Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (1) (2024), 305 – 313
DOI : 10.15672/hujms.1187608

Research Article

Unit and idempotent additive maps over countable linear transformations

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Abstract

Let V be a countably generated right vector space over a field F and $\sigma \in End(V_F)$ be a shift operator. We show that there exist a unit u and an idempotent e in $End(V_F)$ such that $1-u, \sigma-u$ are units in $End(V_F)$ and $1-e, \sigma-e$ are idempotents in $End(V_F)$. We also obtain that if D is a division ring $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$ and V_D is a D-module, then for every $\alpha \in End(V_D)$ there exists a unit $u \in End(V_D)$ such that $1-u, \alpha-u$ are units in $End(V_D)$.

Mathematics Subject Classification (2020). 15A57, 15A27

Keywords. unit, shift operator, idempotent matrix, tripotent matrix, semilocal ring, division ring

1. Introduction

Let R be an associative ring with unity. Given a map $f : R \to R$, f is said to be *unit-additive* if $f(u+v) = f(u) + f(v)$, for all units $u, v \in R$. Moreover, if $f(uv) = f(u)f(v)$ for all units $u, v \in R$, then the ring R is called *unit-homomorphic* [6]. In [6], the authors proved that each unit additive map of a semilocal ring R is additive if and only if either R has no a homomorphic image isomorphic to $\mathbb{Z}_2$ or$R/J(R) \cong \mathbb{Z}_2$, where $J(R)$ denotes the Jacobson radical and $\mathbb{Z}_n$ is the ring of integers modulo n.

The study of rings satisfying the 2-sum property (i.e. rings such that each of their elements is a sum of two units) was introduced by Wolfson [12] and Zelinsky [13]. They, independently, proved that the endomorphism ring of a vector space V over a division ring D satisfies the 2-sum property, except that $dim(V) = 1$ and $D = \mathbb{F}_2$. A ring R is said to have *unit sum number* n, if for any $r \in R$ there exist units $u_1, \cdots, u_n$ of R such that $r = u_1 + \cdots + u_n$. According to [7], a ring R is said to satisfy the *binary 2-sum property* if for any $a, b \in R$ there exist units u_1, u_2, u_3 of R such that $a = u_1 + u_2$ and $b = u_1 + u_3$. Recall that a semilocal ring R has unit sum number 2 if and only if no factor ring of R is isomorphic to $\mathbb{F}_2$ [4]. Recently, the author of [7] provides a similar characterization of semilocal rings with the binary 2-sum property: a semilocal ring R

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Received: 11.10.2022; Accepted: 27.01.2023

satisfies the binary 2-sum property if and only if no factor ring of R is isomorphic to $\mathbb{F}_2$, $\mathbb{F}_3$, or the 2×2 matrix ring $\mathbb{M}_2(\mathbb{F}_2)$. They also obtained in [7, Corollary 19] that if R is an exchange ring with primitive factors Artinian (e.g. a semilocal ring), then R satisfies the binary 2-sum property if R satisfies the GoodearlMenal property (two elements $a, b \in R$ are said to satisfy *the Goodearl-Menal condition*, in case there exists a unit u in R such that $a-u, u^{-1}$ is a unit. A ring R is said to satisfy the *Gooodearl-Menal* if every elements $a, b \in R$ satisfies this property [5], [8].

