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Well-posedness and general energy decay of solutions for a nonlinear damping piezoelectric beams system with thermal and magnetic effects

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Abstract

In this article, we study the piezoelectric beams with thermal and magnetic effects in the presence of a nonlinear damping term acting on the mechanical equation. First, we prove that the system is well-posed in the sense of semigroup theory. And by constructing a suitable Liapunov functional, we show a general decay result of the solution for the system from which the polynomial and exponential decay are only special cases. Furthermore, our result does not depend on any relationship between system parameters.

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1. Introduction

In recent years, we have seen a large number of published works on piezoelectric materials [12,24]. Piezoelectric materials such as quartz, Rochelle salt, and barium titanate have an important property of converting mechanical energy to electromagnetic energy with the effect of mechanical stress. This phenomenon is known by the direct piezoelectric effect that was discovered by the brothers Pierre and Jacques Curie in 1880. Reciprocally, the same materials have the ability to convert electromagnetic energy to mechanical energy and this phenomena is well called the reverse piezoelectric effect that was discovered by Gabriel Lippmann [27] in 1881. There are many applications of piezoelectric materials in real life like in: civil engineering, industrial, automotive, aeronautical and space structures. Also these materials have been widely used as sensors and actuators in the area of structures and intelligent systems [2, 3]. Furthermore, these smart materials can be

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used in many fields, especially when dealing with piezoelectric motors, sonars and injection mechanisms. The activity of these materials is related to the fact that they exhibit microscopic polarization due to the presence of a dipole moment caused by the absence of central symmetry. In addition, during the transformation of mechanical energy into electric one, it also turns a small portion of it into magnetic energy [18]. This last energy has a relatively small effect on the general dynamics, and there exist models that neglect magnetic effects such as piezoelectric beams. However, this magnetic contribution may limit the system performance. For example, the magnetic effect can cause oscillations in the output, which leads to system instability in closed loop [21, 29]. Other problems related to piezoelectric systems can be found in the following references [5, 6, 17, 26, 28]. On the other hand, in the references [13, 14, 30, 31] a great deal of attention has been given to the study of differential variational-hemivariational inequalities.

Morris et al. [18] using a variational approach to introduce the following coupled model of piezoelectric beams with magnetic effects

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,\infty), \end{cases} \tag{1.1}$$

where the positive parameters ρ, α, γ, μ, β, L represent, respectively, the mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, water resistance coefficient of the beam and the length of the beam. In addition, the relationship is considered

$$\alpha = \alpha_1 + \gamma^2\beta \text{ with } \alpha_1 > 0. \tag{1.2}$$

The system (1.1) is subjected to the following initial and boundary conditions

$$\begin{cases} v(0,t) = p(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \\ \beta p_x(L,t) - \gamma\beta v_x(L,t) = -\dfrac{V(t)}{h}, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ p(x,0) = p_0(x),\ p_t(x,0) = p_1(x), \end{cases} \tag{1.3}$$

where h is the thickness of the beam and $V(t)$ is the voltage applied at the electrode. Here the functions v and p are used to denote the transverse displacement of the beam and the total load of the electric displacement along the transverse direction at each point x respectively. Ramos et al. [23] studied the following piezoelectric beams system with magnetic effects

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t = 0 \text{ in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,T), \end{cases} \tag{1.4}$$

and the system (1.4) is equipped by the following initial and boundary conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0,\ 0 \le t \le T, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0,\ 0 \le t \le T, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ p(x,0) = p_0(x),\ p_t(x,0) = p_1(x),\ 0 \le x \le L, \end{cases}$$

they investigated the exponential decay of the total energy and some numerical aspects related to the dissipative piezoelectric beams system with magnetic effects. And also, they proved that the dissipation produced by damping δv_t, acting in the mechanical equation, is strong enough to stabilize exponentially the system solution (1.4) for whatever the physical parameters of the model. In addition, they presented results of numerical simulations using the explicit finite difference method. Ramos et al. [22] studied the one-dimensional piezoelectric beams system with magnetic effects given by

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 \text{ in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,T), \end{cases}$$

with the following initial and boundary conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) + \xi_1 \frac{v_t(L,t)}{h} = 0, \ 0 < t < T, \\ p(0,t) = \beta p_x(L,t) - \gamma\beta v_x(L,t) + \xi_2 \frac{p_t(L,t)}{h} = 0, \ 0 < t < T, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \ 0 < x < L, \end{cases}$$

where $\xi_1, \xi_2 > 0$, and they showed that the system is exponentially stable regardless of any condition on the coefficients of the system, and exponential stability is equivalent to exact observability at the boundary. In [25], Soufyane et al. considered the following piezoelectric beams with magnetic effects, nonlinear damping and nonlinear delay terms

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \alpha_1 \mathfrak{g}_1(v_t) + \alpha_2 \mathfrak{g}_2(v_t(x,t-\tau)) = 0, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \end{cases}$$

with the following initial and boundary conditions

$$\begin{cases} v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \\ v(0,t) = v_x(1,t) = p(0,t) = p_x(1,t) = 0, \\ v_t(x,t-\tau) = g_0(x,t-\tau), \ x \in (0,1), \ 0 < t < \tau. \end{cases}$$

Under appropriate assumptions on the weight of the delay, the authors established an energy decay rate by using a perturbed energy method and some properties of convex functions. Freitas et al. [8] studied the following piezoelectric beams system with thermal and magnetic effects, and with friction damping

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + g_1(v,p) = h_1 \text{ in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + A^v p_t + g_2(v,p) = h_2 \text{ in } (0,L)\times(0,T), \\ c\theta_t - \kappa\theta_{xx} + \delta v_{tx} = 0 \text{ in } (0,L)\times(0,T), \end{cases} \tag{1.5}$$

with the following initial and boundary conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \ t > 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, \ t > 0, \\ \theta(0,t) = \theta(L,t) = 0, \ t > 0, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ 0 < x < L, \\ p_t(x,0) = p_1(x), \ \theta(x,0) = \theta_0(x), \ 0 < x < L, \end{cases} \tag{1.6}$$

where the physical constants ρ, α, β, γ, δ, κ, μ and c are positive constants, g_1 and g_2 are nonlinear source terms, h_1 and h_2 are external forces. Moreover, we consider the relationship

$$\alpha = \alpha_1 + \gamma^2\beta \text{ with } \alpha_1 > 0,$$

$A : D(A) \subset L^2(0,L) \to L^2(0,L)$ is the one-dimensional Laplacian operator defined by

$$A = -\partial_{xx} \text{ with domain } D(A) = \left\{ v \in H^2(0,L) \cap H^1_*(0,L) : v_x(L) = 0 \right\},$$

where $H^1_*(0,L) := \{u \in H^1(0,L) : u(0) = 0\}$ and $A^v : D(A^v) \subset L^2(0,L) \to L^2(0,L)$ is the fractional power associated with the operator A of order $v \in (0,1/2)$. The authors used the variational approach for model of vibrations on piezoelectric beams with fractional damping depending on $v \in (0,1/2)$. Also, they showed that the dynamical system generated by the problem (1.5)–(1.6) has a smooth global attractor with a finite fractal dimension by the theory of quasi-stability [4], the authors obtained the existence of a generalized exponential attractor in a scale of fractional spaces, and they established the stability of global attractors on the perturbation of the fractional exponent. Freitas et al. [7] studied the following nonlinear piezoelectric beams system with a delay term

