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Some Nordhaus - Gaddum Type Relations On Strong Efficient Dominating Sets

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Abstract – Let G = (V, E) be a simple graph with p vertices and q edges. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every v∈V(G), $|N_s[v]\cap S|=1(|N_w[v]\cap S|=1)$.$N_s(v) = \{u\in V(G)$ $uv\in E(G)$, $deg(u) \geq deg(v)\}$.The minimum cardinality of a strong (weak) efficient dominating set G is called strong (weak) efficient domination number of G and is denoted by $\gamma_{se}(G)(\gamma_{we}(G))$. A graph G is strong efficient if there exists a strong efficient dominating set of G. In this paper, the authors introduced a new parameter called the number of strong efficient dominating sets of a graph G denoted by # γ_{se} (G) and studied some Nordhaus- Gaddum type relations on strong efficient domination number of a graph and its derived graph. The relation between the number of strong efficient dominating sets of a graph and its derived graph is also studied.

Keywords - *Strong efficient dominating sets, Strong efficient domination number and number of strong efficient dominating sets*

1. Introduction

Throughout this paper, only finite, undirected and simple graphs are considered. Let G = (V,E) be a graph with p vertices and q edges. The degree of any vertex u in G is the number of edges incident with u and is denoted by deg u. The minimum and maximum degree of a vertex is denoted by δ(G) and Δ(G) respectively. A vertex of degree 0 in G is called an isolated vertex and a vertex of degree 1 in G is called a pendant vertex. A subset S of V(G) of a graph G is called a dominating set of G if every vertex in V(G) \ S is adjacent to a vertex in S [3]. The domination number γ(G) is the minimum cardinality of a dominating set of G. Sampathkumar and Pushpalatha introduced the concepts of strong and weak domination in graphs [6]. A subset S of V(G) is called a strong dominating set of G if for every v∈ V – S there exists a u∈S such that u and v are adjacent and deg u ≥ deg v. A subset S of V(G) is called an efficient dominating set of G if for every v∈V(G), $|N[v]\cap S|$ = 1[1]. The concept of strong (weak) efficient domination in graphs was introduced by Meena, Subramanian and Swaminathan [4]. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every v ∈ V(G), $|N_s[v]\cap S| = 1(|N_w[v]\cap S| = 1)$. $N_s(v) = \{ u \in V(G) : uv \in E(G), deg(u) \geq deg(v) \}$.The minimum cardinality of a strong (weak) efficient dominating set is called strong (weak) efficient domination number and is denoted by $\gamma_{se}(G)(\gamma_{we}(G))$. A graph G is strong efficient if there exists a strong efficient dominating set of G. In this paper, the authors introduced a new parameter called the number of strong efficient dominating sets of a graph G denoted by # γ_{se} (G) and studied some Nordhaus- Gaddum type relations on strong efficient domination number of a graph and its derived graph. They also studied the Nordhaus- Gaddum type relations on the number of strong efficient dominating sets of a graph and its derived graph. For all graph theoretic terminologies and notations, Harary [2] is followed.

2. Basic Definitions and Results

The following basic definitions and results are necessary for the present study.

Definition 2.1: A graph G with vertex set $V(G) = \{v, v_1, v_2, \ldots, v_n\}$ for $n \geq 3$ and edge set $E(G) = \{vv_i / 1 \leq i \leq n\} \cup \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_n v_1\}$is called a wheel graph of length n and is denoted by W_n. The vertex v is called the axial or central vertex of the wheel graph.

Definition 2.2: A gear graph G_n is obtained from the wheel graph W_n by adding a vertex between every pair of adjacent vertices in the cycle.

Definition 2.3: The Bistar $D_{m,n}$ is the graph obtained from K_2 by joining m pendant edges to one end vertex of K_2 and n pendant edges to the other end of K_2. The edge K_2 is called the central edge of $D_{m,n}$ and the vertices of K_2 are called the central vertices of K_2.

Definition 2.4: The H-graph of a path P_n is the graph obtained from two copies of $P_n$with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n}{2}}$ and $u_{\frac{n}{2}+1}$ if n is even.

Definition 2.5: The complement $\bar{G}$ of a graph G has $V(G)$ as its vertex set and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

Definition 2.6 [7]: A vertex switching G_v of a graph G is obtained by taking a vertex v of G, removing all edges incident to v and adding edges joining v to every vertex which are not adjacent to v in G.

Result 2.7 [4]: $\gamma_{se}(K_{1,n}) = 1$, $n \in N$

Result 2.8 [4]: For any path P_m, $\gamma_{se}(P_m) = \begin{cases} n \text{ if } m = 3n, n \in N \\ n+1 \text{ if } m = 3n+1, n \in N \\ n+2 \text{ if } m = 3n+2, n \in N \end{cases}$

Result 2.9 [4]:$\gamma_{se}(C_{3n}) = n$, $n \in N$

Result 2.10 [4]:$\gamma_{se}(K_n) = 1$, $n \in N$

Result 2.11 [4]:$\gamma_{se}(D_{r,s}) = r + 1$ where $r \leq s$

Result 2.12 [5]: A graph G does not admit a strong efficient dominating set if the distance between any two maximum degree vertices is exactly two.

Result 2.13 [4]: Any strong efficient dominating set is independent.

Result 2.14: If G_1 and G_2 are strong efficient graphs, then $G_1 \cup G_2$ is strong efficient.

3. Main Results

In this section, the authors studied the strong efficient domination number of some graphs and their derived graphs and also derived some Nordhaus- Gaddum type relations between them. It was found that there are several strong efficient dominating sets for a given graph. This motivated the authors to define a new parameter called the number of strong efficient dominating sets denoted by $\#\gamma_{se}(G)$. Using this parameter, the authors studied the Nordhaus- Gaddum type relations on the number of strong efficient dominating sets of a graph and its derived graph.

Definition 3.1: Let G be a graph with a strong efficient dominating number $\gamma_{se}(G)$. The number of distinct strong efficient dominating sets of a graph G is denoted by $\#\gamma_{se}(G)$.

Theorem 3.2: $\gamma_{se}(K_{1,n}) + \gamma_{se}(\overline{K_{1,n}}) = 3$ for all n ≥ 1 and $\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}(\overline{K_{1,n}})$ = n+1 for all n ≥ 1

For: $\overline{K_{1,n}}$ is the graph $K_n \cup K_1$. Therefore $\gamma_{se}(\overline{K_{1,n}}) = 2$ and $\#\gamma_{se}(\overline{K_{1,n}})$ = n. Hence the theorem.