Let V be a countably generated right vector space over a division ring D. In 2010, Chen [2] generalized a result of Zelinsky [13] and proved that for any endomorphism f of V there exists an automorphism g of V with $f+g$ and $f-g^{-1}$ both automorphisms of V if $D \neq \mathbb{Z}_2, \mathbb{Z}_3$. We also notice that this result is extended to an Artinian right R-module over a semilocal ring R that contains 1/2 and 1/3 [11]. In [10, Theorem], Nicholson and Varadarjan proved that every countable linear transformation over a division ring is *clean* (every element of a ring is a sum of an idempotent and a unit [9]). Let V be a countably generated vector space over a division ring D such that $|D| \neq 2, 3$, and let $End_D(V)$ denote the ring of linear transformations on V. Chen [3] also obtained two interesting decompositions in $End_D(V)$: (1) For any $f \in End_D(V)$, there exists an automorphism g on V such that $f-g$ and $f-g^{-1}$ are both automorphisms on V. Thus, $End_D(V)$ satisfies a special case of the Goodearl-Menal condition. (2) For any $f \in End_D(V)$, there exists an automorphism g on V such that f^2-g^2 is an automorphism on V. In [1], Camillo and Simon also applied the Nicholson-Varadarajan theorem on clean linear transformations and they used the tool: the shift operator. For a countably infinite dimensional right vector space V_D, a linear transformation $f \in End(V_D)$ is called a *shift operator* if there exists a basis $\{v_1, v_2, \cdots, v_n, \cdots\}$ of V such that $f(v_i) = v_{i+1}$ for all i.

Vidinli Hüseyin Tevfik Pasha (1832-1901), also widely known as General Hussein in America, was the most important mathematician, lecturer, scientist, bureaucrat and member of army of the late modern period of the Ottoman Empire. Even for today his book Linear Algebra (1882) is a basic source for the related area. His notion was originated from a perspective to generalize the notion of multiplication to lines in the two and three dimensional case. Note that the matrix representation of the shift operator f over basis $\{v_i\}_i$ is of the form

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

The main purpose of this study is to obtain the following new results on sum decompositions using a new tool, namely *idempotent additive* maps taking idempotents instead of units in a unit additive map:

(1) Let V be a countably generated right vector space over a field F and $\sigma \in S = End(V_F)$ be a shift operator. Then there exist a unit $u \in S$ and an idempotent $e \in S$ such that $1-u, \sigma-u$ are units in s and $1-e, \sigma-e$ are idempotents in s. (Theorem 2.4);
(2) If D is a division ring and $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$, then there exists a unit $u \in End(V_D)$ for which $1-u, \alpha-u \in U(End(V_D))$ for any $\alpha \in End(V_D)$ (Theorem 2.9);
(3) If D is a division ring and $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$, and f is an unit additive map in $S := End(V_D)$ such that $f(0) = 0$, then f is additive (Corollary 2.10).

2. Results

We will denote by $U(R)$ the set of all units and by $Id(R)$ a set of all idempotents of a ring R.

Definition 2.1. Let R be a ring. A map $\sigma : R \to R$ is called an (a) *idempotent* (*unit*) *additive map* if σ is additive on idempotents (units) of R, i.e

$$\sigma(a+b) = \sigma(a) + \sigma(b),$$

for every idempotents (units) $a, b \in R$.

For convenience, we fix a notation: for $a, b \in R$, we write

$a \leftrightsquigarrow b$ (or $a \overset{u}{\leftrightsquigarrow} b$, to emphasize the element u) if $a-u, b-u \in U(R)$ for some $u \in U(R)$,

$a \rightleftharpoons b$ (or $a \overset{e}{\rightleftharpoons} b$ to emphasize the element e) if $a-e, b-e \in Id(R)$ for some $e \in Id(R)$,

$a \longleftrightarrow b$ (or $a \overset{u}{\longleftrightarrow} b$ to emphasize the unit u), if there exists $u \in U(R)$ such that $a-u, b-u^{-1} \in U(R)$ (Goodearl-Menal condition [5]).

We list some properties of notations in the following observations.

