http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 4, Pages 867–879 (2023)
DOI:10.31801/cfsuasmas.1187854
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: October 12, 2022; Accepted: July 2, 2023

ON THE CURVES LYING ON PARALLEL-LIKE SURFACES OF THE RULED SURFACE IN E^3

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Abstract. In this paper, it has been researched curves lying on parallel-like surfaces M^f of the ruled surface M in E^3. Using the definition of parallel-like surfaces it has been found parametric expressions of parallel-like surface of the ruled surface and image curve of the directrix curve of the base surface. Moreover, obtaining Darboux frames of curves lying on surfaces M and M^f, it has been compared the geodesic curvatures, the normal curvatures and the geodesic torsions of these curves.

1. Introduction

Ruled surfaces are surfaces that attract the attention of many researchers, especially differential geometers, and make them work on these surfaces from past to present. These surfaces have been obtained by the motion of a straight line on a curve in space, also called the directrix of the surface. Since ruled surfaces have a particularly simple structure, they are important for use in many fields such as architecture, engineering, mechanics, kinematics and CAD computer aided design, etc. as well as differential geometry [1, 3, 6, 7, 15].

Parallel-like surfaces are the surfaces obtained as a result of the generalization of parallel surfaces. These surfaces were first described by Tarakcı and Hacısalihoğlu in 2002 and named with surfaces at a constant distance from the edge of regression on a surface [17]. The authors have obtained by considering a surface instead of a curve in the paper written by Hans Vogler. He has defined notion of curve at a constant distance from the edge of regression on a curve. In 2004, Tarakcı and Hacısalihoğlu have computed for parallel-like surfaces some properties and theorems given for parallel surfaces [16]. After this work, it's made many articles by different authors on parallel-like surfaces. In 2010, Sağlam and Kalkan have

2020 *Mathematics Subject Classification.* 53A04, 53A05.
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searched parallel-like surfaces in E_1^3 Minkowski space [12]. In 2015, Yurttançıkmaz and Tarakcı have established a relationship between focal surfaces and parallel-like surfaces. They have obtained focal surfaces using parallel-like surfaces, that is, an alternative way of finding focal surfaces of any surface via parallel-like surfaces was demonstrated [18]. From this point of view, the geometric structure of parallel-like surfaces has an important place in the field of line congruence and therefore in the field of visualization(see [5]). And finally, in 2023, Yurttançıkmaz and Tarakcı have researced image curves on the parallel-like Surfaces in E^3 [19].

In this study, the parametric expression of the parallel like surface of the ruled surface was obtained and the differences between these surfaces were examined in terms of differential geometric properties. In general, it has been shown that the parallel like surface of the ruled surface is also not a ruled surface, but if the directrix curve is selected as the parameter curve, this surface also preserves its characteristics of being a ruled surface.

Since curves lying on the surface have an important place in terms of the theory of curves, it has attracted the attention of many geometers [8, 10, 13, 14]. In the theory of surfaces, the Darboux frame constructed at any non-umbilical point of the surface can be viewed as an analog of the Frenet frame. In here, obtaining β image curve lying on M^f parallel-like surface of M ruled surface, Darboux frames of curve-surface pairs (α, M) and $\left(\beta, M^f\right)$ have been calculated at any points P on M and $f(P)$ on M^f. Finally, it has been compared the geodesic curvatures, the normal curvatures, the geodesic torsions of reference curve α on M and its image curve β on M^f and expressed the relationships between these two curves.

2. Preliminaries

Let α be a unit speed curve lying on a surface M in E^3 and s be arc length of the curve α, i.e. $\left\|\alpha^{'}(s)\right\| = 1$. Suppose that Z is a unit normal vector of the surface M and T is unit tangent vector field of the curve α. Considering the vector field Y defined by $Y = Z \times T$, set of $\{T, Y, Z\}$ create orthonormal frame which is called Darboux frame for partner of curve-surface (α, M).

