrow \mathcal{P}(\Gamma)$ defined by $F(a) = \mathbb{Q}(\sqrt{a}, \sqrt{a^3})$ and $G(b) = \mathbb{Q}(\sqrt{b})$; where $\mathbb{Q}(\sqrt{a}, \sqrt{a^3})$ is the smallest subfield of $\mathbb{R}$ containing $\mathbb{Q}$, $\sqrt{a}$ and $\sqrt{a^3}$. Then (F, A) and (G, B) are soft fields over Γ and $(G, B) \widetilde{\leq} (F, A)$ since $G(b) = \mathbb{Q}(\sqrt{b}) \leq F(b) = \mathbb{Q}(\sqrt{b}, \sqrt{b^3})$ for all $b \in B$.

Definition 3.19. *Let (F, A) and (G, B) be soft fields over Γ such that $(G, B)\widetilde{\leq}(F, A)$. Then (F, A) is called a soft algebraic extension of (G, B) if $F(x)$ is an algebraic extension of $G(x)$ for all $x \in B$.*

Definition 3.20. *Let (F, A) and (G, B) be soft fields over Γ such that $(G, B)\widetilde{\leq}(F, A)$. Then (F, A) is called a soft finite extension of (G, B) if $F(x)$ is a finite extension of $G(x)$ for all $x \in B$.*

As a consequence of Definition 3.19 and Definition 3.20, since every finite extension of a field is an algebraic extension, we see that every soft finite extension (F, A) of (G, B) over Γ is a soft algebraic extension.

Example 3.21. Let $A = \mathbb{Z}^+$, $B \subset A$ and $\Gamma = \mathbb{R}$. We can define the functions $F : A \longrightarrow \mathcal{P}(\Gamma)$ and $G : B \longrightarrow \mathcal{P}(\Gamma)$ given by $F(a) = \mathbb{Q}(\sqrt{a})$ and $G(b) = \mathbb{Q}$. We immediately see that (F, A) and (G, B) are soft fields over Γ. Since $F(b) = \mathbb{Q}(\sqrt{b})$ is both finite and algebraic extension of $\mathbb{Q}$ for all $b \in B$, (F, A) is both soft finite and soft algebraic extension of (G, B).

Definition 3.22. *Let (F, A) be a soft field over Γ. Then (F, A) is called an algebraically closed soft field if the algebraic closure $\overline{F(x)}$ of $F(x)$ is equal to Γ for all $x \in A$.*

Example 3.23. Let $A = \mathbb{Z}^+$ and $\Gamma = \overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic over } \mathbb{Q}\}$ which is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Consider the function $F : A \longrightarrow \mathcal{P}(\Gamma)$ defined by $F(a) = \mathbb{Q}(\sqrt{a})$. Since $\overline{F(a)} = \overline{\mathbb{Q}(\sqrt{a})} = \overline{\mathbb{Q}}(\sqrt{a}) = \overline{\mathbb{Q}} = \Gamma$ for all $a \in A$, (F, A) is an algebraically closed soft field.

4. Some Results on Soft ideals, Soft Quotient Rings and Idealistic Soft Rings

Firstly, we define some special soft ideals of soft rings and present some results on them.

Definition 4.1. *Let (F, A) and (G, B) be soft rings over R such that $(G, B)\widetilde{\lhd}(F, A)$. (G, B) is called a soft prime (maximal) ideal of (F, A) if $G(x)$ is a prime (maximal) ideal of $F(x)$ for all $x \in B$.*

Example 4.2. Let $A = R = \mathbb{Z}$ and $B = \{p \in \mathbb{Z} \mid p \textit{ is prime}\}$. Consider the functions $F : A \longrightarrow \mathcal{P}(R)$ and $G : B \longrightarrow \mathcal{P}(R)$ defined by $F(a) = \mathbb{Z}$ and $G(p) = p\mathbb{Z}$. Then (F, A) is a commutative soft ring with identity and (G, B) is a commutative soft ring over R and $(G, B)\widetilde{\lhd}(F, A)$. Since $G(p) = p\mathbb{Z}$ is both maximal and prime ideal of $F(p) = \mathbb{Z}$ for all prime numbers $p \in B$, (G, B) is both soft prime and soft maximal ideal of (F, A).