Lemma 2.2. *The followings hold for a ring R and elements $a, b \in R$, $u, x, y \in U(R)$.*

(1) *Let σ be a unit-additive map of R. If $-a \leftrightsquigarrow u$, then $\sigma(a+u) = \sigma(a) + \sigma(u)$.*
(2) *If $1 \leftrightsquigarrow c$ for all $c \in R$, then every unit-additive map of R is additive.*
(3) *Let σ be an automorphism or anti-automorphism of R. Then:*
  (a) *$a \overset{u}{\leftrightsquigarrow} b$ iff $\sigma(a) \overset{\sigma(u)}{\leftrightsquigarrow} \sigma(b)$.*
  (b) *$a \overset{u}{\leftrightsquigarrow} b$ iff $xay \overset{xuy}{\leftrightsquigarrow} xby$.*
(4) (a) *$1 \overset{u}{\leftrightsquigarrow} a$ iff $1 \overset{u^{-1}}{\longleftrightarrow} a$.*
  (b) *$1 \leftrightsquigarrow x$ for all $x \in R$ iff $v \leftrightsquigarrow x$ for all $x \in R$ and all $v \in U(R)$.*
  (c) *$1 \longleftrightarrow x$ for all $x \in R$ iff $v \longleftrightarrow x$ for all $x \in R$ and all $v \in U(R)$.*
  (d) *$v \leftrightsquigarrow x$ for all $x \in R$ and all $v \in U(R)$ iff $v \longleftrightarrow x$ for all $x \in R$ and all $v \in U(R)$.*

Proof. (1) and (2) See [6, Lemmas 2.3 and 2.4].
(3) and (4) See [7, Lemmas 2.7 and 2.8]. □

Lemma 2.3. *The following conditions hold for a ring R and $r \in R$.*

(1) *Let σ be an idempotent-additive map of R and $e \in Id(R)$. If $-r \rightleftharpoons e$, then $\sigma(r+e) = \sigma(r) + \sigma(e)$.*
(2) *If $1 \rightleftharpoons x$ for all $x \in R$, then every idempotent-additive map of R is additive.*
(3) *$r \rightleftharpoons 1$ if and only if there exist $e, f \in Id(R)$ such that $r = e + f$,*
(4) *Let σ be a ring automorphisms of R. Then $r \rightleftharpoons 1$ if and only if $\sigma(r) \rightleftharpoons 1$*

Proof. (1) and (2) The proofs are similar to the proofs of Lemma 2.2 (1) and (2).
(3) If there exists $e \in Id(R)$ such that $r-e, 1-e \in Id(R)$, then it is enough to put $f := r-e$. The converse follow from the fact that $1-e \in Id(R)$ for an arbitrary idempotent e.
(4) This is clear since $\sigma(e) \in Id(R)$ for each $e \in Id(R)$. □

Now we are ready to prove our first main theorem.

Theorem 2.4. *Let V be a countably generated right vector space over a field F and $\sigma \in S = End(V_F)$ be a shift operator. Then*

(1) *$1 \rightleftharpoons \sigma$,*
(2) *$1 \leftrightsquigarrow \sigma$.*

Proof. (1) Let $E_1 := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $E_2 := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $0_{i\times j}$ be a zero matrix of type $i \times j$ and $(u_i)_{i<\omega}$ be a basis of V. Define an infinite block-diagonal matrices

$$B = \begin{pmatrix} E_1 & 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \dots \\ 0_{2\times 2} & E_1 & 0_{2\times 2} & 0_{2\times 2} & \dots \\ 0_{2\times 2} & 0_{2\times 2} & E_1 & 0_{2\times 2} & \dots \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & E_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0_{1\times 1} & 0_{1\times 2} & 0_{1\times 2} & 0_{1\times 2} & 0_{1\times 2} & \dots \\ 0_{2\times 1} & E_2 & 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \dots \\ 0_{2\times 1} & 0_{2\times 2} & E_2 & 0_{2\times 2} & 0_{2\times 2} & \dots \\ 0_{2\times 1} & 0_{2\times 2} & 0_{2\times 2} & E_2 & 0_{2\times 2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

and endomorphisms $e, f \in End(V)$ such that B is the matrix of e and C is the matrix of f with respect to the basis $(u_i)_{i<\omega}$, i.e.

$$e(u_{2i-1}) = e(u_{2i}) = u_{2i},$$

$$f(u_{2i-1}) = 0,\ f(u_{2i}) = u_{2i} + u_{2i+1}$$

for each $i \geq 1$. Then

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 2 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \end{pmatrix}.$$

is the matrix of $e+f$ and it is easy to see that $e, f \in Id(End(V))$ as $E_1^2 = E_1$ and $E_2^2 = E_2$.