Thus, the geodesic curvature κ_g, the normal curvature κ_n, the geodesic torsion t_r of the curve $\alpha(s)$ can be calculated as follows

$$\kappa_g = \left\langle \alpha^{''}(s), Y \right\rangle \tag{1}$$

$$\kappa_n = \left\langle \alpha^{''}(s), Z_{\alpha(s)} \right\rangle \tag{2}$$

$$t_r = -\left\langle Z^{'}_{\alpha(s)}, Y \right\rangle. \tag{3}$$

Besides, the derivative formulas of the Darboux frame of (α, M) is given by

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & t_r \\ -\kappa_n & -t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}. \tag{4}$$

In addition**,** given an arbitrary curve $\beta(s)$ on the surface M under the condition $\left\|\beta'(s)\right\| = C$, the geodesic curvature κ_g, the normal curvature κ_n, the geodesic torsion t_r of the curve $\beta(s)$ can be calculated as follows

$$\kappa_g = \frac{1}{C^2}\left\langle \beta''(s), Y \right\rangle \tag{5}$$

$$\kappa_n = \frac{1}{C^2}\left\langle \beta''(s), Z_{\beta(s)} \right\rangle \tag{6}$$

$$t_r = -\frac{1}{C}\left\langle Z'_{\beta(s)}, Y \right\rangle. \tag{7}$$

Furthermore, in the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M the followings are well-known

i) $\alpha(s)$ is a geodesic curve $\Longleftrightarrow \kappa_g = 0$,

ii) $\alpha(s)$ is an asymptotic curve $\Longleftrightarrow \kappa_n = 0$,

iii) $\alpha(s)$ is a principal line $\Longleftrightarrow t_r = 0$ [2, 9, 11].

3. Parallel-Like Surfaces

Definition 1. *Let M and M^f be two surfaces in Euclidean space E^3 and Z_P be a unit normal vector and T_PM be tangent space at point P of the surface M and $\{X_P, Y_P\}$ be an orthonormal bases of T_PM. Take a unit vector $E_P = d_1X_P + d_2Y_P + d_3Z_P$, where $d_1, d_2, d_3 \in \mathbb{R}$ are constants and $d_1^2 + d_2^2 + d_3^2 = 1$. If there is a function f defined by,*

$$f : M \to M^f, \qquad f(P) = P + rE_P$$

where $r \in \mathbb{R}$, then the surface M^f is called parallel-like surface of the surface M.

Here, if $d_1 = d_2 = 0$, then $E_P = Z_P$ and so M and M^f are parallel surfaces. Now, we represent parametrization for parallel-like surface of the surface M . Let (ϕ, U) be a parametrization of M, so we can write that

$$\begin{aligned} \phi \;\; : \;\; & U \subset E^2 \mapsto M. \\ & (u,v) \;\; \mapsto \phi(u,v) \end{aligned}$$

In the case $\{\phi_u, \phi_v\}$ is a bases of T_PM, then we can write that $E_P = d_1\phi_u + d_2\phi_v + d_3Z_P$. Where, ϕ_u, ϕ_v are respectively partial derivatives of ϕ according to u and v. Since $M^f = \{f(P) : \; f(P) = P + rE_P\}$, a parametric representation of M^f is

$$\psi(u,v) = \phi(u,v) + rE(u,v).$$

Thus, it's obtained

$$M^f = \{\psi(u,v) : \; \psi(u,v) = \phi(u,v) + r\left(d_1\phi_u(u,v) + d_2\phi_v(u,v) + d_3Z(u,v)\right)\}$$

and if we get $rd_1 = \lambda_1$, $rd_2 = \lambda_2$, $rd_3 = \lambda_3$, then we have

$$M^f = \left\{ \begin{array}{c} \psi(u,v): \ \psi(u,v) = \phi(u,v) + \lambda_1\phi_u(u,v) + \lambda_2\phi_v(u,v) + \lambda_3 Z(u,v), \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = r^2 \end{array} \right\}$$

Calculation of ψ_u and ψ_v gives us that

$$\begin{aligned} \psi_u &= \phi_u + \lambda_1\phi_{uu} + \lambda_2\phi_{vu} + \lambda_3 Z_u \\ \psi_v &= \phi_v + \lambda_1\phi_{uv} + \lambda_2\phi_{vv} + \lambda_3 Z_v. \end{aligned} \tag{8}$$

Here $\phi_{uu}, \phi_{vu}, \phi_{uv}, \phi_{vv}, Z_u, Z_v$ are calculated as like as [17]. Suppose that parameter curves are curvature lines of M and let u and v be arc length of these curves. Thus, following equations are obtained

$$\begin{array}{c} \phi_{uu} = -\kappa_1 Z \\ \phi_{vv} = -\kappa_2 Z \\ \phi_{uv} = \phi_{vu} = 0 \\ Z_u = \kappa_1\phi_u \\ Z_v = \kappa_2\phi_v. \end{array} \tag{9}$$