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + g_1(v,p) + v_t = h_1, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + g_2(v,p) + \mu_1 p_t + \mu_2 p_t(x,t-\tau) = h_2, \end{cases} \tag{1.7}$$

subject to the following initial and boundary conditions

$$\begin{cases} v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = 0, \ t \geq 0, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \ x \in (0, L), \end{cases}$$

where $(x,t) \in (0,L) \times (0,T)$, the functions $g_1(v,p)$ and $g_2(v,p)$ represent nonlinear source terms, h_1 and h_2 are external forces, whereas p_t and v_t denote magnetic current and damping in displacement, respectively. They discussed its long time behavior through the related dynamical system. The authors also showed that the system is asymptotically smooth. In addition, they established a stabilizability inequality to get the quasi-stability of the system and therefore obtain the finite fractal dimension of the global attractor and exponential attractors.

In this article, motivated and inspired by the above papers, we consider the following system

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + \chi(t) g(v_t) = 0 \text{ in } (0,L) \times (0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L) \times (0,\infty), \\ c\theta_t - \kappa\theta_{xx} + \delta v_{tx} = 0 \text{ in } (0,L) \times (0,\infty). \end{cases} \tag{1.8}$$

This system is subjected to the following initial and boundary conditions

$$\begin{cases} v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ p(x,0) = p_0(x), \ x \in (0,L), \\ p_t(x,0) = p_1(x), \ \theta(x,0) = \theta_0(x), \ x \in (0,L), \\ v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \ t \in (0,\infty), \end{cases}$$

where ρ, α, β, γ, δ, κ, μ and c are positive constants, the functions p, v and θ represent, respectively, the total load of the electric displacement along the transverse direction at each point x, the longitudinal displacement of the center line, and temperature. The term $\chi(t) g(v_t)$ is the nonlinear damping term where the functions χ, g are specified later, v_0, v_1, p_0, p_1, θ_0 are the initial data. Other systems with nonlinear terms [1, 11, 16]. However, it remains with great importance in the asymptotic behavior study of the solution for different types of systems that can be found in the following papers [9, 10, 19, 32–34]. Throughout this article, we will suppose that (1.2) is satisfied and χ and g satisfy the following assumptions:

(A1) $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable non-increasing function.

(A2) $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing C^0-function, such that there exist positive constants c_1, c_2, ε and a strictly increasing function $\Phi \in C^1([0,+\infty))$ with $\Phi(0) = 0$, and Φ is linear or strictly convex C^2-function on $(0,\varepsilon]$ such that

$$\begin{cases} s^2 + g^2(s) \leq \Phi^{-1}(sg(s)) \text{ for all } |s| \leq \varepsilon, \\ c_1 |s| \leq |g(s)| \leq c_2 |s| \text{ for all } |s| \geq \varepsilon, \end{cases}$$

which means that $sg(s) > 0$ for all $s \neq 0$.

(A3) The function g satisfies the following condition

$$|g(u_2) - g(u_1)| \leq k_0 (|u_1|^\rho + |u_2|^\rho) |u_1 - u_2|, \ u_1, u_2 \in \mathbb{R}, \tag{1.9}$$

where $k_0 > 0$, $\rho > 0$.

Outline of the article. To show our goals, this article takes the following route. In Section 2, by using semigroup techniques, we study the existence and uniqueness of solutions for the system (1.8). Next, in Section 3, we give some technical lemmas, which will be used in the proof of our stability results. In Section 4, we present the proofs of our stability results. Furthermore, throughout this work we use c to denote a generic positive constant.

2. The well-posedness of the problem

In this section, by using the semigroup theory [15, 20], we prove that the system (1.8) is well-posed. So, if $\mathbb{U} = (v, u, p, q, \theta)^T$ with $u = v_t$ and $q = p_t$, then, we can write the system (1.8) as

$$\begin{cases} \frac{d\mathbb{U}}{dt} - \mathbb{A}\mathbb{U} = \mathbb{G}(\mathbb{U}), \ t > 0, \\ \mathbb{U}(x,0) = \mathbb{U}_0(x) = (v_0, v_1, p_0, p_1, \theta_0)^T, \end{cases} \tag{2.1}$$

where the linear operator $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{H} \to \mathbb{H}$ is defined by

$$\mathbb{A}\mathbb{U} = \begin{pmatrix} u \\ \frac{\alpha}{\rho} v_{xx} - \frac{\gamma\beta}{\rho} p_{xx} - \frac{\delta}{\rho}\theta_x \\ q \\ \frac{\beta}{\mu} p_{xx} - \frac{\gamma\beta}{\mu} v_{xx} \\ \frac{\kappa}{c}\theta_{xx} - \frac{\delta}{c} u_x \end{pmatrix},$$

and the nonlinear operator $\mathbb{G} : \mathbb{H} \longrightarrow \mathbb{H}$ is defined by

$$\mathbb{G}(\mathbb{U}) = \begin{pmatrix} 0 \\ -\frac{\chi(t)}{\rho} g(u) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We consider the following spaces

$$\begin{aligned} \tilde{H}^1(0,L) &= \left\{ g \in H^1(0,L) : g(0) = 0 \right\}, \\ \tilde{H}^2(0,L) &= H^2(0,L) \cap \tilde{H}^1(0,L), \end{aligned}$$

and $\mathbb{H}$ is the energy space given by

$$\mathbb{H} = \tilde{H}^1(0,L) \times L^2(0,L) \times \tilde{H}^1(0,L) \times L^2(0,L) \times L^2(0,L),$$

equipped with the inner product

$$\begin{aligned} \left\langle \mathbb{U}, \tilde{\mathbb{U}} \right\rangle_{\mathbb{H}} &= \rho \int_0^L u\tilde{u}dx + \mu \int_0^L q\tilde{q}dx + c\int_0^L \theta\tilde{\theta}dx + \alpha_1 \int_0^L v_x \tilde{v}_x dx \\ &+ \beta \int_0^L (\gamma v_x - p_x)(\gamma \tilde{v}_x - \tilde{p}_x)\, dx. \end{aligned} \tag{2.2}$$

The domain $D(\mathbb{A})$ of $\mathbb{A}$ is given by

$$\begin{aligned} D(\mathbb{A}) = \Big\{ &\mathbb{U} \in \mathbb{H} : v \in \tilde{H}^2(0,L),\ u \in \tilde{H}^1(0,L),\ p \in \tilde{H}^2(0,L), \\ &q \in \tilde{H}^1(0,L),\ \theta \in H^2(0,L) \cap H_0^1(0,L),\ v_x(L) = p_x(L) = 0 \Big\}. \end{aligned}$$

Clearly, $D(\mathbb{A})$ is dense in $\mathbb{H}$.