Let us denote $g := e + f$ and we will construct a basis $(v_i)_{i<\omega}$ which witnesses that g is a shift operator, i.e. that $g(v_i) = v_{i+1}$. First, put $v_1 = u_1$ and $v_2 = u_2$. Then

$$\mathrm{Span}(v_1, v_2) = \mathrm{Span}(u_1, u_2),$$

$$g(v_1) = v_2$$

and

$$g(v_2) \in \mathrm{Span}(v_1, v_2, u_3) \setminus \mathrm{Span}(v_1, v_2).$$

So we have $v_1, \dots, v_i$ such that

$$\mathrm{Span}(v_1, \dots, v_i) = \mathrm{Span}(u_1, \dots, u_i),$$

$$g(v_{i-1}) = v_i$$

and

$$g(v_i) \in \mathrm{Span}(v_1, \dots, v_i, u_{i+1}) \setminus \mathrm{Span}(v_1, \dots, v_i).$$

Define $v_{i+1} := g(v_i)$. By the induction hypotheses $v_1, \dots, v_{i+1}$ is linearly independent, which implies

$$\mathrm{Span}(v_1, \dots, v_{i+1}) = \mathrm{Span}(u_1, \dots, u_{i+1}).$$

Hence, it is clear from the matrix A that $g(v_{i+1}) \in \mathrm{Span}(v_1, \dots, v_{i+1}, u_{i+2}) \setminus \mathrm{Span}(v_1, \dots, v_{i+1})$.

Since $(v_i)_{i<\omega}$ is a basis satisfying $[e + f](v_i) = v_{i+1}$ for each i, we have already obtained that $e + f$ is a shift operator, which implies $1 \rightleftharpoons e + f$ by Lemma 2.3(3). As there exists an invertible operator, say $a \in End(V)$, such that $e + f = a^{-1}\sigma a$, the assertion follows from Lemma 2.3(4).

(2) Denote by $(v_i)_{i<\omega}$ a basis of V such that $\sigma(v_i) = v_{i+1}$. First, suppose that characteristic of F is not 2. Let $U_1 := \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, $U_2 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $U_3 := \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$. Remark that

all these matrices are invertible. We denote by u an operator such that its matrix with respect to the basis $(v_i)_{i<\omega}$ is

$$[u]_{(v_i)}\begin{pmatrix} U_1 & 0 & 0 & 0 & \dots \\ 0 & U_1 & 0 & 0 & \dots \\ 0 & 0 & U_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

Now we easily compute matrices

$$[1-u]_{(v_i)} = \begin{pmatrix} U_3 & 0 & 0 & 0 & \dots \\ 0 & U_3 & 0 & 0 & \dots \\ 0 & 0 & U_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \quad \text{and} \quad [\sigma - u]_{(v_i)} = \begin{pmatrix} 1_{1\times 1} & 0 & 0 & 0 & \dots \\ 0 & U_2 & 0 & 0 & \dots \\ 0 & 0 & U_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

Since all these matrices are invertible, we can see that $u, 1-u, \sigma - u \in U(S)$.

Now, let $1+1=0$ and consider the matrix

$$A = \begin{pmatrix} U & 0 & 0 & 0 & \dots \\ 0 & U & 0 & 0 & \dots \\ 0 & 0 & U & 0 & \dots \\ 0 & 0 & 0 & U & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

where $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is an invertible matrix with the inverse $U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. Clearly, the matrices A and $A+I$ are invertible with the inverses

$$A^{-1} = \begin{pmatrix} U^{-1} & 0 & 0 & 0 & \dots \\ 0 & U^{-1} & 0 & 0 & \dots \\ 0 & 0 & U^{-1} & 0 & \dots \\ 0 & 0 & 0 & U^{-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

and

$$(A+I)^{-1} = \begin{pmatrix} (U+I_3)^{-1} & 0 & 0 & 0 & \dots \\ 0 & (U+I_3)^{-1} & 0 & 0 & \dots \\ 0 & 0 & (U+I_3)^{-1} & 0 & \dots \\ 0 & 0 & 0 & (U+I_3)^{-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

where $(U+I_3)^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Let A be the matrix of an operator u with respect to the basis $(v_i)_{i<\omega}$. Hence u and $1+u$ are invertible operators.