From 8 and 9, we find

$$\begin{aligned} \psi_u &= (1 + \lambda_3\kappa_1)\,\phi_u - \lambda_1\kappa_1 Z \\ \psi_v &= (1 + \lambda_3\kappa_2)\,\phi_v - \lambda_2\kappa_2 Z \end{aligned}$$

and $\{\psi_u, \psi_v\}$ be a bases of $\chi(M^f)$. If we denote by Z^f unit normal vector of M^f, then Z^f is

$$Z^f = \frac{\psi_u \times \psi_v}{\|\psi_u \times \psi_v\|} = \frac{\lambda_1\kappa_1(1+\lambda_3\kappa_2)\,\phi_u + \lambda_2\kappa_2(1+\lambda_3\kappa_1)\,\phi_v + (1+\lambda_3\kappa_1)(1+\lambda_3\kappa_2)\,Z}{\sqrt{\lambda_1^2\kappa_1^2(1+\lambda_3\kappa_2)^2 + \lambda_2^2\kappa_2^2(1+\lambda_3\kappa_1)^2 + (1+\lambda_3\kappa_1)^2(1+\lambda_3\kappa_2)^2}}$$

where, κ_1, κ_2 are principal curvatures of the surface M. If

$$A = \sqrt{\lambda_1^2\kappa_1^2(1+\lambda_3\kappa_2)^2 + \lambda_2^2\kappa_2^2(1+\lambda_3\kappa_1)^2 + (1+\lambda_3\kappa_1)^2(1+\lambda_3\kappa_2)^2}$$

we can write

$$Z^f = \frac{\lambda_1\kappa_1(1+\lambda_3\kappa_2)}{A}\phi_u + \frac{\lambda_2\kappa_2(1+\lambda_3\kappa_1)}{A}\phi_v + \frac{(1+\lambda_3\kappa_1)(1+\lambda_3\kappa_2)}{A}Z.$$

Here in case of $\kappa_1 = \kappa_2$ and $\lambda_3 = -\frac{1}{\kappa_1} = -\frac{1}{\kappa_2}$ since ψ_u and ψ_v are not linear independent, M^f is not regular surface. We will not consider this case [17].

4. Parallel-Like Surfaces of Ruled-Surfaces

Ruled surfaces are surfaces formed by the movement of a straight line based on a curve called the directrix curve in space. This moving straight line is called a "generator". The parametric expression of the ruled surface M with the directrix curve $\alpha(u)$ and the direction vector $X(u)$ along the generator is

$$\phi(u,v)=\alpha(u)+vX(u). \tag{10}$$

Considering the definition of the parallel-like surface, it is clear that parametrization of the parallel-like surface of the ruled surface will be

$$\begin{aligned}\psi(u,v) &= \phi(u,v)+\lambda_1\phi_u(u,v)+\lambda_2\phi_v(u,v)+\lambda_3 Z(u,v) \qquad (11)\\ &= \alpha(u)+vX(u)+\lambda_1\left(\alpha^{'}(u)+vX^{'}(u)\right)+\lambda_2X(u)+\lambda_3\frac{\phi_u\times\phi_v}{\|\phi_u\times\phi_v\|}.\end{aligned}$$

Let's assume that the directrix curve α of the ruled surface M is given by the arc parameter and the tangent vector X on the directrix is the unit vector for any parameter u. If the unit tangent vector of the directrix curve α is $\alpha^{'}(u)=\frac{d\alpha}{du}=T$, the unit normal of the ruled surface M is Z and the curve α is chosen with the condition $\langle T,X\rangle=0$, the triple $\{T,X,Z\}$ becomes an orthonormal basis along the directrix curve. In order to express the parameterization of the parallel-like surface of the ruled surface more simply, we will find the variation of the triple $\{T,X,Z\}$ along the curve α, that is, the covariant derivatives of each tangent vector with respect to T. If the covariant derivatives of both sides of the equations $\langle T,T\rangle=\langle X,X\rangle=\langle Z,Z\rangle=1$ along the curve α is calculated with respect to T, it has been obtained the equations $\langle D_TT,T\rangle+\langle T,D_TT\rangle=0\Rightarrow 2\langle D_TT,T\rangle=0\Rightarrow\langle D_TT,T\rangle=0$ and similarly $\langle D_TX,X\rangle=0$ and $\langle D_TZ,Z\rangle=0$. Here, if $a=\langle D_TT,X\rangle$, $b=\langle D_TT,Z\rangle$, $c=\langle D_TX,Z\rangle$, so it is achieved following matrix [4]

$$\begin{bmatrix} D_TT\\ D_TX\\ D_TZ\end{bmatrix}=\begin{bmatrix}0&a&b\\-a&0&c\\-b&-c&0\end{bmatrix}\begin{bmatrix}T\\X\\Z\end{bmatrix}. \tag{12}$$