Theorem 3.3: $\overline{P_n}$ is strong efficient if and only if $n \leq 4$. Also

$$\#\gamma_{se}(P_n) + \#\gamma_{se}(\overline{P_n}) = \begin{cases} 3 \text{ when } n = 2 \text{ or } 3 \\ 4 \text{ when } n = 4 \end{cases}$$

Proof: Let $\overline{P_n}$ be strong efficient. Suppose n > 4. Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of $\overline{P_n}$. Let S be a strong efficient dominating set of $\overline{P_n}$. Deg(v_1) = deg(v_n) = n-2 and deg(v_i) = n-3 for $2 \leq i \leq n-1$. Since S is independent, v_1 and v_n are adjacent, either $v_1 \in S$ or $v_n \in S$. Suppose $v_1 \in S$.Then v_1 strongly dominates all v_i where $3 \leq i \leq n$ and v_2 must be an element of S. Then $|N_s[v_i] \cap S| \geq 2$ for $4 \leq i \leq n-1$, a contradiction. The proof is similar if $v_n \in S$. Further, when n = 2, $\overline{P_2}$ is $2K_1$ and when n = 3, $\overline{P_3}$ is $K_2 \cup K_1$ which are obviously strong efficient with $\gamma_{se}(\overline{P_2})$ =2, # $\gamma_{se}(\overline{P_2})$ =1, $\gamma_{se}(\overline{P_3})$ =2 and # $\gamma_{se}(\overline{P_3})$ =2. In $\overline{P_4}$, $\{v_1, v_2\}$ and $\{v_3, v_4\}$ are the strong efficient dominating sets. Hence $\gamma_{se}(\overline{P_4})$ =2 and # $\gamma_{se}(\overline{P_4})$ =2.

Thus $$\#\gamma_{se}(P_n) + \#\gamma_{se}(\overline{P_n}) = \begin{cases} 3 \text{ when } n = 2 \text{ or } 3 \\ 4 \text{ when } n = 4 \end{cases}$$

Theorem 3.4: $\overline{C_n}$ is strong efficient if and only if $n \leq 4$

Proof: Let $\overline{C_n}$ be strong efficient. Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of $\overline{C_n}$. Suppose n > 4. Then $\overline{C_n}$ is a regular graph of degree n-3. Let S be a strong efficient dominating set of $\overline{C_n}$. Let v_i be an element of S. v_i strongly dominates all the vertices other than v_{i-1} and v_{i+1}. Since S is independent and v_{i-1} and v_{i+1} are adjacent, either $v_{i-1} \in S$ or $v_{i+1} \in S$. Suppose $v_{i-1} \in S$. Then $|N_s[v_j] \cap S| \geq 2, i+2 \leq j$, a contradiction. The proof is similar if $v_{i+1} \in S$. Further, suppose n = 3. Then $\overline{C_3}$ is $3K_1$ which has a unique strong efficient dominating set with strong efficient domination number 3. On the other hand, suppose n = 4. Then $\overline{C_4}$ is $2K_2$ which is also strong efficient with strong efficient domination number 2 and $\#\gamma_{se}(\overline{C_4})$ = 4. Therefore $\#\gamma_{se}(C_n) + \#\gamma_{se}(\overline{C_n})$ = 4 when n = 3

Result 3.5: There exists a graph G for which $\gamma_{se}(G) + \gamma_{se}(\bar{G}) = \#\gamma_{se}(G) + \#\gamma_{se}(\bar{G})$

Example: Let G = K_n.

For, $\overline{K_n}$ is nK_1 which has a unique strong efficient dominating set with strong efficient domination number n. So $\gamma_{se}(K_n) + \gamma_{se}(\overline{K_n})$ = n+1 and $\#\gamma_{se}(K_n) + \#\gamma_{se}(\overline{K_n})$ = n + 1 for all n ≥ 1.

Theorem 3.6: The bistar $D_{r,s}$ is strong efficient with $\gamma_{se}(D_{r,s}) + \gamma_{se}(\overline{D_{r,s}}) = r + 3$ and $\#\gamma_{se}(D_{r,s}) + \#\gamma_{se}(\overline{D_{r,s}}) = r + s + 1$ for all r, s ≥ 1 and where r < s

Proof: Let $V(\overline{D_{r,s}}) = \{u_1, u_2, v_i / 1 \le i \le r+s\}$ and
$E(\overline{D_{r,s}}) = \{u_1 v_i, u_2 v_j \,;\, r+1 \le i \le r+s, 1 \le j \le r\} \cup \{v_i v_j / 1 \le i \ne j \le r+s\}$
In $\overline{D_{r,s}}$, u_1 is adjacent with $v_{r+1}, v_{r+2}, \dots, v_{r+s}$, $u_2$is adjacent with $v_1, v_2, \dots, v_r$, $v_i$is adjacent with all the vertices other than u_1; $1 \le i \le r$ and $v_j$is adjacent with all the vertices other than u_2; $r+1 \le j \le r+s$. $u_1$and u_2 are nonadjacent. Deg$(u_1) = s$, deg(u_2) = r = $\delta(\overline{D_{r,s}})$, deg$(v_i)$ = r + $s = \Delta(\overline{D_{r,s}})$; $1 \le i \le r+s$. $\{v_1, u_1\}, \{v_2, u_1\}, \dots, \{v_r, u_1\}, \{v_{r+1}, u_2\}, \{v_{r+2}, u_2\}, \dots, \{v_{r+s}, u_2\}$ are the strong efficient dominating sets of $\overline{D_{r,s}}$. So $\gamma_{se}(\overline{D_{r,s}}) = 2$ whereas $\#\gamma_{se}(\overline{D_{r,s}}) = r + s$. Hence the theorem.

Corollary 3.7: $\gamma_{se}(D_{r,r}) + \gamma_{se}(\overline{D_{r,r}})$ = r + 3 and $\#\gamma_{se}(D_{r,r}) + \#\gamma_{se}(\overline{D_{r,r}})$ = 2r +1

Theorem 3.8: $\overline{W_n}$ is strong efficient if and only if $n \le 4$

Proof: Suppose n ≥ 5. Let $\overline{W_n}$ be strong efficient. Let S be a strong efficient dominating set of $\overline{W_n}$. Let v,$v_1, v_2, \dots, v_n$ be the vertices of $\overline{W_n}$. v is isolated in $\overline{W_n}$. v_i is adjacent with all the vertices other than v_{i-1} and v_{i+1}. As in the proof of theorem 2.4, a contradiction arises. Hence n < 5. Conversely, suppose n = 3. Then $\overline{W_3}$ is $4K_1$ which has a unique strong efficient dominating set. Thus $\gamma_{se}(\overline{W_3})$ = 4 and $\#\gamma_{se}(\overline{W_3}) = 1$. Suppose n = 4. Then $\overline{W_4}$ **is** $K_1 \cup 2K_2$ which is also strong efficient in which $\{v, v_1, v_2\}, \{v, v_1, v_4\}, \{v, v_3, v_2\}$ and $\{v, v_3, v_4\}$ are strong efficient dominating sets. Thus $\gamma_{se}(\overline{W_4})$ = 3 and $\#\gamma_{se}(\overline{W_4}) = 4$.