Example 4.3. Let $A = \mathbb{Z} - \{0\}$, $R = \mathbb{Q}[x]$ and $B = \{p \in \mathbb{Z} \mid \text{p is prime}\}$. Consider the functions $F : A \longrightarrow \mathcal{P}(R)$ and $G : B \longrightarrow \mathcal{P}(R)$ given by $F(a) = \langle a \rangle = \mathbb{Q}[x]$ and $G(p) = \langle x^2 - p \rangle = \{(x^2 - p)f(x) \mid f(x) \in \mathbb{Q}[x]\}$. Then (F, A) is a commutative soft ring with identity and (G, B) is a commutative soft ring over R and $(G, B)\widetilde{\triangleleft}(F, A)$. Since $G(p) = \langle x^2 - p \rangle$ is both maximal and prime ideal of $F(a) = \langle a \rangle = \mathbb{Q}[x]$ for all prime numbers $p \in B$, (G, B) is both soft prime and soft maximal ideal of (F, A). Moreover; since $\mathbb{Q}[x]$ is an Euclidean domain, a principal ideal domain and a unique factorization domain; (F, A) is a soft Euclidean domain, a soft principal ideal domain and a soft unique factorization domain.

Theorem 4.4. *Let (F, A), (γ_1, B) and (γ_2, C) be soft rings over R such that (γ_1, B) and (γ_2, C) are prime (maximal) ideals of (F, A). If $B \cap C = \emptyset$, then $(\gamma_1, B) \widetilde{\bigcup} (\gamma_2, C)$ is a soft prime (maximal) ideal of (F, A).*

Proof. According to Definition 2.6, $(\gamma_1, B) \widetilde{\bigcup} (\gamma_2, C) = (\gamma, D)$ where $B \cup C = D$. For all $x \in D$, we have

$$\gamma(x) = \begin{cases} \gamma_1(x) & \text{if } x \in B - C, \\ \gamma_2(x) & \text{if } x \in C - B, \\ \gamma_1(x) \cup \gamma_2(x) & \text{if } x \in B \cap C. \end{cases}$$

$$= \begin{cases} \gamma_1(x) & \text{if } x \in B, \\ \gamma_2(x) & \text{if } x \in C. \end{cases}$$

As we see, the result follows directly since, for $x \in B$ and $y \in C$, $\gamma_1(x)$ and $\gamma_2(y)$ are prime (maximal) ideals of R. $\square$

Every maximal ideal of a commutative ring R with identity is a prime ideal. Using this fact, we have the following result.

Theorem 4.5. *Let (F, A) be a commutative soft ring with identity over R. Then every soft maximal ideal of (F, A) is a soft prime ideal.*

Proof. Let (F, A) be a commutative soft ring with identity over R and (γ, B) be a soft maximal ideal of (F, A). Then $\gamma(x)$ is a maximal ideal of $F(x)$ for all $x \in B$. So, $\gamma(x)$ is a prime ideal of $F(x)$ for all $x \in B$. Consequently, (γ, B) is a soft prime ideal of (F, A). $\square$

Note that, as in the classical algebra, the converse of Theorem 4.5 is not true. But, for a finite commutative ring R with identity, the converse of this theorem is also true. Because, if R is a finite commutative ring with identity, then every prime ideal of R is a maximal ideal.

Now, we define a soft quotient ring (or soft factor ring) as follows.

Definition 4.6. *Let (F, A) be a soft ring over R, $I \lhd R$ and (γ, B) be a soft ideal of (F, A), where $\gamma(x) = I$ for all $x \in B$. Then we define the soft quotient ring (or soft factor ring) of (F, A) modulo (γ, B) over the ring R/I as $(F, A)/(\gamma, B) = (H, A)$ where $H : A \longrightarrow \mathcal{P}(R/I)$ is the mapping given by $H(x) = F(x)/I$ for all $x \in A$.*

Example 4.7. Let $A = \{1, 2, 3, 6\}$, $B = \{3, 6\}$ and $R = \mathbb{Z}$. Consider the functions $F : A \longrightarrow \mathcal{P}(R)$ and $\gamma : B \longrightarrow \mathcal{P}(R)$ defined by $F(a) = a\mathbb{Z}$ for all $a \in A$ and $\gamma(b) = 6\mathbb{Z}$ for all $b \in B$. Then (F, A) and (G, B) are commutative soft rings over R and $(\gamma, B)\widetilde{\lhd}(F, A)$. The soft quotient ring of (F, A) modulo (γ, B) is the soft ring $(F, A)/(\gamma, B) = (H, A)$, where the map $H : A \longrightarrow \mathcal{P}(\mathbb{Z}/6\mathbb{Z})$ is given by $H(a) = a\mathbb{Z}/6\mathbb{Z}$ such that $H(1) = 1\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}_6$, $H(2) = 2\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}_3$, $H(3) = 3\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}_2$ and $H(6) = 6\mathbb{Z}/6\mathbb{Z} \simeq \{0\}$.