Finally, the operator $u+\sigma$ is invertible since it has a matrix with respect to $(v_i)_{i<\omega}$

$$\begin{pmatrix} B & 0 & 0 & 0 & \dots \\ E_{13} & B & 0 & 0 & \dots \\ 0 & E_{13} & B & 0 & \dots \\ 0 & 0 & E_{13} & B & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

with the inverse

$$\begin{pmatrix} B^{-1} & 0 & 0 & 0 & \dots \\ C & B^{-1} & 0 & 0 & \dots \\ 0 & C & B^{-1} & 0 & \dots \\ 0 & 0 & C & B^{-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

where $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. □

$GL_n(D)$ denotes the n-dimensional general linear group over a division ring D and $\mathbb{M}_n(D)$ denotes the ring of all $n \times n$ matrices over D with an identity I_n.

Recall that the matrices a and b are *equivalent* if there exists a regular matrix p such that $a = p^{-1}bp$.

Lemma 2.5. *Let D be a division ring of characteristic different from 2, $n \in \mathbb{N}$ and $b \in \mathbb{M}_n(D)$. Then the following conditions are equivalent.*

(1) $b \rightleftharpoons I_n$

(2) *b is equivalent to a block matrix* $\begin{pmatrix} 2I_r & a_{12} & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}_n(D)$ *where I_r, I_s, I_t are identity matrices, and $a_{i,j}$ and 0 are matrices.*

Proof. Recall that $b \rightleftharpoons I_n$ if and only if there exist $e, f \in Id(\mathbb{M}_n(D))$ such that $b = e + f$ by Lemma 2.3(3). Since

$$\begin{pmatrix} 2I_r & a_{12} & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_r & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I_r & 0 & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where both the matrices on the right side are idempotents, we get that (2) ⇒ (1) holds.

Let $b = e + f$ for idempotent matrices e, f and let us identify all matrices with linear operator on D^n given by the matrix multiplication. Let us denote by B the basis of $\mathrm{im}(e) \cap \mathrm{im}(f)$ which could be completed to bases of $\mathrm{im}(e)$ and $\mathrm{im}(f)$ by E and F, i.e. $B \cup E$ is a basis of $\mathrm{im}(e)$ and $B \cup F$ is a basis of $\mathrm{im}(f)$. Since e and f are idempotents, we get $e(u) = u$ for each $u \in B \cup E$ and $f(u) = u$ for each $u \in B \cup E$. Hence $e(v) \in \mathrm{Span}(B \cup E)$ and $f(v) \in \mathrm{Span}(B \cup F)$ for all $v \in D^n$.

Finally let K be a basis of $\ker(b)$ and let $k \in \ker(b)$. Then $0 = b(k) = e(k) + f(k)$ and so $e(k) = f(-k) \in \mathrm{im}(e) \cap \mathrm{im}(f) = \mathrm{Span}(B)$. Hence $k = e(k) = f(-k) = -k$ which implies that $k = 0$ and $\ker(b) \subseteq \ker(e) \cap \ker(f)$. It means that the matrix of operator $b = e + f$ with respect to the basis $B \cup E \cup F \cup K$ is of the form

$$\begin{pmatrix} I_r & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I_r & 0 & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2I_r & a_{12} & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is equivalent to the matrix b. □

Theorem 2.6. *Let D be a division ring.*

(1) *Let the characteristic of D be different from 2 and $b \in \mathbb{M}_2(D)$. Then $b \rightleftharpoons I_2$ if and only if b is equivalent to one of the matrices:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & c \\ d & 1 \end{pmatrix}, \begin{pmatrix} 2 & c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

for some $c, d \in D$.