Result 3.9: Complement of a strong efficient graph need not be strong efficient.

Example: Consider the Gear graph G_n. Let V(G_n) = {v, $v_1, v_2, v_3 \dots, v_{2n}$ }. The vertex v strongly dominates the vertices $v_1, v_3, \dots, v_{2n-1}$. Hence {v, $v_2, v_4, v_6 \dots, v_{2n}$ }is the unique strong efficient dominating set of G_n. Therefore G_n is strong efficient. In $\overline{G_n}$, v is adjacent with v_{2i} where $1 \le i \le n$. Deg(v) = n. Each v_{2j-1} is adjacent with all the vertices other than v_{2j-2} , v_{2j} and v where $2 \le j \le n$. v_1 is adjacent with all the vertices other than v_{2n}, v_2 and v. v_{2j} is adjacent with all the vertices other than v_{2j-1} and v_{2j+1}. Deg$(v_{2j-1}) = 2n - 3; 1 \le j \le n$ and deg$(v_{2i}) = 2n - 2; 1 \le i \le n$. Suppose $\overline{G_n}$ is strong efficient. Let S be a strong efficient dominating set of $\overline{G_n}$. Since degree of any v_{2i}

is maximum for some i, $v_{2i} \in S$ for some i. v_{2i} strongly dominates all the vertices other than v_{2i-1} and v_{2i+1}. Since v_{2i-1} and $v_{2i+1}$are adjacent and S is independent, either $v_{2i-1} \in S$ or $v_{2i+1} \in S$. If $v_{2i-1} \in S$, then v_{2i+3} is strongly dominated by both v_{2i-1} and v_{2i}. $|N_s[v_{2i+3}] \cap S| \geq 2$, a contradiction. Proof is similar if $v_{2i+1} \in S$. Hence $\overline{G_n}$ is not strong efficient.

Theorem 3.10: The complement of the H-graph $\overline{H_n}$ is strong efficient if and only if $n \leq 4$.

Proof: Suppose $\overline{H_n}$ is strong efficient. Let $v_1, v_2, v_3 \dots, v_n, u_1, u_2, u_3 \dots, u_n$ be the vertices of $\overline{H_n}$. Suppose n > 4. Since the vertices v_1, v_n, u_1 and u_n are pendant vertices in H_n, their degree is maximum in $\overline{H_n}$ which is 2n-2. Let S be a strong efficient dominating set of $\overline{H_n}$. Since the maximum degree vertices are mutually adjacent with each other S contains exactly one of them. Without loss of generality, let $v_1 \in S$. v_1 strongly dominates all the vertices other than v_2. Therefore $v_2 \in S$. But u_2 is adjacent with both v_1 and v_2 in $\overline{H_n}$ and deg(v_2) = deg(u_2). $|N_s[u_2] \cap S| = 2 > 1$. Therefore n ≤ 4. Conversely suppose n = 3. $\overline{H_3}$ is given in the figure 1.

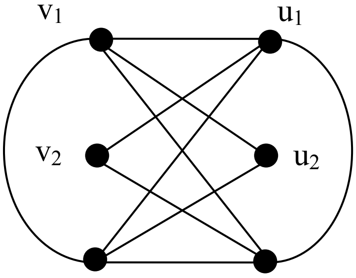

Figure 1

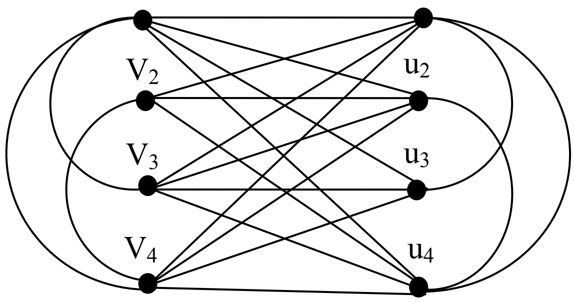

Figure 2

$\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{u_1, u_2\}$ and $\{u_2, u_3\}$ are strong efficient dominating sets of $\overline{(H_3)}$. Therefore $\gamma_{se}\,\overline{(H_3)}$ = 2 and # $\gamma_{se}\,\overline{(H_3)}$ = 4. Suppose n = 4. $\overline{H_4}$ is given in figure 2. $\{v_1, v_2\}$ and $\{u_3, u_4\}$ are strong efficient dominating sets of $\overline{(H_4)}$. Therefore $\gamma_{se}\,\overline{(H_4)}$ = 2 and # $\gamma_{se}\,\overline{(H_4)}$ = 2.

Theorem 3.11: The graph $P_{m\,[v_i]}$ where v_i is an end vertex of the path P_m and m ≥ 2 is strong efficient with

$$\gamma_{se}(P_m) + \gamma_{se}[P_{m\,[v_i]}] = \begin{cases} 3 \text{ when } m = 2 \\ 2 \text{ when } m = 3 \\ n+2 \text{ when } m = 3n, n \geq 2, n \in N \\ n+3 \text{ when } m = 3n+1, n \in N \\ n+4 \text{ when } m = 3n+2, n \in N \end{cases}$$

$$\#\gamma_{se}(P_m) + \#\gamma_{se}[P_{m[v_i]}] = \begin{cases} 3 \text{ when } m = 2 \\ 2 \text{ when } m = 3n \text{ and } m = 3n+2, n \in N \\ 3 \text{ when } m = 3n+1, n \in N \end{cases}$$

Proof: Let $v_1, v_2, v_3, \dots, v_m$ be the vertices of the path P_m. Let i = 1. Let $P_{m\,[v_1]}$ be the graph obtained by switching the end vertex v_1 of the path P_m.

Case(i): Suppose m = 2. Then $P_{2[v_1]}$ is $2K_1$ which is strong efficient with a unique strong efficient dominating set $\{v_1, v_2\}$. $\gamma_{se}(P_2) + \gamma_{se}[P_{2[v_i]}] = 3$ and $\#\gamma_{se}(P_2) + \#\gamma_{se}[P_{2[v_i]}] = 3$

Case(ii): Suppose m = 3. Then $P_{3[v_1]}$ is P_3 which is strong efficient with a unique strong efficient dominating set $\{v_3\}$. In this case, $\gamma_{se}[P_{3[v_i]}] = 1$ and $\#\gamma_{se}[P_{3[v_i]}] = 1$. $\gamma_{se}(P_3) + \gamma_{se}[P_{3[v_i]}] = 2$.