Let R be a commutative ring with identity and $I \neq R$ be an ideal of R. We know that I is a prime ideal of R if and only if R/I is an integral domain and I is a maximal ideal of R if and only if R/I is a field. Using these facts, we can state the following two theorems.

Theorem 4.8. *Let R be a commutative ring with identity, I be an ideal of R, (F, A) be a soft ring over R and $(\gamma, B)\widetilde{\lhd}(F, A)$ such that $\gamma(x) = I$ for all $x \in B$. Then R/I is an integral domain and $(F, A)/(\gamma, B)$ is a soft integral domain over R/I if and only if I is a prime ideal of R and (γ, B) is a soft prime ideal of (F, A).*

Proof. Result of Definition 4.1 and Definition 4.6. □

Theorem 4.9. *Let R be a commutative ring with identity, I be an ideal of R, (F, A) be a soft ring over R and $(\gamma, B)\widetilde{\lhd}(F, A)$ such that $\gamma(x) = I$ for all $x \in B$. Then R/I is a field and $(F, A)/(\gamma, B)$ is a soft field over R/I if and only if I is a maximal ideal of R and (γ, B) is a soft maximal ideal of (F, A).*

Proof. Direct result of Definition 4.1 and Definition 4.6. □

Now, we shall present some results on idealistic soft rings.

Theorem 4.10. *Let (F, A) and (G, B) be idealistic soft rings over R. Then we have the followings:*
(1) $(F, A) \Cap (G, B)$ is an idealistic soft ring over R whenever $A \cap B \neq \emptyset$.
(2) $(F, A) \wedge (G, B)$ is an idealistic soft ring over R.
(3) $(F, A) \,\widetilde{\cup}\, (G, B)$ is an idealistic soft ring over R provided that $A \cap B = \emptyset$..
(4) $(F, A) \sqcap_{\varepsilon} (G, B)$ is an idealistic soft ring over R.
(5) $(F, A) \cup_{\Re} (G, B)$ is an idealistic soft ring over R if $C = A \cap B \neq \emptyset$ and $F(x)$ and $G(x)$ are ordered by the set inclusion for all $x \in Supp((F, A) \cup_{\Re} (G, B))$.
(6) $(F, A) \vee (G, B)$ is an idealistic soft ring over R if $F(x)$ and $G(y)$ are ordered by the set inclusion for all $(x, y) \in Supp((F, A) \vee (G, B))$.

Proof. (1), (2) and (3) follow from [1].
(4) By definition, we have $(F, A) \sqcap_\varepsilon (G, B) = (H, C)$, where $C = A \cup B$ and

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B, \\ G(x) & \text{if } x \in B - A, \\ F(x) \cap G(x) & \text{if } x \in A \cap B. \end{cases}$$

Let $x \in Supp(H, C)$. If $x \in A - B$ then $H(x) = F(x)$ is an ideal of R and if $x \in B - A$ then $H(x) = G(x)$ is an ideal of R. If $x \in A \cap B$, then $H(x) = F(x) \cap G(x)$ is an ideal of R since the intersection of any number of ideals of a ring is an ideal of itself. Consequently, (H, C) is an idealistic soft ring over R.
(5) By definition, we have $(F, A) \cup_\Re (G, B) = (H, C)$, where $H(x) = F(x) \cup G(x)$ for all $x \in C = A \cap B$. If $x \in Supp(H, C)$, then $H(x) = F(x) \cup G(x)$. Since $F(x)$ and $G(x)$ are ordered by the set inclusion we have $F(x) \cup G(x) = F(x)$ or $F(x) \cup G(x) = G(x)$. While both $F(x)$ and $G(x)$ are ideals of R, $H(x)$ is also and ideal of R. Hence, (H, C) is an idealistic soft ring over R.
(6) $(F, A) \vee (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \cup G(y)$ for all $(x, y) \in A \times B$. Since $F(x)$ and $G(y)$ are ordered by the set inclusion for all $(x, y) \in Supp(H, A \times B)$, $F(x) \cup G(y) = F(x)$ or $F(x) \cup G(y) = G(y)$ which means that $H(x, y)$ is an ideal of R for all $(x, y) \in A \times B$. Consequently, $(H, A \times B)$ is an idealistic soft ring over R. □