(2) *If $D \not\cong \mathbb{Z}_2, \mathbb{Z}_3$ and $n \in \mathbb{N}$, then*

(i) *for any $a, b \in \mathbb{M}_n(D)$, there exists $c \in GL_n(D)$ such that $b \overset{c}{\leftrightsquigarrow} a$.*

(ii) $b \overset{c}{\leftrightsquigarrow} I_n$.

Proof. (1) This follows from Lemma 2.5.
(2) Assuming $D \not\cong \mathbb{Z}_2, \mathbb{Z}_3$ implies that $|D| \geq 4$. Let $x, y \in D$. We have the following three cases.

If $x = 0$, then we choose a nonzero element $u \in D$ such that $u \neq y$. Hence $y - u \neq 0$.

If $y = 0$, then we choose a nonzero element $u \in D$ such that $u \neq x$. Hence $x - u \neq 0$.

If $x \neq$ and $y \neq 0$, then we choose a nonzero element $u \in D$ such that $u \neq x$ and $u \neq y$. As a result we obtain that $x \overset{u}{\leftrightsquigarrow} u$.

Let $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{M}_n(D)$ and $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{M}_n(D)$, where $a_{11}, b_{11} \in D$, $a_{12}, b_{12} \in \mathbb{M}_{1\times(n-1)}(D)$, $a_{21}, b_{21} \in \mathbb{M}_{(n-1)\times 1}(D)$ and $a_{22}, b_{22} \in \mathbb{M}_{(n-1)\times(n-1)}(D)$. Note that there exists $0 \neq x \in D$ such that $a_{11} - x = u_1 \neq 0$ and $b_{11} - x = u_2 \neq 0$. Since $a_{22} - a_{21}u_1^{-1}a_{12} \in \mathbb{M}_{(n-1)}(D)$ and $b_{22} - b_{21}u_1^{-1}b_{12} \in \mathbb{M}_{(n-1)}(D)$, we can obtain $y \in GL_{n-1}(D)$ such that $a_{22} - a_{21}u_1^{-1}a_{12} - y = v_1 \in GL_{n-1}(D)$ and $b_{22} - b_{21}u_1^{-1}b_{12} - y \in GL_{n-1}(D)$. They imply that

$$a - diag(x, y) = \begin{pmatrix} u_1 & a_{12} \\ a_{21} & v_1 + a_{21}u_1^{-1}a_{12} \end{pmatrix}$$

and

$$b - diag(x, y) = \begin{pmatrix} u_2 & b_{12} \\ b_{21} & v_2 + b_{21}u_1^{-1}b_{12} \end{pmatrix}.$$

Since

$$\begin{pmatrix} u_1 & a_{12} \\ a_{21} & v_1 + a_{21}u_1^{-1}a_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_{21}u_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} u_1 & a_{12} \\ 0 & v_1 \end{pmatrix}$$

and

$$\begin{pmatrix} u_2 & b_{12} \\ b_{21} & v_2 + b_{21}u_1^{-1}b_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_{21}u_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} u_2 & b_{12} \\ 0 & v_2 \end{pmatrix},$$

we get $\begin{pmatrix} u_1 & a_{12} \\ a_{21} & v_1 + a_{21}u_1^{-1}a_{12} \end{pmatrix}, \begin{pmatrix} u_2 & b_{12} \\ b_{21} & v_2 + b_{21}u_1^{-1}b_{12} \end{pmatrix} \in GL_n(D)$ as desired. □

For the last main theorem we need the following a series of lemmas.

Lemma 2.7. *Let D be a division ring and $\alpha \in End(V_D)$ such that V_D is spanned by $\{y, \alpha(y), \alpha^2(y), \cdots\}$ for some $y \in V$. If $D \not\cong \mathbb{Z}_2, \mathbb{Z}_3$, then*

(1) $1 \leftrightsquigarrow \alpha$.