Case(iii): Suppose m ≥ 4. In $P_{m[v_1]}$, v_1 is adjacent with all the vertices other than v_2. Deg (v_1) = m - 2 = $\Delta(P_{m[v_1]})$, $\deg(v_2) = 1$, $\deg(v_i) = 3$; $3 \leq i \leq m-1$ and $\deg(v_m)$ = 2. Now $v_1$strongly dominates all the vertices of P_m other than v_2. Therefore $\{v_1, v_2\}$ is the unique strong efficient dominating set of $P_{m[v_1]}$.Therefore $\gamma_{se}[P_{m[v_1]}] = 2$ and $\#\gamma_{se}[P_{m[v_1]}] = 1$. Proof is similar if i = m. Hence the result.

Theorem 3.12: The graph $P_{m[v_1,v_m]}$ where v_1 and v_m are the end vertices of the path P_m is strong efficient if and only if m $\neq 4$. More over

$$\gamma_{se}(P_m) + \gamma_{se}[P_{m[v_1,v_m]}] = \begin{cases} 3 \text{ when } m = 2 \text{ or } 3 \\ 4 \text{ when } m = 5 \text{ or } 6 \\ n+2 \text{ when } m = 3n \text{ and } n > 2 \\ n+3 \text{ when } m = 3n+1, n > 1 \\ n+4 \text{ when } m = 3n+2 \text{ and } n > 1 \end{cases}$$

$$\#\gamma_{se}(P_m) + \#\gamma_{se}[P_{m[v_1,v_m]}] = \begin{cases} 3 \text{ when } m = 2 \text{ or } 3 \text{ or } m = 3n \text{ or } m = 3n+2, n > 1 \\ 2 \text{ when } m = 5 \\ 4 \text{ when } m = 3n+1, n > 1 \\ 6 \text{ when } m = 6 \end{cases}$$

Proof: Let $v_1, v_2, v_3, \ldots, v_m$ be the vertices of the path P_m.

Case(i): Suppose m = 2. The graph $P_{2[v_1,v_2]}$ is $2K_1$ which has the unique strong efficient dominating set $\{v_1, v_2\}$. $\gamma_{se}(P_2) + \gamma_{se}[P_{2[v_1,v_2]}] = 3$ and $\#\gamma_{se}(P_2) + \#\gamma_{se}[P_{2[v_1,v_2]}] = 3$

Case (ii): Suppose m = 3. The graph $P_{3[v_1,v_3]}$ is $K_2 \cup K_1$ which is obviously strong efficient with $\gamma_{se}[P_{3[v_1,v_3]}] = 2$ and $\#\gamma_{se}[P_{3[v_1,v_3]}] = 2$

Case(iii): Suppose m = 5. In $P_{5[v_1,v_5]}$, v_3 is adjacent with all the vertices. Deg $(v_3) = \Delta(P_{5[v_1,v_5]})$. There fore v_3 strongly dominates all the vertices and $\{v_3\}$ is the unique strong efficient dominating set. Hence $\gamma_{se}[P_{5[v_1,v_5]}] = 1$ and $\#\gamma_{se}[P_{5[v_1,v_5]}] = 1$.

Case(iv): Suppose m $= 6$. In the graph $P_{6[v_1,v_6]}$, v_1 and v_4 are adjacent with all the vertices other than v_2, v_3 and v_6 are adjacent with all the vertices other than v_2. Deg (v_1) = $\deg(v_3) = \deg(v_4) = \deg(v_6) = 4 = \Delta(P_{6[v_1,v_6]})$, $\deg(v_2) = \deg(v_5) = 2 = \delta(P_{6[v_1,v_6]})$. v_2 and v_5 are non adjacent. Therefore $\{v_1, v_2\}$, $\{v_3, v_5\}$, $\{v_4, v_2\}$ and $\{v_6, v_5\}$ are strong efficient dominating sets of $P_{6[v_1,v_6]}$. Therefore $\gamma_{se}[P_{6[v_1,v_6]}] = 2$ and $\#\gamma_{se}[P_{6[v_1,v_6]}] = 4$.

Case(v): Suppose m > 6. In the graph $P_{m[v_1,v_m]}$, v_1 is adjacent with all the vertices other than v_2. Similarly v_n is adjacent with all the vertices other than v_{n-1}. Deg (v_1) = deg (v_n) = n-2 = $\Delta(P_{m[v_1,v_m]})$, deg (v_2) = deg $(v_{n-1}) = 2$ and $\deg(v_i)$ = n-3; $3 \leq i \leq n-2$.

v_1 strongly dominates all the vertices other than v_2. Therefore$\{v_1, v_2\}$ is a strong efficient dominating set of $P_{m[v_1,v_m]}$ where m > 6. Similarly $\{v_{m-1}, v_m\}$ is also a strong efficient dominating set of $P_{m[v_1,v_m]}$. So $\gamma_{se}\left[P_{m[v_1,v_m]},\right] = 2$ and $\#\gamma_{se}\left[P_{m[v_1,v_m]},\right] = 2$. Hence the result.

Conversely suppose m = 4. Then $P_{4[v_1,v_4]}$ is the cycle C_4 which is not strong efficient.

Theorem 3.13: The graph $P_{m[v_1,v_2]}$ is strong efficient if and only if $m \leq 6$. Also

$\gamma_{se}(P_m) + \gamma_{se}\left[P_{m[v_1,v_2]}\right] = 3$ when $m \leq 4$

$\gamma_{se}(P_m) + \gamma_{se}\left[P_{m[v_1,v_2]}\right] = 4$ when m = 5,6

$\#\gamma_{se}(P_m) + \#\gamma_{se}\left[P_{m[v_1,v_2]}\right] = 3$ when $m \leq 4$

$\#\gamma_{se}(P_m) + \#\gamma_{se}\left[P_{m[v_1,v_2]}\right] = 2$ when m = 5, 6

Proof: Let $P_{m[v_1,v_2]}$ be strong efficient. Let $v_1, v_2, v_3, \ldots, v_m$ be the vertices of the path P_m. Suppose m ≥ 7. Now v_1 is adjacent with all the vertices other than v_2 and v_2 is adjacent with the vertices other than v_1 and v_3. Deg(v_1) is m-2 and deg(v_2) is m-3, deg$(v_3) = 2$, deg(v_n) = 3, deg (v_i)= 4; $4 \leq i \leq n-1$. Let S be a strong efficient dominating set of $P_{m[v_1,v_2]}$ where m ≥ 7. v_1 strongly dominates all the vertices other than v_2. So $v_1 \in S$. v_1 and v_2 are non adjacent. This is a contradiction since $|N_s[v_j] \cap S| = 2 > 1$for $4 \leq j \leq n-1$. Hence $P_{m[v_1,v_2]}$ is not strong efficient when $m \geq 7$.