Next, we prove the existence results. So, we show that the operator $\mathbb{A}$ is maximal dissipative.

Theorem 2.1. *Let $\mathbb{U}_0 \in \mathbb{H}$ and assume that (A1)–(A3) hold. Then, there exists a unique solution $\mathbb{U} \in C(\mathbb{R}_+, \mathbb{H})$ of the problem (2.1). Moreover, if $\mathbb{U}_0 \in D(\mathbb{A})$ then*

$$\mathbb{U} \in C(\mathbb{R}_+, D(\mathbb{A})) \cap C^1(\mathbb{R}_+, \mathbb{H}).$$

Proof. First, we show that the operator $\mathbb{A}$ is maximal dissipative. For any $\mathbb{U} \in D(\mathbb{A})$ and by using the inner product, we have

$$\langle \mathbb{A}\mathbb{U}, \mathbb{U} \rangle_{\mathbb{H}} = \left\langle \begin{pmatrix} u \\ \frac{\alpha}{\rho} v_{xx} - \frac{\gamma\beta}{\rho} p_{xx} - \frac{\delta}{\rho}\theta_x \\ q \\ \frac{\beta}{\mu} p_{xx} - \frac{\gamma\beta}{\mu} v_{xx} \\ \frac{\kappa}{c}\theta_{xx} - \frac{\delta}{c} u_x \end{pmatrix}, \begin{pmatrix} v \\ u \\ p \\ q \\ \theta \end{pmatrix} \right\rangle,$$

we obtain

$$\langle \mathbb{A}\mathbb{U}, \mathbb{U}\rangle_{\mathbb{H}} = -\kappa \int_0^L \theta_x^2 dx \leq 0. \tag{2.3}$$

So, the operator $\mathbb{A}$ is dissipative. Now, we show that $R(I - \mathbb{A}) = \mathbb{H}$. For this, it is sufficient to prove that for $S = (s_1, s_2, s_3, s_4, s_5)^T \in \mathbb{H}$, there exists $\mathbb{U} = (v, u, p, q, \theta)^T \in D(\mathbb{A})$ such that

$$(I - \mathbb{A})\mathbb{U} = S. \tag{2.4}$$

That is,

$$\begin{cases} v - u = s_1 \in \tilde{H}^1(0, L), \\ \rho u - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x = \rho s_2 \in L^2(0, L), \\ p - q = s_3 \in \tilde{H}^1(0, L), \\ \mu q - \beta p_{xx} + \gamma\beta v_{xx} = \mu s_4 \in L^2(0, L), \\ c\theta - \kappa\theta_{xx} + \delta u_x = cs_5 \in L^2(0, L). \end{cases} \tag{2.5}$$

Inserting $u = v - s_1$ in $(2.5)_2$, $(2.5)_5$ and $q = p - s_3$ in $(2.5)_4$, we get

$$\begin{cases} \rho v - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x = h \in L^2(0, L), \\ \mu p - \beta p_{xx} + \gamma\beta v_{xx} = J \in L^2(0, L), \\ c\theta - \kappa\theta_{xx} + \delta v_x = Q \in L^2(0, L), \end{cases} \tag{2.6}$$

where

$$h = \rho(s_1 + s_2), \ J = \mu(s_3 + s_4), \ Q = cs_5 + \delta s_{1x}.$$

For solve (2.6), we introduce the variational formulation as follows

$$\mathbb{B}((v, p, \theta), (v_1, p_1, \theta_1)) = \mathbb{L}(v_1, p_1, \theta_1), \ \forall (v_1, p_1, \theta_1) \in W, \tag{2.7}$$

where $W = \tilde{H}^1(0, L) \times \tilde{H}^1(0, L) \times H_0^1(0, L)$, the bilinear form $\mathbb{B} : W \times W \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned} &\mathbb{B}((v, p, \theta), (v_1, p_1, \theta_1)) \\ &= \rho \int_0^L v v_1 dx + \alpha_1 \int_0^L v_x v_{1x} dx \\ &+ \mu \int_0^L p p_1 dx + \beta \int_0^L (\gamma v_x - p_x)(\gamma v_{1x} - p_{1x}) dx \\ &+ \delta \int_0^L (\theta_x v_1 + v_x \theta_1) dx + c \int_0^L \theta\theta_1 dx + \kappa \int_0^L \theta_x \theta_{1x} dx, \end{aligned}$$

and the linear form $\mathbb{L} : W \longrightarrow \mathbb{R}$ is given by

$$\mathbb{L}(v_1, p_1, \theta_1) = \int_0^L h v_1 dx + \int_0^L J p_1 dx + \int_0^L Q\theta_1 dx.$$

Now, for $W = \tilde{H}^1(0, L) \times \tilde{H}^1(0, L) \times H_0^1(0, L)$ equipped with the norm

$$\|(v, p, \theta)\|_W^2 = \|v\|_2^2 + \|v_x\|_2^2 + \|p\|_2^2 + \|\gamma v_x - p_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2.$$

Then, we have

$$\mathbb{B}((v, p, \theta), (v, p, \theta)) = \rho\|v\|_2^2 + \alpha_1\|v_x\|_2^2 + \mu\|p\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2 + c\|\theta\|_2^2 + \kappa\|\theta_x\|_2^2.$$

So, for some $M > 0$, we get

$$\mathbb{B}((v, p, \theta), (v, p, \theta)) \geq M\|(v, p, \theta)\|_W^2.$$

Then, the operator $\mathbb{B}$ is coercive.