(2) *If V_D is infinitely generated, then $1 \rightleftharpoons \alpha$.*

Proof. (1) We may assume that $V_D \neq 0$. If $\alpha^n(y) \notin yD + \alpha(y)D + \cdots + \alpha^{n-1}(y)D$ for all $n \geq 1$, then $\{y, \alpha(y), \alpha^2(y), \cdots\}$ is a basis of V_D. Since α is a shift operator with respect to the basis $\{y, \alpha(y), \alpha^2(y), \cdots\}$, we get $1 \leftrightsquigarrow \alpha$ by Theorem 2.4(2). Now suppose that there exists $n \in \mathbb{N}$ such that $\alpha^n(y) \notin yD + \alpha(y)D + \cdots + \alpha^{n-1}(y)D$. If n is minimal with respect to this property, then $\{y, \alpha(y), \alpha^2(y), \cdots\}$ forms a basis for V_D. Hence $End_D(V_D) \cong \mathbb{M}_n(D)$. By Lemma 2.3(2), we obtain that $1 \leftrightsquigarrow \alpha$.
(2) This follows from Theorem 2.4(1) using the arguments of (1). □

Lemma 2.8. *Let D be a division ring such that $D \not\cong \mathbb{Z}_2, \mathbb{Z}_3$, $\alpha \in End(V_D)$ and U be an α-invariant subspace of V_D. Assume that there exists a vector $y \in U \setminus V$ such that $V = U + \sum_{i\geq 0} \alpha^i(y)D$. If the restriction $\alpha|_U$ satisfies $1 \leftrightsquigarrow \alpha|_U$, then $1 \leftrightsquigarrow \alpha$*

Proof. Let $V = M \oplus U$ where M is a subspace which contains y. Define

$$\widetilde{\alpha} : V/U \to V/U$$
$$\overline{v} \to \overline{\alpha(v)}$$

(see [10, Lemma 4]).

Clearly,

$$\overline{\alpha^n(y)} = \widetilde{\alpha^n}(\overline{v})$$

and there exists a D-subisomorphism $\theta_0 : V/U \to M$ given by $\theta_0(\overline{v}) = \theta(v)$ by [10, Lemma 4] where θ is an idempotent in $End_D(V)$ satisfying $\theta(V) = M$ and $Ker(\theta) = U$. By [10, Lemma 4], we have the endomorphism ring of M as:

$$\beta := \theta_0 \widetilde{\alpha} \theta_0^{-1} : M \to V/U \to V/U \to M.$$

By the hypothesis, $\{\overline{y}, \overline{\alpha(Y)}, \cdots\}$ spans V/U. Hence $\{\overline{y}, \widetilde{\alpha}(\overline{y}), \cdots\}$ spans V/U since $\overline{\alpha^n(y)} = \widetilde{\alpha^n}(\overline{v})$. Now it is easy to see that $\{\theta_0[\overline{y}], \theta_0[\widetilde{\alpha}(\overline{y})], \cdots\}$ spans M. By Lemma 2.7, we get $\beta \leftrightsquigarrow 1$. Then $\beta - v_1 = a_1$ and $1 - v_1 = b_1$ for some units v_1, a_1, b_1 of $End(M)$. By hypothesis, $1 \leftrightsquigarrow \alpha|_U$, we have $\alpha|_U - v_2 = a_2$ and $1 - v_2 = b_2$ for some units v_2, a_2, b_2 of $End(M)$. Since $V = M \oplus U$, we can define

$$v^*(v) = v^*(m + u) = v_1(m) + [\alpha(m) - \beta(m) + v_2(u)].$$

v^* is an automorphism of V: Since $v^*(m+u) = 0$ implies $v_1(m) = 0$ and $[\alpha(m) - \beta(m)] + v_2(u) = 0$, whence $m = u = 0$, we get v^* is monic. As $u = v_2(u_0) = v^*(0 + u_0)$ for some $u_0 \in U$, we obtain $U \subseteq Im(v^*)$. If $m \in M$, we write $m = v_1(m_1)$ for $m_1 \in M$, then $\alpha(m_1) - \beta(m_1) = -v_2(u_0)$. Then $v^*(m_1 + u_0) = v_1(m_1) + [\alpha(m_1) - \beta(m_1) + v_2(u_0)]$ which implies that $M \subseteq Im(v^*)$. Hence v^* is epic.
$\alpha - v^*$ is an automorphism: Firstly,