Further, suppose m = 2. $P_{2[v_1,v_2]}$ is $2K_1$ is strong efficient with a unique strong efficient dominating set $\{v_1, v_2\}$. Suppose m = 3. $P_{3[v_1,v_2]}$ is $K_2 \cup K_1$ which is strong efficient with strong efficient dominating sets $\{v_1, v_2\}$ and $\{v_3, v_2\}$. Suppose m = 4 or 5. In both $P_{4[v_1,v_2]}$ and in $P_{5[v_1,v_2]}$, the vertex v_4 is adjacent with all the vertices. There fore $\{v_4\}$ is the unique strong efficient dominating set. Suppose m = 6. In $P_{6[v_1,v_2]}$, deg(v_1) = deg(v_4) = deg(v_5) = 4, deg(v_2) = deg(v_6) = 3 and deg(v_3) = 2. v_5 is adjacent with all the vertices other than v_3. Hence $\{v_5, v_3\}$ is the unique strong efficient dominating set. Hence the result.

Theorem 3.14: The graph $P_{m[v_1,v_3]}$,m ≥ 3 is strong efficient. Also

$$\gamma_{se}(P_m) + \gamma_{se}\left[P_{m[v_1,v_3]}\right] = \begin{cases} n+2 \text{ when } m = 3n, n \in N \\ n+3 \text{ when } m = 3n+1, n \in N \\ n+4 \text{ when } m = 3n+2, n \in N \end{cases}$$

$$\#\gamma_{se}(P_m) + \#\gamma_{se}\left[P_{m[v_1,v_3]}\right] = \begin{cases} 3 \text{ when } m = 3 \text{ or } 6 \text{ or } m = 3n+1, n \in N \\ 2 \text{ when } m = 3n, n \geq 3 \text{ or } m = 3n+2, n \in N \end{cases}$$

Proof: Case(i): Suppose m = 3. Then $P_{3[v_1,v_3]}$ is $K_2 \cup K_1$ which is strong efficient with strong efficient dominating sets $\{v_1, v_2\}$ and $\{v_2, v_3\}$.

Case(ii): Let m = 6. Then in $P_{6[v_1,v_3]}$, v_1 and v_5 are the maximum degree vertices which are mutually adjacent and adjacent with all the vertices other than v_2. Hence $\{v_1, v_2\}$ and $\{v_2, v_5\}$ are strong efficient dominating sets of $P_{6[v_1,v_3]}$.

Case(iii): Let m $\neq$ 3 or 6. Then in $P_{m[v_1,v_3]}$, v_1 is adjacent with all the vertices other than v_2 and deg (v_1) = n-2 = $\Delta(P_{m[v_1,v_3]})$. v_2 is isolated. Therefore $\{v_1, v_2\}$is the unique strong efficient dominating set of $P_{m[v_1,v_3]}$. Hence the result.

Theorem 3.15: The graph $C_{3n[v_i]}$ is strong efficient with $\gamma_{se}(C_{3n}) + \gamma_{se}[C_{3n[v_i]}] = n + 3$ and $\#\gamma_{se}(C_{3n}) + \#\gamma_{se}[C_{3n[v_i]}] = \begin{cases} 5 \text{ for n } = 2 \\ 4 \text{ for n } \neq 2 \end{cases}$

Proof: Let $v_1, v_2, v_3, \ldots, v_{3n}$ be the vertices of the cycle C_{3n}.

Case(i): Suppose n = 1. $C_{3[v_1]}$ is $K_1 \cup K_2$ which is strong efficient with the strong efficient dominating sets $\{v_1, v_2\}$ and $\{v_1, v_3\}$.

Case(ii): Suppose n = 2. In $C_{6[v_1]}$, $v_1$strongly dominates all the vertices of C_6 except v_2 and v_6. Clearly $\{v_1, v_2, v_6\}$and $\{v_4, v_2, v_6\}$ are strong efficient dominating sets.

Case(iii): Suppose n ≥ 3. In $C_{3n[v_1]}$, v_1 is adjacent with all the vertices other than v_2 and v_{3n}. Deg(v_1) = 3n-3 = $\Delta(C_{3n[v_1]})$, deg$(v_i) = 3$; $3 \leq i \leq 3n - 1$, deg(v_2) = deg $(v_{3n}) = 1$. Hence $\{v_1, v_2, v_{3n}\}$ is the unique strong efficient dominating set of $C_{3n[v_1]}$. Hence the result.

Proposition 3.16: The graph $K_{1,n[v]}$ where v is the central vertex of the star $K_{1,n}$ is strong efficient. Also $\gamma_{se}(K_{1,n}) + \gamma_{se}[K_{1,n[v]}]$ = n + 2 and # $\gamma_{se}(K_{1,n}) + \#\gamma_{se}[K_{1,n[v]}]$ = 2, n ≥ 1

Proof: Let $v, v_1, v_2, \ldots, v_n$ be the vertices of the star $K_{1,n}$. $K_{1,n[v]}$ is the graph (n+1)K_1. So $\gamma_{se}[K_{1,n[v]}] = n + 1$ and $\#\gamma_{se}[K_{1,n[v]}] =$ 1.Therefore$\gamma_{se}(K_{1,n}) + \gamma_{se}[K_{1,n[v]}]$ = n + 2 and # $\gamma_{se}(K_{1,n}) + \#\gamma_{se}[K_{1,n[v]}]$ = 2, n ≥ 1

Proposition 3.17: The graph $K_{1,n[v_i]}$ where v_i is any pendant vertex of the star $K_{1,n}$ is strong efficient if and only if n = 1, 2

Proof: Let $v, v_1, v_2, \ldots, v_n$ be the vertices of the star$K_{1,n}$. Let $K_{1,n[v_1]}$ is the graph obtained by switching the pendant vertex v_1 of the star $K_{1,n}$. When n = 1, the graph $K_{1,1[v_1]}$ is $2K_1$ which has the unique strong efficient dominating set $\{v, v_1\}$. When n = 2, the graph $K_{1,2[v_1]}$ is the path P_3 which has the unique strong efficient dominating set $\{v_2\}$.
Conversely suppose n ≥ 3. Let S be a strong efficient dominating set of $K_{1,n[v_i]}$.

Case(i): Suppose n = 3. Then the graph $K_{1,3[v_1]}$ is the cycle C_4 which is not strong efficient.