Now, by using the Cauchy-Schwartz inequality, we have

$$|\mathbb{B}((v, p, \theta), (v_1, p_1, \theta_1))| \leq n\|(v, p, \theta)\|_W \|(v_1, p_1, \theta_1)\|_W.$$

Similarly

$$|\mathbb{L}(v_1, p_1, \theta_1)| \leq l\|(v_1, p_1, \theta_1)\|_W.$$

Then, by using the Lax-Milgram theorem, we prove the existence of a unique

$$(v,p,\theta)\in \tilde{H}^1(0,L)\times \tilde{H}^1(0,L)\times H_0^1(0,L),$$

satisfying

$$\mathbb{B}\left((v,p,\theta),(v_1,p_1,\theta_1)\right)=\mathbb{L}\left(v_1,p_1,\theta_1\right),\ \forall\left(v_1,p_1,\theta_1\right)\in W.$$

By substituting v into $(2.5)_1$ and p into $(2.5)_3$, we obtain

$$(u,q)\in \tilde{H}^1(0,L)\times \tilde{H}^1(0,L).$$

Furthermore, if we take $(v_1,\theta_1)=(0,0)\in \tilde{H}^1(0,L)\times H_0^1(0,L)$ in (2.7), then we obtain

$$\mu\int_0^L pp_1dx+\beta\int_0^L p_xp_{1x}dx-\gamma\beta\int_0^L v_xp_{1x}dx=\int_0^L Jp_1dx,\ \forall p_1\in \tilde{H}^1(0,L). \tag{2.8}$$

By multiplying $(2.6)_1$ and $(2.6)_2$ by $\gamma\beta$ and α respectively, and by adding the obtained results, we get

$$p_{xx}=\frac{\gamma\rho}{\alpha_1}v+\frac{\alpha\mu}{\beta\alpha_1}p+\frac{\gamma\delta}{\alpha_1}\theta_x-\frac{\gamma}{\alpha_1}h-\frac{\alpha}{\beta\alpha_1}J\in L^2(0,L).$$

Consequently, we obtain

$$p\in \tilde{H}^2(0,L).$$

In the same way, if we take $(p_1,\theta_1)=(0,0)\in \tilde{H}^1(0,L)\times H_0^1(0,L)$ in (2.7), we get

$$\rho\int_0^L vv_1dx+\alpha\int_0^L v_xv_{1x}dx-\gamma\beta\int_0^L p_xv_{1x}dx+\delta\int_0^L \theta_xv_1dx=\int_0^L hv_1dx,\ \forall v_1\in \tilde{H}^1(0,L). \tag{2.9}$$

Multiplying $(2.6)_2$ by γ and adding with $(2.6)_1$, we obtain

$$v_{xx}=\frac{\rho}{\alpha_1}v+\frac{\gamma\mu}{\alpha_1}p+\frac{\delta}{\alpha_1}\theta_x-\frac{1}{\alpha_1}h-\frac{\gamma}{\alpha_1}J\in L^2(0,L).$$

Consequently, we obtain

$$v\in \tilde{H}^2(0,L).$$

Similarly, if we take $(v_1,p_1)=(0,0)\in \tilde{H}^1(0,L)\times \tilde{H}^1(0,L)$ in (2.7), then we have

$$c\int_0^L \theta\theta_1dx+\kappa\int_0^L \theta_x\theta_{1x}dx+\delta\int_0^L v_x\theta_1dx=\int_0^L Q\theta_1dx,\ \forall\theta_1\in H_0^1(0,L). \tag{2.10}$$

By exploiting $(2.6)_3$, we obtain

$$\theta_{xx}=\frac{c}{\kappa}\theta+\frac{\delta}{\kappa}v_x-\frac{1}{\kappa}Q\in L^2(0,L).$$

Consequently, we obtain

$$\theta\in H^2(0,L)\cap H_0^1(0,L).$$

Thus, by integrating (2.8) and (2.9) by parts and exploiting $(2.6)_1$, $(2.6)_2$, then we obtain

$$\begin{cases}\left(\beta p_x(L)-\gamma\beta v_x(L)\right)p_1(L)-\left(\beta p_x(0)-\gamma\beta v_x(0)\right)p_1(0)=0,\\ \left(\alpha v_x(L)-\gamma\beta p_x(L)\right)v_1(L)-\left(\alpha v_x(0)-\gamma\beta p_x(0)\right)v_1(0)=0.\end{cases}$$

Furthermore, if we take $p_1=\frac{\gamma x}{L}$ and $v_1=\frac{x}{L}$, then we get

$$\begin{cases}\gamma\beta p_x(L)-\gamma^2\beta v_x(L)=0,\\ \alpha v_x(L)-\gamma\beta p_x(L)=0.\end{cases} \tag{2.11}$$

By performing some calculations on the above expression (2.11), we get

$$\left(\alpha-\gamma^2\beta\right)v_x(L)=0,$$

and as $(\alpha-\gamma^2\beta)=\alpha_1$, then we find

$$\alpha_1v_x(L)=0.$$

Since $\alpha_1 > 0$, then we obtain

$$v_x(L) = 0. \tag{2.12}$$

By substituting the value of (2.12) into (2.11), then we get

$$p_x(L) = 0.$$

Therefore,

$$v_x(L) = p_x(L) = 0.$$

Then, there exists a unique $\mathbb{U} \in D(\mathbb{A})$, such that (2.4) is satisfied. Hence, the operator $\mathbb{A}$ is maximal dissipative.

Next, we show that the operator $\mathbb{G}$ defined in (2.1) is locally Lipschitz in $\mathbb{H}$. Let $\mathbb{U} = (v, u, p, q, \theta)^T \in \mathbb{H}$ and $\mathbb{U}_1 = (v_1, u_1, p_1, q_1, \theta_1)^T \in \mathbb{H}$, then we have

$$\|\mathbb{G}(\mathbb{U}) - \mathbb{G}(\mathbb{U}_1)\|_{\mathbb{H}} \leq \eta \|g(u) - g(u_1)\|_{L^2}.$$

By exploiting (1.9), Hölder inequality, we can obtain

$$\|g(u) - g(u_1)\|_{L^2} \leq k_0 \left(\|u\|_{2\rho}^{\rho} + \|u_1\|_{2\rho}^{\rho}\right) \|u - u_1\|_{L^2} \leq \eta_1 \|u - u_1\|_{L^2},$$

which gives us

$$\|\mathbb{G}(\mathbb{U}) - \mathbb{G}(\mathbb{U}_1)\|_{\mathbb{H}} \leq \eta_2 \|\mathbb{U} - \mathbb{U}_1\|_{\mathbb{H}}.$$

So, $\mathbb{G}$ is locally Lipschitz operator in $\mathbb{H}$.

Therefore, by using the Hille-Yosida theorem, we obtain the well-posedness result. $\square$

3. Technical lemmas

In this section, by using the multiplier technique, we prove and state our stability results for the solution of the system (1.8).