Theorem 4.11. *Let (F, A) and (G, B) be idealistic soft rings over the rings R_1 and R_2 respectively. Then $(F, A) \times (G, B)$ is an idealistic soft ring over $R_1 \times R_2$.*

Proof. By definition, we have $(F, A) \times (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \times G(y)$ for all $(x, y) \in A \times B$. Since $F(x)$ is an ideal of R_1 and $G(y)$ is an ideal of R_2, $H(x, y)$ is an ideal of $R_1 \times R_2$ for all $(x, y) \in Supp(H, A \times B)$. Therefore, $(H, A \times B)$ is an idealistic soft ring over $R_1 \times R_2$. □

Definition 4.12. *Let (F, I_1) and (G, I_2) be idealistic soft rings over the ring $(R, +, \cdot)$. Then the sum $(F, I_1) + (G, I_2)$ of idealistic soft rings (F, I_1) and (G, I_2) is defined as the soft set $(F, I_1) + (G, I_2) = (H, I_1 \times I_2)$ over R, where $H(x, y) = F(x) + G(y)$ for all $(x, y) \in Supp(H, I_1 \times I_2)$.*

Theorem 4.13. *Let (F, I_1) and (G, I_2) be idealistic soft rings over the ring $(R, +, \cdot)$. Then $(F, I_1) + (G, I_2)$ is an idealistic soft ring over R.*

Proof. We have $(F, I_1) + (G, I_2) = (H, I_1 \times I_2)$, where $H(x, y) = F(x) + G(y)$ for all $(x, y) \in I_1 \times I_2$. Since $F(x)$ and $G(y)$ are ideals of R, $H(x, y)$ is an ideal of R for all $(x, y) \in Supp(H, I_1 \times I_2)$. As a result, $(H, I_1 \times I_2)$ is an idealistic soft ring over R. □

Definition 4.14. *Let (F, A) be an idealistic soft ring over R. Then (F, A) is called a trivial idealistic soft ring (resp. an improper idealistic soft ring) over R if $F(x) = \{0\}$ (resp. $F(x) = R$) for all $x \in Supp(F, A)$.*

Theorem 4.15. *Let R be a ring with identity and (F, A) an idealistic soft ring over R such that $F(x)$ contains a unit element of R for all $x \in Supp(F, A)$. Then (F, A) is an improper idealistic soft ring over R.*

Proof. This is clear since if I is an ideal of R which contains a unit element of R, then $I = R$. □

Corollary 4.16. *Let R be a field and let (F, A) be an idealistic soft ring over R. Then (F, A) is an improper or a trivial idealistic soft ring over R.*

Proof. This follows directly from the fact that any field has only two ideals, namely zero ideal and itself. □

Theorem 4.17. *Let (F, A) and (G, B) be idealistic soft rings over R. Then we have the followings:*
(i) If (F, A) and (G, B) are trivial idealistic soft rings over R, then $(F, A) \Cap (G, B)$ is a trivial idealistic soft ring over R.
(ii) If (F, A) and (G, B) are improper idealistic soft rings over R, then $(F, A) \Cap (G, B)$ is an improper idealistic soft ring over R.
(iii) If (F, A) is trivial idealistic soft ring over R and (G, B) is an improper idealistic soft ring over R, then $(F, A) \Cap (G, B)$ is a trivial idealistic soft ring over R.

Proof. Straightforward. □

Theorem 4.18. *Let (F, I_1) and (G, I_2) be idealistic soft rings over the rings R_1 and R_2 respectively. Then we have the followings:*
(i) If (F, I_1) and (G, I_2) are trivial idealistic soft rings over R_1 and R_2 respectively, then $(F, I_1) \times (G, I_2)$ is a trivial idealistic soft ring over $R_1 \times R_2$.
(i) If (F, I_1) and (G, I_2) are improper idealistic soft rings over R_1 and R_2 respectively, then $(F, I_1) \times (G, I_2)$ is an improper idealistic soft ring over $R_1 \times R_2$.