Case(ii): Suppose n ≥ 4. $K_{1,n[v_1]}$ is the graph in which v and v_1 are adjacent with $v_2, v_3, \ldots, v_n$. Deg(v) = deg(v_1) = n-1, deg(v_i) = 2 for 2 $\leq i \leq n$. Let $v_1 \in S$. v and v_1 are non adjacent so that $\{v_1, v\} \subseteq S$ for every strong efficient dominating set S. $|N_s[v_j] \cap S| = 2 > 1$, a contradiction. From both cases (i) and (ii), n = 1, 2.

Theorem 3.18: The gear graph G_n is strong efficient for all n ≥ 3.

Proof: Let v, $v_1, v_2, \ldots, v_{2n}$ be the vertices of the gear graph G_n. The vertex v is adjacent with v_{2i-1}; $1 \leq i \leq n$. Deg(v) = n = $\Delta(G_n)$, deg(v_{2i-1}) = 3; $1 \leq i \leq n$, deg(v_{2i}) = 2; $1 \leq i \leq n$. v strongly dominates all the vertices v_{2i-1}; $1 \leq i \leq n$. The verticesv_{2i} and v_{2i-1} are mutually non adjacent with each other. Therefore $\{v, v_2, v_4, \ldots, v_{2n}\}$ is the unique strong efficient dominating set of G_n.Thus $\gamma_{se}(G_n)$ = n+1 and $\#\gamma_{se}(G_n)$ = 1.

Theorem 3.19: The graph $G_{n[v_{2i-1}]}$, $1 \leq i \leq n$ is strong efficient if and only if n = 4.

Proof: Let v, $v_1, v_2, \dots, v_{2n}$ be the vertices of the graph G_n. Suppose n = 4. Let i = 1. Let v,$v_1, v_2, \dots, v_8$ be the vertices of $G_{4[v_1]}$. v_1 is adjacent with all the vertices other than v, v_2 and v_8. Deg(v_1) = 5 = $\Delta(G_{4[v_1]})$. Therefore v_1 strongly dominates all the vertices other than v, v_2 and v_8 and these vertices are mutually non adjacent with each other. Deg(v_2) = Deg(v_8) = 1. Deg(v) = 3. v is adjacent with v_3, v_7 and v_5. Deg(v_3) = Deg(v_7) = Deg(v_5) = 4. v_2 is adjacent with v_3 and v_8 is adjacent with v_7. Hence $\{v, v_1, v_2, v_8\}$ is the unique strong efficient dominating set of $G_{4[v_1]}$. Proof is similar if $2 \leq i \leq 4$. Thus $G_{4[v_{2i-1}]}$, $1 \leq i \leq 4$ is strong efficient Hence $\gamma_{se}[G_{4[v_1]}] = 4$ and $\#\gamma_{se}[G_{4[v_1]}] = 1$. Conversely, let $G_{n[v_{2i-1}]}$, $1 \leq i \leq n$ be strong efficient. Suppose $n \neq 4$.

Case(i): Let n = 3. In $G_{3[v_1]}$, deg(v_3) = deg$(v_5) = 4 = \Delta(G_{3[v_1]})$ and d$(v_3, v_5) = 2$. Hence by the result 1.12, $G_{3[v_1]}$ is not strong efficient.

Case(ii): Suppose $n \geq 5$. Let i = 1. Let S be a strong efficient dominating set of $G_{n[v_1]}$. In $G_{n[v_1]}$, the vertex v_1 is adjacent with all the vertices other than v_2, v_{2n} and v. Deg$(v_1) = 2n - 3 = \Delta(G_{n[v_1]})$. deg$(v_{2i-1})$=4;$2 \leq i \leq n$, deg$(v_{2i})$ =3;$2 \leq i \leq n-1$, deg(v) = n−1. The vertex v_1 is the unique maximum degree vertex and it strongly dominates all the vertices other than v_2, v_{2n} and v. Therefore $v_1 \in S$. The vertices v_2, v_{2n} and v are mutually non adjacent with each other. Hence they belong to S. But $|N_s[v_{2j-1}] \cap S| = |\{v, v_1\}| = 2 > 1$, for every $i \neq j$, a contradiction. Hence $G_{n[v_1]}$ is not strong efficient. Proof is similar for other values of i.

Theorem 3.20: The graph$G_{n[v_{2i}]}$ $1 \leq i \leq n$, is strong efficient. Further

$\gamma_{se}(G_n) + \gamma_{se}[G_{n[v_{2i}]}] = n + 4$ and

$$\#\gamma_{se}(G_n) + \#\gamma_{se}[G_{n[v_{2i}]}] = \begin{cases} 3 \text{ when } n = 3 \\ 2 \text{ when } n \geq 4 \end{cases}$$

Proof: Let v,v_1,v_2,...,v_{2n} be the vertices of the graph G_n. Let i = 1.

Case(i): Suppose n = 3. In $G_{3[v_2]}$, v_2 and v_5 are mutually adjacent and are adjacent with all the vertices other than v_1 and v_3. Deg(v_2) = deg(v_5) = 4 = $\Delta(G_{3[v_2]})$, deg(v_1) = deg$(v_3) = 2$. v_2 and v_5 strongly dominates all the vertices other than v_1 and v_3. Therefore $\{v_2, v_1, v_3\}$ and $\{v_5, v_1, v_3\}$are strong efficient dominating sets of $G_{3[v_2]}$. Hence $\gamma_{se}[G_{3[v_2]}]$ = 3 and $\#\gamma_{se}[G_{3[v_2]}] = 2$.

Case(ii): Suppose $n \geq 4$. In $G_{n[v_2]}$, v_2 is adjacent with all the vertices other than v_1 and v_3. Deg(v_2)= $2(n-1) = \Delta(G_{n[v_2]})$. v_2 is the unique maximum degree vertex. deg(v_1) = deg(v_3) = 2 = $\delta(G_{n[v_2]})$. v_2 strongly dominates all the vertices other than v_1 and v_3 which are non adjacent. Hence $\{v_2, v_1, v_3\}$ is the unique strong efficient dominating set of $G_{n[v_2]}$. Proof is similar for other values of i. Hence the result.

Theorem 3.21: The graph $G_{n[v]}$ where v is the central vertex in a gear graph G_n is strong efficient. Further $\gamma_{se}(G_n) + \gamma_{se}[G_{n[v]}] = 2n + 2$ and $\#\gamma_{se}(G_n) + \#\gamma_{se}[G_{n[v]}] = 2$

Proof: Let v,v_1,v_2,...,v_{2n} be the vertices of the graph G_n. In $G_{n[v]}$, v is adjacent with all the vertices $v_{2i}; 1 \leq i \leq n$ and non adjacent with all the vertices $v_{2i-1}; 1 \leq i \leq n$. $G_n \cong G_{n[v]}$. Therefore $G_{n[v]}$ is strong efficient. Hence the result.