Lemma 3.1. *If (v, p, θ) is a solution of (1.8), then the energy functional defined by*

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L \left[\rho v_t^2 + \mu p_t^2 + \alpha_1 v_x^2 + \beta(\gamma v_x - p_x)^2 + c\theta^2\right] dx, \tag{3.1}$$

satisfies

$$\mathcal{E}'(t) = -\kappa \int_0^L \theta_x^2 dx - \chi(t) \int_0^L v_t g(v_t)\, dx \leq 0. \tag{3.2}$$

Proof. Multiplying the first equation in (1.8) by v_t, the second one by p_t and the third one by θ, respectively. Then, integrating over $(0, L)$, applying integration by parts and the boundary conditions, and adding the obtained results, we get (3.2). $\square$

Lemma 3.2. *If (v, p, θ) is a solution of (1.8), then the functional*

$$\mathcal{F}_1(t) = \rho \int_0^L v v_t dx + \gamma\mu \int_0^L v p_t dx, \ t \geq 0,$$

satisfies

$$\begin{aligned} \mathcal{F}_1'(t) \leq & -\frac{\alpha_1}{4}\int_0^L v_x^2 dx + \left[\rho + \frac{\gamma\mu}{4\varepsilon_1}\right]\int_0^L v_t^2 dx + \gamma\mu\varepsilon_1 \int_0^L p_t^2 dx \\ & + \frac{\delta^2 c}{\alpha_1}\int_0^L \theta_x^2 dx + \frac{\chi^2(0)\, c}{2\alpha_1}\int_0^L g^2(v_t)\, dx. \end{aligned} \tag{3.3}$$

Proof. By differentiating $\mathcal{F}_1(t)$, using $(1.8)_1$, $(1.8)_2$ and integrating by parts together with the boundary conditions, we obtain

$$\mathcal{F}_1'(t) = -\alpha_1 \int_0^L v_x^2 dx + \rho \int_0^L v_t^2 dx + \gamma\mu \int_0^L v_t p_t dx - \delta \int_0^L \theta_x v dx - \chi(t) \int_0^L v g(v_t)\, dx. \tag{3.4}$$

By using the Young and Poincaré inequalities, we get

$$\gamma\mu\int_0^L v_t p_t dx \leq \gamma\mu\varepsilon_1\int_0^L p_t^2 + \frac{\gamma\mu}{4\varepsilon_1}\int_0^L v_t^2 dx, \tag{3.5}$$

$$-\chi(t)\int_0^L vg(v_t)\,dx \leq \frac{\alpha_1}{2}\int_0^L v_x^2 dx + \frac{\chi^2(0)\,c}{2\alpha_1}\int_0^L g^2(v_t)\,dx, \tag{3.6}$$

$$-\delta\int_0^L \theta_x v dx \leq \frac{\alpha_1}{4}\int_0^L v_x^2 dx + \frac{\delta^2 c}{\alpha_1}\int_0^L \theta_x^2 dx. \tag{3.7}$$

Substituting (3.5), (3.6) and (3.7), in (3.4), we get (3.3). □

Lemma 3.3. *If (v, p, θ) is a solution of (1.8), then the functional*

$$\mathcal{F}_2(t) = \int_0^L (\rho v_t + \gamma\mu p_t)(\gamma v - p)\,dx,\ t \geq 0,$$

satisfies

$$\begin{aligned}\mathcal{F}_2'(t) \leq &-\frac{\gamma\mu}{2}\int_0^L p_t^2 dx + \alpha_1\varepsilon_3\int_0^L (\gamma v_x - p_x)^2 dx + \frac{\alpha_1}{4\varepsilon_3}\int_0^L v_x^2 dx\\ &+\delta\varepsilon_5 c\int_0^L (\gamma v_x - p_x)^2 dx + \frac{\delta}{4\varepsilon_5}\int_0^L \theta_x^2 dx + c\chi(0)\varepsilon_4\int_0^L (\gamma v_x - p_x)^2 dx\\ &+\frac{\chi(0)}{4\varepsilon_4}\int_0^L g^2(v_t)\,dx + \left[\rho\gamma + \frac{(\gamma^2\mu - \rho)^2}{2\gamma\mu}\right]\int_0^L v_t^2 dx.\end{aligned} \tag{3.8}$$

Proof. By differentiating $\mathcal{F}_2(t)$, using $(1.8)_1$, $(1.8)_2$ and integrating by parts together with the boundary conditions, we get

$$\begin{aligned}\mathcal{F}_2'(t) = &-\gamma\mu\int_0^L p_t^2 dx - \alpha_1\int_0^L v_x(\gamma v_x - p_x)\,dx - \delta\int_0^L \theta_x(\gamma v - p)\,dx\\ &-\chi(t)\int_0^L (\gamma v - p)\,g(v_t)\,dx + \rho\gamma\int_0^L v_t^2 dx + \left(\gamma^2\mu - \rho\right)\int_0^L p_t v_t dx.\end{aligned} \tag{3.9}$$

By applying the Young and Poincaré inequalities, we get

$$-\alpha_1\int_0^L v_x(\gamma v_x - p_x)\,dx \leq \alpha_1\varepsilon_3\int_0^L (\gamma v_x - p_x)^2 dx + \frac{\alpha_1}{4\varepsilon_3}\int_0^L v_x^2 dx, \tag{3.10}$$

$$-\chi(t)\int_0^L (\gamma v - p)\,g(v_t)\,dx \leq c\chi(0)\varepsilon_4\int_0^L (\gamma v_x - p_x)^2 dx + \frac{\chi(0)}{4\varepsilon_4}\int_0^L g^2(v_t)\,dx, \tag{3.11}$$

and

$$-\delta\int_0^L \theta_x(\gamma v - p)\,dx \leq \delta\varepsilon_5 c\int_0^L (\gamma v_x - p_x)^2 dx + \frac{\delta}{4\varepsilon_5}\int_0^L \theta_x^2 dx. \tag{3.12}$$

By using the Young inequality again, we obtain

$$\left(\gamma^2\mu - \rho\right)\int_0^L p_t v_t dx \leq \frac{\gamma\mu}{2}\int_0^L p_t^2 dx + \frac{(\gamma^2\mu - \rho)^2}{2\gamma\mu}\int_0^L v_t^2 dx. \tag{3.13}$$

By substituting (3.10)–(3.13) in (3.9), we get (3.8). □

Lemma 3.4. *If (v, p, θ) is a solution of (1.8), then the functional*

$$\mathcal{F}_3(t) = \rho\int_0^L v v_t dx + \mu\int_0^L p p_t dx,\ t \geq 0,$$

satisfies

$$\begin{aligned}\mathcal{F}_3'(t) \leq &-\beta\int_0^L (\gamma v_x - p_x)^2 dx + \rho\int_0^L v_t^2 dx + \delta c\varepsilon_7\int_0^L v_x^2 dx + \frac{\delta}{4\varepsilon_7}\int_0^L \theta_x^2 dx\\ &+\mu\int_0^L p_t^2 dx + \chi(0)\,c\varepsilon_6\int_0^L v_x^2 dx + \frac{\chi(0)}{4\varepsilon_6}\int_0^L g^2(v_t)\,dx.\end{aligned} \tag{3.14}$$