$$\begin{aligned}
(\alpha - v^*)(m + u) &= \alpha(m + u) - v^*(m + u) \\
&= \alpha(m) + \alpha(u) - v_1(m) - [\alpha(m) - \beta(m) - v_2(u)] \\
&= \alpha_{|_u}(u) - v_2(u) - v_1(m) + \beta(m) \\
&= b_2(u) + b_1(m).
\end{aligned}$$

Now, by a similar technic of previous proof, we can obtain that $\alpha - v^*$ is monic and epic.
$1 - v^*$ is an automorphism: Firstly,

$$\begin{aligned}
(1 - v^*)(m + u) &= 1(m + u) - v^*(m + u) \\
&= \alpha(m) + \alpha(u) - v_1(m) - [\alpha(m) - \beta(m) - v_2(u)] \\
&= 1(m) + 1(u) - v_1(m) - [\alpha(m) - \beta(m) + v_2(u)] \\
&= 1(m) - v_1(m) + 1(u) - v_2(u) + \beta(m) - \alpha(m) \\
&= b_1(m) + [b_2(u) + \beta(m) - \alpha(m)].
\end{aligned}$$

Finally, the same argument as for $\alpha - v^*$ shows that $1 - v^*$ is monic and epic. □

Theorem 2.9. *Let D be a division ring and $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$. Then $1 \leftrightsquigarrow \alpha$ for any $\alpha \in End(V_D)$.*

Proof. Fix $\alpha \in End(V_D)$. Define

$$\chi = \{(U, v) : U_D \subseteq V \text{ is a } \alpha-\text{invariant and } \alpha_{|_u} \overset{v}{\leftrightsquigarrow} 1\}.$$

Note that $(0, 0) \in \chi$. Now we define $(U, v) \leq (U', v')$ by $U \subseteq U'$ and $v'_{|_u} = v$ is a partial order of χ. By Zorn's Lemma, there exists a maximal element, say (U, v) in χ.

Assume $U \neq V$. Then, take $y \in V \setminus U$ and let $K := \sum_{i \geq 0} \alpha^i(y)D$. Hence we write $V_0 = U + K$. Clearly, V_0 and K are α-invariant subspaces, $\alpha \in End(V_0)$ and $\alpha_{|_U} \overset{v}{\leftrightsquigarrow} 1$ because $(U, v) \in \chi$. By Lemma 2.8, we get $\alpha \leftrightsquigarrow 1$ which contradicts the maximality of $(U, v) \in \chi$. □

Corollary 2.10. *Let D be a division ring different from $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and V_D a countably generated right vector space over D. If f is an unit additive map in $S := End(V_D)$ such that $f(0) = 0$, then f is additive.*

Proof. Observe that $f(0) = 0$ so $f(-a) = -f(a)$ for every $a \in U(S)$.

Let $\alpha, \beta \in S$. By Theorem , there exists invertible $u \in S$ such that $1 - u, \alpha - u, \beta + u$ are invertible. Hence,

$$\begin{aligned} f(\alpha+\beta) &= f(\alpha-1) + f(1+\beta) \\ &= f(\alpha-u+u-1) + f(1-u+u+\beta) \\ &= f(\alpha-u) + f(u-1) + f(1-u) + f(u+\beta) \\ &= f(\alpha-u) + f(u) - f(1) + f(1) + f(-u) + f(u+\beta) \\ &= f(\alpha-u+u) + f(-u+u+\beta) \\ &= f(\alpha) + f(\beta), \end{aligned}$$

as desired. □

Acknowledgment. The authors are very grateful to the editor and the referee, who suggested to Corollary 2.10, for their valuable comments and suggestions to improve this paper.

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