Theorem 3.22: The graph obtained by switching any one of the central vertices of the bistar $D_{r,s}$, r,s≥2 is not strong efficient.

Proof: Let u,v,$v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_{r+s}$ be the vertices of the bistar $D_{r,s}$, r,s≥2. $v_1, v_2, \dots, v_r$ are the pendant vertices adjacent with u and $v_{r+1}, \dots, v_{r+s}$ are the pendant vertices adjacent with v. In $D_{r,s[u]}$, both u and v are adjacent with the vertices $v_{r+1}, \dots, v_{r+s}$. But u and v are non adjacent. Deg(u) = deg(v) = $\Delta\left(D_{r,s[u]}\right)$ and d(u, v) = 2. Therefore by result 2.12, $D_{r,s[u]}$ is not strong efficient.

Corollary 3.23:$D_{1,s[v]}$, s ≥ 2 where v is defined in theorem 3.22 is strong efficient.

Proof: Let v_1 be the pendant vertex adjacent with the central vertex u and $v_2, v_3, \dots, v_{s+1}$ be the pendant vertices adjacent with v. $D_{1,s[v]}$ is the graph $P_3 \cup sK_1$ which is strong efficient. $\{v_1, v_2, \dots, v_s, v_{s+1}\}$ is the unique strong efficient dominating set of $D_{1,s[v]}$.

Corollary 3.24: $D_{1,s[u]}$, s ≥ 2 where u is defined in theorem 3.22 is not strong efficient.

Proof: In $D_{1,s[u]}$, u and v are adjacent with the vertices $v_2, v_3, \dots, v_s, v_{s+1}$. Also u and v are non adjacent. Deg(u) = deg(v) = s = $\Delta\left(D_{1,s[u]}\right)$. Since d$(u, v)$ = 2,by result 2.12, the graph $D_{1,s[u]}$ is not strong efficient.

Theorem 3.25: Let $D_{r,s[u,v]}$ be the graph obtained by switching both the central vertices u and v of the bistar $D_{r,s}$. Then

$\gamma_{se}\left(D_{r,s}\right) + \gamma_{se}\left[D_{r,s[u,v]}\right]$ = r +3 when r ≤ s

$\#\gamma_{se}\left(D_{r,s}\right) + \#\gamma_{se}\left[D_{r,s[u,v]}\right]$ = 2 when r < s

= 3 when r = s

Proof: Let u, v, $v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_{r+s}$ be the vertices of the bistar $D_{r,s}$. $v_1, v_2, \dots, v_r$ are the pendant vertices adjacent with u and $v_{r+1}, \dots, v_{r+s}$ are the pendant vertices adjacent with v. The graph $D_{r,s[u,v]}$ is $K_{1,r} \cup K_{1,s}$ which is strong efficient. $\{u, v\}$ is the unique strong efficient dominating set of $D_{r,s[u,v]}$. Hence the result.

Theorem 3.26: The graph $D_{r,s[v_i]}$ where 1≤ $i \leq r + s$ obtained by switching a pendant vertex of the bistar $D_{r,s}$ is strong efficient if and only if r = 1 and i =1 or s = 1 and i = r+1 or both r,s = 1.

Proof: Case(i): Let r,s > 2 and r < s. Suppose $D_{r,s[v_i]}$; 1 ≤ $i \leq r + s$ be strong efficient. Let S be a strong efficient dominating set of $D_{r,s[v_i]}$.

Subcase $i(a)$: Suppose 1≤ $i \leq r$. In $D_{r,s[v_i]}$, v_i is adjacent with all the vertices other than u. Deg(v_i) = r+s = $\Delta\left(D_{r,s[v_i]}\right)$, deg$(v)$ = s+2, deg(u) = r, deg(v_j) = 2 for j≠ i. v_i strongly dominates all the vertices other than u. Hence $v_i, u \in S$. $|N_S[v_k] \cap S| = 2 > 1, k \neq i, 1 \leq k \leq r$. This is a contradiction.

Subcase $i(b)$: Suppose r+1 $\leq$ i $\leq$ r+s. In $D_{r,s[v_i]}$, v_i is adjacent with all the vertices other than v. Deg(v_i) = r+s = $\Delta(D_{r,s[v_i]})$, deg(v) = s, deg(u) = r+2 and deg(v_j) = 2 for j$\neq i$. v_i strongly dominates all the vertices other than v. Hence v_i , v $\in S$. $|N_S[v_k] \cap S| = 2 > 1, k \neq i, r+1 \leq k \leq r+s$. This is a contradiction.

Case(ii): Suppose r,s > 2 and r = s. Proof is similar to that of subcase $i(a)$.

Case(iii): Suppose r, s = 2 and 1$\leq i \leq 2$. In $D_{2,2[v_i]}$, v_i is adjacent with all the vertices other than u. Deg(v_i) = deg(v) = 4 = $\Delta(D_{2,2[v_i]})$, deg(u) = 2,deg(v_j) = 2 for j$\neq i$. Also d(u, v) = 2. Hence by result 1.12, the graph $D_{2,2[v_i]}$ is not strong efficient. Proof is similar if 3$\leq i \leq 4$.

Case(iv): Suppose r =1 and i ≥ 2. In $D_{1,s[v_i]}$, v_i is adjacent with all the vertices other than v. Deg(v_i) = s+1 = $\Delta(D_{1,s[v_i]})$, deg(v) = s, deg(v_j) = 2 for j $\neq i$. The vertex v_i strongly dominates all the vertices other than v. Hence v_i,v $\in S$. $|N_S[v_k] \cap S| = 2 > 1, k \neq i, 2 \leq k \leq s+1$. This is a contradiction.

Case(v): Suppose s = 1 and 1$\leq$ i $\leq r$.Proof is similar to that of case(iv). Conversely

Case(i): Let r =1 and i =1. In $D_{1,s[v_1]}$, the vertex v_1 is adjacent with all the vertices other than u and v is the full degree vertex.$\{v\}$ is the unique strong efficient dominating set of $D_{1,s[v_1]}$.

Case(ii): Let s =1 and i = r+1. In $D_{1,s[v_{r+1}]}$, the vertex v_{r+1} is adjacent with all the vertices other than v and u is the full degree vertex.$\{u\}$ is the unique strong efficient dominating set of $D_{1,s[v_{r+1}]}$.

Case(iii): Let r = s =1. Proof is similar to that of case(i) and case(ii).Hence from all the above cases, the graph $D_{r,s[v_i]}$ is strong efficient.

Theorem 3.27: The graph $H_{n[u_i]}$ where u_i is the pendant vertex of the H- graph H_n is strong efficient if and only if n $\neq 3$.