Proof. By differentiating $\mathcal{F}_3(t)$, using $(1.8)_1$, $(1.8)_2$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned}\mathcal{F}_3'(t) &= -\beta\int_0^L (\gamma v_x - p_x)^2\,dx - \alpha_1\int_0^L v_x^2dx + \rho\int_0^L v_t^2dx\\&- \delta\int_0^L \theta_x v dx + \mu\int_0^L p_t^2 dx - \chi(t)\int_0^L g(v_t)\,v dx. \end{aligned}\tag{3.15}$$

By using the Young and Poincaré, inequalities, we obtain

$$-\chi(t)\int_0^L g(v_t)\,v dx \le \chi(0)\,c\varepsilon_6\int_0^L v_x^2dx + \frac{\chi(0)}{4\varepsilon_6}\int_0^L g^2(v_t)\,dx, \tag{3.16}$$

and

$$-\delta\int_0^L \theta_x v dx \le \delta c\varepsilon_7\int_0^L v_x^2dx + \frac{\delta}{4\varepsilon_7}\int_0^L \theta_x^2 dx. \tag{3.17}$$

Substituting (3.16), (3.17) in (3.15), we get (3.14). □

Now, we define the Liapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) := \mathcal{N}\mathcal{E}(t) + \sum_{i=1}^3 \mathcal{N}_i\mathcal{F}_i(t), \tag{3.18}$$

where $\mathcal{N}$, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ are positive constants.

Lemma 3.5. *If (v,p,θ) be a solution of (1.8), then there are two positive constants τ_1 and τ_2 such that the Liapunov functional (3.18) satisfies*

$$\tau_1\mathcal{E}(t) \le \mathcal{L}(t) \le \tau_2\mathcal{E}(t),\ \forall t\ge 0, \tag{3.19}$$

and

$$\mathcal{L}'(t) \le -\beta_1\mathcal{E}(t) + c\int_0^L \left(v_t^2 + g^2(v_t)\right)dx,\ \forall t\ge 0. \tag{3.20}$$

Proof. From (3.18), we have

$$\begin{aligned}|\mathcal{L}(t) - \mathcal{N}\mathcal{E}(t)| &\le \rho\mathcal{N}_1\int_0^L |v_t v|\,dx + \gamma\mu\mathcal{N}_1\int_0^L |vp_t|\,dx\\&+ \mathcal{N}_2\int_0^L |(\rho v_t + \gamma\mu p_t)|\,|(\gamma v - p)|\,dx\\&+ \rho\mathcal{N}_3\int_0^L |vv_t|\,dx + \mu\mathcal{N}_3\int_0^L |pp_t|\,dx.\end{aligned}$$

By using the Young, Poincaré and Cauchy-Schwartz inequalities, we obtain

$$|\mathcal{L}(t) - \mathcal{N}\mathcal{E}(t)| \le c\mathcal{E}(t),$$

which yields

$$(\mathcal{N} - c)\,\mathcal{E}(t) \le \mathcal{L}(t) \le (\mathcal{N} + c)\,\mathcal{E}(t).$$

By choosing $\mathcal{N}$ (depending on $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$) sufficiently large we get (3.19). Now, By differentiating $\mathcal{L}(t)$, using (3.2), (3.3), (3.8), and (3.14), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq &-\left[\frac{\mathcal{N}_1\alpha_1}{4} - \frac{\mathcal{N}_2\alpha_1}{4\varepsilon_3} - \mathcal{N}_3\chi(0)\, c\varepsilon_6 - \delta c\varepsilon_7\mathcal{N}_3\right]\int_0^L v_x^2 dx \\
&- \left[\frac{\mathcal{N}_2\gamma\mu}{2} - \mathcal{N}_1\gamma\mu\varepsilon_1 - \mu\mathcal{N}_3\right]\int_0^L p_t^2 dx \\
&- \left[\beta\mathcal{N}_3 - \mathcal{N}_2\left(\alpha_1\varepsilon_3 + c\chi(0)\varepsilon_4 + \delta\varepsilon_5 c\right)\right]\int_0^L (\gamma v_x - p_x)^2\, dx \\
&- \left[\kappa\mathcal{N} - \frac{\mathcal{N}_1\delta^2 c}{\alpha_1} - \frac{\mathcal{N}_2\delta}{4\varepsilon_5} - \frac{\delta\mathcal{N}_3}{4\varepsilon_7}\right]\int_0^L \theta_x^2 dx \\
&+ \left[\mathcal{N}\chi(0)\,\varepsilon_2 + \mathcal{N}_1\left(\rho + \frac{\gamma\mu}{4\varepsilon_1}\right) + \mathcal{N}_2\left(\frac{(\gamma^2\mu-\rho)^2}{2\gamma\mu} + \rho\gamma\right) + \mathcal{N}_3\rho\right]\int_0^L v_t^2 dx \\
&+ \left[\frac{\mathcal{N}\chi(0)}{4\varepsilon_2} + \frac{\mathcal{N}_1\chi^2(0)c}{2\alpha_1} + \frac{\mathcal{N}_2\chi(0)}{4\varepsilon_4} + \frac{\mathcal{N}_3\chi(0)}{4\varepsilon_6}\right]\int_0^L g^2(v_t)\, dx.
\end{aligned}$$

By setting $\varepsilon_1 = \frac{1}{\mathcal{N}_1}$, $\varepsilon_2 = \frac{1}{\mathcal{N}}$, $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \frac{1}{\mathcal{N}_2}$, $\varepsilon_6 = \varepsilon_7 = \frac{1}{\mathcal{N}_3}$

$$\begin{aligned}
\mathcal{L}'(t) \leq &-\left[\frac{\mathcal{N}_1\alpha_1}{4} - \frac{\mathcal{N}_2^2\alpha_1}{4} - \chi(0)\, c - \delta c\right]\int_0^L v_x^2 dx \\
&- \left[\frac{\mathcal{N}_2\gamma\mu}{2} - \mu\mathcal{N}_3 - \gamma\mu\right]\int_0^L p_t^2 dx \\
&- \left[\beta\mathcal{N}_3 - (\alpha_1 + c\chi(0) + \delta c)\right]\int_0^L (\gamma v_x - p_x)^2\, dx \\
&- \left[\kappa\mathcal{N} - \frac{\mathcal{N}_1\delta^2 c}{\alpha_1} - \frac{\mathcal{N}_2^2\delta}{4} - \frac{\delta\mathcal{N}_3^2}{4}\right]\int_0^L \theta_x^2 dx \\
&+ \left[\chi(0) + \mathcal{N}_1\left(\rho + \frac{\mathcal{N}_1\gamma\mu}{4}\right) + \mathcal{N}_2\left(\frac{(\gamma^2\mu-\rho)^2}{2\gamma\mu} + \rho\gamma\right) + \mathcal{N}_3\rho\right]\int_0^L v_t^2 dx \\
&+ \left[\frac{\mathcal{N}^2\chi(0)}{4} + \frac{\mathcal{N}_1\chi^2(0)c}{2\alpha_1} + \frac{\mathcal{N}_2^2\chi(0)}{4} + \frac{\mathcal{N}_3^2\chi(0)}{4}\right]\int_0^L g^2(v_t)\, dx.
\end{aligned}$$

Now, we select our parameters appropriately as follows.