If these equations are substituted in equation 12, parametric expression of parallel-like surface of ruled surface is as follows

$$\begin{aligned}\psi(u,v) &= \alpha(u)+vX+\lambda_1\left(T+v\left(-aT+cZ\right)\right)+\lambda_2X+\lambda_3\left(\frac{-cvT+(1-av)Z}{\sqrt{c^2v^2+(1-av)^2}}\right)\\ &= \alpha(u)+\left((1-av)\lambda_1-\frac{cv\lambda_3}{\sqrt{c^2v^2+(1-av)^2}}\right)T\\ &\quad+(v+\lambda_2)X+\left(cv\lambda_1+\frac{\lambda_3(1-av)}{\sqrt{c^2v^2+(1-av)^2}}\right)Z\end{aligned}$$

and in other words

$$\begin{aligned}\psi(u,v) &= \alpha\left(u\right)+\left(X-a\lambda_1 T+c\lambda_1 Z-\frac{c\lambda_3 T+a\lambda_3 Z}{\sqrt{c^2v^2+\left(1-av\right)^2}}\right)v \\ &\quad +\lambda_1 T+\lambda_2 X+\frac{\lambda_3}{\sqrt{c^2v^2+\left(1-av\right)^2}}Z\end{aligned} \tag{13}$$

is obtained. Is equation 13 also a ruled surface? It will now be investigated whether the parallel-like surface of the ruled surface is also a ruled surface or can be under certain conditions.

Consider the ruled surface given by equation $\phi(u,v)=\alpha\left(u\right)+vX(u)$. If the directrix curve α of the ruled surface M is chosen as the first parameter curve, then it is obtained that (see chapter 5)

$$\phi_u=\alpha^{'}\left(u\right)=T. \tag{14}$$

Moreover, when the partial derivative of the general equation of the ruled surface given by eq.10 is calculated with respect to u, following expressions are found

$$\begin{aligned}\phi_u &= \alpha^{'}\left(u\right)+vX^{'}\left(u\right)\\ &= T+v\left(-aT+cZ\right)\\ &= \left(1-av\right)T+cvZ.\end{aligned} \tag{15}$$

Here, if equations 14 and 15 are compared, it is concluded that $a=0$, $c=0$. Substituting these results in the equation 13, then the parallel-like surface of the ruled surface will be

$$\psi(u,v)=\alpha\left(u\right)+\lambda_1 T+\lambda_2 X+\lambda_3 Z+vX(u). \tag{16}$$

Considering that the image of a curve on a parallel-like surface is

$$\alpha^f(u)=\beta(u)=\alpha(u)+\lambda_1 T+\lambda_2 X+\lambda_3 Z \tag{17}$$

and substituting eq.17 in eq.16, so we get

$$\psi(u,v)=\beta(u)+vX(u). \tag{18}$$

From here, it is clear that the equation 18 also denotes a ruled surface. Thus, the following result can be written.

Corollary 1. *If the directrix curve α of the ruled surface M is chosen as the first parameter curve, then the parallel-like surface of this ruled surface is also ruled surface.*

Now, in case the vector E_P in the definition of the parallel-like surface is chosen more specifically, it will be investigated the change in the parametric expression of the parallel-like surface of the ruled surface and in which cases the obtained surface will also be the ruled surface.

Proposition 1. *In case $\lambda_1 = 0$ in the definition of parallel-like surface, that is, if the vector E_P is lying in the plane $Sp\{\phi_v, Z\}$, then parametric expression of parallel-like surface of the ruled surface is*

$$\psi(u,v) = \alpha(u) - \frac{cv\lambda_3}{\sqrt{c^2v^2 + (1-av)^2}}T + (v+\lambda_2)X + \frac{\lambda_3(1-av)}{\sqrt{c^2v^2+(1-av)^2}}Z.$$