Proof: Let u_i,v_i where 1$\leq i \leq n$ be the vertices of the graph H_n. Suppose n ≥ 4. In $H_{n[u_1]}$, the vertex u_1 is adjacent with all the vertices other than u_2. u_1 strongly dominates all the vertices all the vertices other than u_2. Deg(u_1) = 2n-2 = $\Delta(H_{n[u_1]})$, deg(u_2) =1. Hence $\{u_1, u_2\}$ is the unique strong efficient dominating set. Similarly $H_{n[u_n]}$, $H_{n[v_1]}$ and $H_{n[v_n]}$ are strong efficient.

Conversely, let $H_{3[u_1]}$ be strong efficient. Let S be a strong efficient dominating set. In $H_{3[u_1]}$, u_1 is adjacent with all the vertices other than u_2 and v_2 is adjacent with all the vertices other than u_3. Deg(u_1) = deg$(v_2) = 4 = \Delta(H_{3[u_1]})$ and u_1,$v_2$are adjacent. Therefore S contains either u_1 or v_2. Deg(v_1) = deg(v_3) = deg(u_2) = deg(u_3)= 2 = $\delta(H_{3[u_1]})$. If $u_1 \in S$, then $u_2 \in S$ and $|N_S[u_3] \cap S| = 2 > 1$. This is a contradiction. If

$v_2 \in S$, then $u_3 \in S$ and $|N_S[u_2] \cap S| = 2 > 1$. This is also a contradiction. Proof is similar for the graphs $H_{3[u_3]}$, $H_{3[v_1]}$ and $H_{3[v_3]}$. Hence the graph $H_{n[u_i]}$ where v_i is the pendant vertex of the H- graph H_n is strong efficient if and only if n $\neq 3$.

Theorem 3.28: (i) $H_{n\left[u_{\frac{n+1}{2}}\right]}$, n ≥ 3 and n is odd is strong efficient if and only if n $\neq 3$.

$(ii) H_{n\left[u_{\frac{n}{2}}\right]}$, n ≥ 4 and n is even is strong efficient f and only if n $\neq 4$.

Proof: Let u_i,v_i where $1 \leq i \leq n$ be the vertices of the graph H_n.

Case(i): Suppose n $\neq 3$ and n be odd. In $H_{n\left[u_{\frac{n+1}{2}}\right]}$, $u_{\frac{n+1}{2}}$ is adjacent with all the vertices other than $u_{\frac{n-1}{2}}$, $u_{\frac{n+3}{2}}$ and $v_{\frac{n+1}{2}}$. Deg $\left(u_{\frac{n+1}{2}}\right)$ = 2n-4 $= \Delta\left(H_{n\left[u_{\frac{n+1}{2}}\right]}\right)$, deg$\left(u_{\frac{n-1}{2}}\right) =$ deg$\left(u_{\frac{n+3}{2}}\right)$ = 1 = $\delta\left(H_{n\left[u_{\frac{n+1}{2}}\right]}\right)$ and deg$\left(v_{\frac{n+1}{2}}\right)$ = 2 = deg(v_1)= deg(v_n)= deg(u_1) = deg(u_n), deg(u_k) = deg(v_k) = 3 for k $\neq \frac{n+1}{2}$,1 and n. Also $u_{\frac{n+1}{2}}$, $u_{\frac{n-1}{2}}$, $u_{\frac{n3}{2}}$ and $v_{\frac{n+1}{2}}$ are mutually non adjacent. Therefore $\left\{u_{\frac{n+1}{2}}, u_{\frac{n-1}{2}}, u_{\frac{n+3}{2}}, v_{\frac{n+1}{2}}\right\}$ is the unique strong efficient dominating set of $H_{n\left[u_{\frac{n+1}{2}}\right]}$. Hence $H_{n\left[u_{\frac{n+1}{2}}\right]}$ is strong efficient.

Conversely suppose n = 3. $H_{3[u_2]}$ is the graph $C_4 \cup 2K_1$. Since C_4 is not strong efficient, $H_{3[u_2]}$ is not strong efficient.

Case(ii): Suppose n $\neq 4$ and n be even. In $H_{n\left[u_{\frac{n}{2}}\right]}$, the vertex $u_{\frac{n}{2}}$ is adjacent with all the vertices other than $u_{\frac{n-2}{2}}$,$u_{\frac{n+2}{2}}$ and $v_{\frac{n+2}{2}}$. Also Deg$\left(u_{\frac{n}{2}}\right)$ = 2n-4 = $\Delta\left(H_{n\left[u_{\frac{n}{2}}\right]}\right)$, deg $\left(u_{\frac{n-2}{2}}\right)$ = deg$\left(u_{\frac{n+2}{2}}\right)$ =1 = $\delta\left(H_{n\left[u_{\frac{n}{2}}\right]}\right)$ and deg$\left(v_{\frac{n+2}{2}}\right)$ = 2 = deg(v_1)= deg(v_n)= deg(u_1) = deg(u_n). All the other vertices are of degree 3. The vertex $u_{\frac{n}{2}}$. Strongly dominates all the vertices other than $u_{\frac{n-2}{2}}$,$u_{\frac{n+2}{2}}$ and $v_{\frac{n+2}{2}}$. Therefore $\left\{u_{\frac{n}{2}}, u_{\frac{n-2}{2}}, u_{\frac{n+2}{2}}, v_{\frac{n+2}{2}}\right\}$ is the unique strong efficient dominating set of $H_{n\left[u_{\frac{n}{2}}\right]}$. Therefore $H_{n\left[u_{\frac{n}{2}}\right]}$ is strong efficient.

Conversely let n = 4. In $H_{4[u_2]}$, deg(u_2) = 4 = $\Delta\left(H_{4\left[u_{\frac{n}{2}}\right]}\right)$. The vertex u_2 strongly dominates all the vertices other than u_1,u_3 and v_3. The vertex u_1 is an isolate. Deg(u_3) = 1, deg(v_3) = deg(v_4) = 2 = deg(v_1), deg(v_2) = 3. Suppose $H_{4[u_2]}$ is strong efficient. Let S be a strong efficient dominating set of $H_{4[u_2]}$. Hence $u_2 \in S$. u_1, v_3, u_3 and u_2 are mutually non adjacent. Hence they belong to S. $|N_S[v_4] \cap S| = |\{u_2, v_3\}| = 2 > 1$. This is a contradiction. Hence the graph $H_{4[u_2]}$ is not strong efficient. Hence the theorem.

4. Conclusion

In this paper, the authors studied some Nordhaus- Gaddum type relations on strong efficient domination number of a graph and its derived graph. They introduced the concept of number of strong efficient dominating sets and studied the relation between the number of strong efficient dominating sets of a graph and its derived graph. Similar studies can be made on this type for various derived graphs

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