First, we choose $\mathcal{N}_3$ large enough so that

$$\delta_1 = \beta\mathcal{N}_3 - (\alpha_1 + c\chi(0) + \delta c) > 0.$$

Then we choose $\mathcal{N}_2$ large enough so that

$$\delta_2 = \frac{\mathcal{N}_2\gamma\mu}{2} - \mu\mathcal{N}_3 - \gamma\mu > 0.$$

Next, we select $\mathcal{N}_1$ so large that

$$\delta_3 = \frac{\mathcal{N}_1\alpha_1}{4} - \frac{\mathcal{N}_2^2\alpha_1}{4} - \chi(0)\, c - \delta c > 0.$$

Finally, we choose $\mathcal{N}$ large enough so that

$$\delta_4 = \kappa\mathcal{N} - \frac{\mathcal{N}_1\delta^2 c}{\alpha_1} - \frac{\mathcal{N}_2^2\delta}{4} - \frac{\delta\mathcal{N}_3^2}{4} > 0.$$

So, we end up with

$$\mathcal{L}'(t) \leq -\delta_1 \int_0^L (\gamma v_x - p_x)^2 \, dx - \delta_2 \int_0^L p_t^2 dx - \delta_3 \int_0^L v_x^2 dx$$
$$- \delta_4 \int_0^L \theta_x^2 dx + c \int_0^L \left(v_t^2 + g^2(v_t)\right) dx.$$

Use the Poincaré inequality to substitute $-\int_0^L \theta_x^2 dx$ by $-\int_0^L \theta^2 dx$, we get

$$\begin{aligned}\mathcal{L}'(t) &\leq -\delta_1 \int_0^L (\gamma v_x - p_x)^2 \, dx - \delta_2 \int_0^L p_t^2 dx - \delta_3 \int_0^L v_x^2 dx \\ &\quad - c\delta_4 \int_0^L \theta^2 dx + c \int_0^L \left(v_t^2 + g^2(v_t)\right) dx \\ &\leq -\min(\delta_1, \delta_2, \delta_3, c\delta_4) \int_0^L \left[v_x^2 + p_t^2 + (\gamma v_x - p_x)^2 + \theta^2\right] dx \\ &\quad + c \int_0^L \left(v_t^2 + g^2(v_t)\right) dx. \end{aligned} \tag{3.21}$$

On the other hand,

$$\mathcal{E}(t) \leq c \int_0^L \left[v_x^2 + p_t^2 + (\gamma v_x - p_x)^2 + v_t^2 + \theta^2\right] dx,$$

which implies that

$$-\int_0^L \left[v_x^2 + p_t^2 + (\gamma v_x - p_x)^2 + \theta^2\right] dx \leq -c'\mathcal{E}(t) + \int_0^L v_t^2 dx. \tag{3.22}$$

The combination of (3.21) and (3.22) gives (3.20). □

4. Stability results

In this section, we state and prove our stability result.

Theorem 4.1. *Suppose (A1)–(A2) hold. Then, there exist positive constants μ_1, μ_2, μ_3, and ε_0 such that the solution of (1.8) satisfies*

$$\mathcal{E}(t) \leq \mu_1 \Phi_1^{-1} \left(\mu_2 \int_0^t \chi(s) \, ds + \mu_3\right), \ t \geq 0, \tag{4.1}$$

where

$$\Phi_1(t) = \int_t^1 \frac{1}{\Phi_0(s)} ds \quad \text{and} \quad \Phi_0(t) = t\Phi'(\varepsilon_0 t), \ \forall \varepsilon_0 \geq 0.$$

Proof. Multiplying (3.20) by $\chi(t)$, we have

$$\chi(t) \mathcal{L}'(t) \leq -\beta_1 \chi(t) \mathcal{E}(t) + c\chi(t) \int_0^L \left(v_t^2 + g^2(v_t)\right) dx. \tag{4.2}$$

Now, we distinguish two cases.

Case 1. Φ is linear on $[0, \varepsilon]$. By exploiting (3.2) and the hypothesis (A2) and note that c is a generic positive constant, then we obtain

$$\begin{aligned}\chi(t) \mathcal{L}'(t) &\leq -\beta_1 \chi(t) \mathcal{E}(t) + c\chi(t) \int_0^L (v_t g(v_t)) \, dx \\ &= -\beta_1 \chi(t) \mathcal{E}(t) - c\mathcal{E}'(t) - c\kappa \int_0^L \theta_x^2 dx \\ &\leq -\beta_1 \chi(t) \mathcal{E}(t) - c\mathcal{E}'(t),\end{aligned}$$

which implies

$$(\chi(t) \mathcal{L}(t) + c\mathcal{E}(t))' - \chi'(t) \mathcal{L}(t) \leq -\beta_1 \chi(t) \mathcal{E}(t). \tag{4.3}$$

Since $\chi'(t) \le 0$, then (4.3) is equivalent to

$$\mathcal{K}_0'(t) \le -\beta_1 \chi(t)\, \mathcal{E}(t),$$

where

$$\mathcal{K}_0(t) := \chi(t)\, \mathcal{L}(t) + c\mathcal{E}(t) \sim \mathcal{E}(t). \tag{4.4}$$

So, for some positive constant λ_1, we obtain

$$\mathcal{K}_0'(t) + \lambda_1 \chi(t)\, \mathcal{K}_0(t) \le 0, \quad \forall t \ge 0. \tag{4.5}$$

The combination of (4.5) and (4.4), gives

$$\mathcal{E}(t) \le \mathcal{E}(0) e^{-\lambda_1 \int_0^t \chi(s)ds} = \mathcal{E}(0)\Phi_1^{-1}\left(\lambda_1 \int_0^t \chi(s)\, ds\right). \tag{4.6}$$

Case 2. Φ is nonlinear on $[0, \varepsilon]$. In this case, we first choose $0 < \varepsilon_1 \le \varepsilon$ such that

$$sg(s) \le \min\{\varepsilon, \Phi(\varepsilon)\}, \ \forall |s| \le \varepsilon_1. \tag{4.7}$$

By using (A2) along with fact that the function g is continuous and $|g(s)| > 0$, for $s \neq 0$, it follows that

$$\begin{cases} s^2 + g^2(s) \le \Phi^{-1}(sg(s)), \ \forall |s| \le \varepsilon_1, \\ c_1 |s| \le |g(s)| \le c_2 |s|, \ \forall |s| \ge \varepsilon_1. \end{cases} \tag{4.8}$$