Proof. Clear. □

Definition 4.19. *Let (F, A) be an idealistic soft ring over the ring R. Then (F, A) is called a maximal idealistic soft ring over R if $F(x)$ is a maximal ideal of R for all $x \in Supp(F, A)$.*

Example 4.20. Let $A = \{0, 3, 5\}$ and $R = \mathbb{Z}_{15}$. Consider the soft set (F, A) over R, where $F : A \longrightarrow \mathcal{P}(R)$ is defined by $F(0) = F(3) = \{0, 3, 6, 9, 12\}$, $F(5) = \{0, 5, 10\}$ which are maximal ideals of $\mathbb{Z}_{15}$. Consequently, (F, A) is a maximal idealistic soft ring over $\mathbb{Z}_{15}$.

Definition 4.21. *Let (F, A) be an idealistic soft ring over R. Then (F, A) is called a prime idealistic soft ring over R if $F(x)$ is a prime ideal of R for all $x \in Supp(F, A)$.*

Example 4.22. Let $A = \{x \in \mathbb{Z} \ : \ x \text{ is prime}\}$ and $R = \mathbb{Z}$. Consider the soft set (F, A) over R, where the function $F : A \longrightarrow \mathcal{P}(\mathbb{Z})$ is given by $F(x) = x\mathbb{Z}$ for all $x \in A$. We see that (F, A) is both maximal and prime idealistic soft ring over R since $F(x) = x\mathbb{Z}$ is both maximal and prime ideal of $\mathbb{Z}$ for all $x \in A$

Example 4.23. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $A = \{(0, 1), (1, 1)\}$. Let (F, A) be the soft set over R, where $F : A \longrightarrow \mathcal{P}(R)$ is defined by $F((0, 1)) = \{(0, 0), (0, 1), (0, 2), (0, 3)\} = H_1$ which is a cyclic group and $F((1, 1)) = \{(0, 0), (1, 0), (0, 2), (1, 2)\} = H_2$ which is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since R/H_1 and R/H_2 are cyclic groups of order two, $F((0, 1))$ and $F((1, 1))$ are maximal and prime ideals of R, Hence, (F, A) is both a maximal and prime idealistic soft ring over R.

Theorem 4.24. *If (F, A) is a prime (or maximal) idealistic soft ring over R and $\emptyset \neq B \subset A$ then so is $(F|_B, B)$, where $F|_B$ is the restriction of F to B.*

Proof. Clear. □

Theorem 4.25. *Let (F, A) and (G, B) be prime (resp. maximal) idealistic soft rings over R, where $A \cap B = \emptyset$. Then $(F, A) \,\widetilde{\cup}\, (G, B)$ is a prime (resp. maximal) idealistic soft ring over R.*

Proof. This follows directly from the definition of $(F, A) \,\widetilde{\cup}\, (G, B)$. □

Theorem 4.26. *Every maximal idealistic soft ring (F, A) over a commutative ring R with unity is a prime idealistic soft ring.*

Proof. Let (F, A) be maximal idealistic soft ring over R. Then, for all $x \in A$, $F(x)$ is a maximal ideal of R and therefore it is a prime ideal of R. So, (F, A) is prime idealistic soft ring over R. □

The converse of Theorem 4.26 is not true. For a finite commutative ring R with unity, the converse of the related theorem is also valid.

Definition 4.27. *Let (F, A) be an idealistic soft ring over a commutative ring R with unity. Then (F, A) is called a principal idealistic soft ring over R if $F(x)$ is a principal ideal of R for all $x \in Supp(F, A)$.*

Example 4.28. Every idealistic soft ring (F, A) over a principal ideal domain D is a principal idealistic soft ring.

5. Conclusion

The soft set concept and some basic algebraic structures on it are introduced by Molodtsov, Aktaş-Çağman, P. K. Maji et al., Y. B. Jun et al., F. Feng et al., etc. Acar et al. introduced soft rings and soft ideals etc. In this paper; we have defined some kinds of soft ideals, soft quotient rings, soft fields and studied some related concepts with soft rings and soft fields. Having these presented results, one can study further algebraic structures of soft rings and soft fields.

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