To estimate the last integral in (4.2), we introduce the following partition of $(0, L)$

$$I_1 = \{x \in (0, L) : |v_t| \le \varepsilon_1\}, \ I_2 = \{x \in (0, L) : |v_t| > \varepsilon_1\}.$$

Now, we define $I(t)$ by

$$I(t) = \int_{I_1} v_t g(v_t) dx,$$

using Jensen inequality (note that Φ^{-1} is concave), we have

$$\Phi^{-1}(I(t)) \ge c \int_{I_1} \Phi^{-1}(v_t g(v_t))\, dx. \tag{4.9}$$

Direct computations using (4.8) and (4.9) yields

$$\begin{aligned}
&\chi(t) \int_0^L \left(v_t^2 + g^2(v_t)\right) dx \\
&= \chi(t) \int_{I_1} \left(v_t^2 + g^2(v_t)\right) dx + \chi(t) \int_{I_2} \left(v_t^2 + g^2(v_t)\right) dx \\
&\le \chi(t) \int_{I_1} \Phi^{-1}(v_t g(v_t))\, dx + c\chi(t) \int_{I_2} v_t g(v_t) dx \\
&\le c\chi(t)\, \Phi^{-1}(I(t)) - c\mathcal{E}'(t) - c\kappa \int_0^L \theta_x^2 dx \\
&\le c\chi(t)\, \Phi^{-1}(I(t)) - c\mathcal{E}'(t).
\end{aligned} \tag{4.10}$$

So, by substituting (4.10) into (4.2) and using (4.4) and (A1), we have

$$\mathcal{K}_0'(t) \le -\beta_1 \chi(t)\, \mathcal{E}(t) + c\chi(t)\, \Phi^{-1}(I(t)), \ \forall t \ge 0. \tag{4.11}$$

Now, for $\varepsilon_0 < \varepsilon$ and $\delta_0 > 0$, using (4.11) and the fact that $\mathcal{E}'(t) \le 0$, $\Phi'(t) > 0$, $\Phi''(t) > 0$ on $(0, \varepsilon]$, we find that the functional $\mathcal{K}_1$, defined by

$$\mathcal{K}_1(t) := \Phi'\left(\varepsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \mathcal{K}_0(t) + \delta_0 \mathcal{E}(t),$$

satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 \mathcal{K}_1(t) \le \mathcal{E}(t) \le \alpha_2 \mathcal{K}_1(t), \tag{4.12}$$

and

$$\begin{aligned}\mathcal{K}_1'(t) &= \varepsilon_0\frac{\mathcal{E}'(t)}{\mathcal{E}(0)}\Phi''\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\mathcal{K}_0(t)+\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\mathcal{K}_0'(t)+\delta_0\mathcal{E}'(t)\\ &\le -\beta_1\chi(t)\,\mathcal{E}(t)\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)+c\chi(t)\,\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\Phi^{-1}(I(t))+\delta_0\mathcal{E}'(t).\end{aligned} \tag{4.13}$$

Let Φ^* be the convex conjugate of Φ defined by

$$\Phi^*(s)=s(\Phi')^{-1}(s)-\Phi\left[(\Phi')^{-1}(s)\right] \text{ if } s\in(0,\Phi'(\varepsilon)],$$

satisfying the following general Young inequality

$$AB\le\Phi^*(A)+\Phi(B) \text{ if } A\in(0,\Phi'(\varepsilon)],\ B\in(0,\varepsilon].$$

Taking $A=\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)$ and $B=\Phi^{-1}(I(t))$, using (4.7), we get

$$c\chi(t)\,\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\Phi^{-1}(I(t))\le c\chi(t)\,\Phi^*\left(\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\right)+c\chi(t)\,I(t).$$

By using (3.2) and the fact that $\Phi^*(s)\le s(\Phi')^{-1}(s)$, we have

$$\begin{aligned}&c\chi(t)\,\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\Phi^{-1}(I(t))\\ &\le c\varepsilon_0\chi(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)-c\mathcal{E}'(t)-c\kappa\int_0^L\theta_x^2dx\\ &\le c\varepsilon_0\chi(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)-c\mathcal{E}'(t).\end{aligned} \tag{4.14}$$

Substituting (4.14) into (4.13), we obtain

$$\begin{aligned}\mathcal{K}_1'(t) &\le -\beta_1\chi(t)\,\mathcal{E}(t)\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\\ &\quad + c\varepsilon_0\chi(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)-c\mathcal{E}'(t)+\delta_0\mathcal{E}'(t)\\ &\le -(\beta_1\mathcal{E}(0)-c\varepsilon_0)\chi(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)+(\delta_0-c)\,\mathcal{E}'(t).\end{aligned}$$

We now choose ε_0 and δ_0 small enough such that

$$k=\beta_1\mathcal{E}(0)-c\varepsilon_0>0 \text{ and } \delta_0-c>0,$$

using that $\mathcal{E}'(t)\le 0$, we get

$$\mathcal{K}_1'(t)\le -k\chi(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\Phi'\left(\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)=-k\chi(t)\,\Phi_0\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right), \tag{4.15}$$

where $\Phi_0(t)=t\Phi'(\varepsilon_0 t)$. Note that

$$\Phi_0'(t)=\Phi'(\varepsilon_0 t)+\varepsilon_0 t\Phi''(\varepsilon_0 t).$$

So, using the strict convexity of Φ on $(0,\varepsilon]$, we find that $\Phi_0(t)>0$, $\Phi_0'(t)>0$ on $(0,1]$. With $\mathcal{K}(t):=\frac{\alpha_1\mathcal{K}_1(t)}{\mathcal{E}(0)}$ it is obvious that $\mathcal{K}(t)\le\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\le 1$. Now, using (4.12) and (4.15), we have

$$\mathcal{K}(t)\sim\mathcal{E}(t), \tag{4.16}$$

and, for some $\mu_2>0$,

$$\mathcal{K}'(t)\le-\mu_2\chi(t)\,\Phi_0(\mathcal{K}(t)). \tag{4.17}$$

Inequality (4.17) implies that $\frac{d}{dt}[\Phi_1(\mathcal{K}(t))]\ge\mu_2\chi(t)$, where

$$\Phi_1(t)=\int_t^1\frac{1}{\Phi_0(s)}ds.$$

So, by integrating over $[0,t]$, we get, for some $\mu_3 > 0$,

$$\mathcal{K}(t) \leq \Phi_1^{-1}\left(\mu_2 \int_0^t \chi(s)\, ds + \mu_3\right). \tag{4.18}$$

Here, we used the fact that Φ_1 is strictly decreasing on $(0,1]$. Therefore, by using (4.16) and (4.18), we get (4.1). $\square$

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