Proposition 2. *In case $\lambda_2 = 0$ in the definition of parallel-like surface, that is, if the vector E_P is lying in the plane $Sp\{\phi_u, Z\}$, then parametric expression of parallel-like surface of the ruled surface is*

$$\begin{aligned}\psi(u,v) &= \alpha(u) + \left((1-av)\lambda_1 - \frac{cv\lambda_3}{\sqrt{c^2v^2+(1-av)^2}}\right)T \\ &\quad + vX + \left(cv\lambda_1 + \frac{\lambda_3(1-av)}{\sqrt{c^2v^2+(1-av)^2}}\right)Z.\end{aligned}$$

Proposition 3. *In case $\lambda_3 = 0$ in the definition of parallel-like surface, that is, if the vector E_P is lying in the plane $Sp\{\phi_u, \phi_v\}$, then parametric expression of parallel-like surface of the ruled surface is*

$$\psi(u,v) = \alpha(u) + \lambda_1(1-av)T + (v+\lambda_2)X + cv\lambda_1 Z.$$

Proposition 4. *In case $\lambda_1 = \lambda_2 = 0$ in the definition of parallel-like surface, that is, if the vector E_P is in the direction of the normal vector Z, then parametric expression of parallel-like surface of the ruled surface is*

$$\psi(u,v) = \alpha(u) - \frac{cv\lambda_3}{\sqrt{c^2v^2+(1-av)^2}}T + vX + \frac{\lambda_3(1-av)}{\sqrt{c^2v^2+(1-av)^2}}Z.$$

Proposition 5. *In case $\lambda_2 = \lambda_3 = 0$ in the definition of parallel-like surface, that is, if the vector E_P is in the direction of the vector ϕ_u, then parametric expression of parallel-like surface of the ruled surface is*

$$\psi(u,v) = \alpha(u) + \lambda_1(1-av)T + vX + cv\lambda_1 Z.$$

Proposition 6. *In case $\lambda_1 = \lambda_3 = 0$ in the definition of parallel-like surface, that is, if the vector E_P is in the direction of the vector ϕ_v, then parametric expression of parallel-like surface of the ruled surface is*

$$\psi(u,v) = \alpha(u) + (v+\lambda_2)X.$$

5. Darboux Frame of Curves Lying on Parallel-Like Surfaces of the Ruled Surfaces

Assuming that the directrix curve α of the ruled surface M given by the equation $\phi(u,v) = \alpha(u) + vX(u)$ is the first parameter curve, then

$$\phi_u = (1-av)T + cvZ$$

and since u and v are arc parameters of the parameter curves on the ruled surface, $\phi_u = \alpha^{'}(u) = T$ and so, $1 - av = 1$, $cv = 0$. As a result from here, $a = c = 0$, that is, in case the directrix curve α of the ruled surface M is chosen as the first parameter curve, it is seen that the directrix curve α is both the principal curvature line and the geodesic curve. Thus, $\kappa_g = 0$ and $t_r = 0$ for the directrix curve α. Principal directions relating to different curvature lines of the ruled surface M are orthogonal, thus we can take as $\phi_u = \alpha^{'}(u) = T$ and $\phi_v = X$. Under these conditions, we can use Darboux frame $\{T, X, Z\}$ in place of orthonormal frame $\{\phi_u, \phi_v, Z\}$. If we consider definition of parallel-like surface of the ruled surface M, parametric representation of the curve β which is image of the curve α is

$$\beta(u) = \alpha(u) + \lambda_1 T + \lambda_2 X + \lambda_3 Z. \tag{19}$$

Now, we calculate Darboux frame $\left\{T^f, X^f, Z^f\right\}$ for partner of curve-surface $\left(\beta, M^f\right)$. It is clear that

$$T^f = \frac{\beta^{'}(u)}{\left\|\beta^{'}(u)\right\|}$$

If we take derivative according to u of eq.19, we find

$$\beta^{'}(u) = \alpha^{'}(u) + \lambda_1 T^{'} + \lambda_2 X^{'} + \lambda_3 Z^{'}$$

and if considering that $\alpha(u)$ is a principal line and so $t_r = 0$ equations 4 are substituted in this equation, we obtain

$$\beta^{'}(u) = \left(1 - \kappa_n \lambda_3\right) T + \kappa_n \lambda_1 Z \tag{20}$$

where $\left\|\beta^{'}(u)\right\| = C = \sqrt{\left(1 - \kappa_n \lambda_3\right)^2 + \lambda_1^2 \kappa_n^2}$. Thus, we find

$$T^f = \frac{\left(1 - \kappa_n \lambda_3\right)}{C} T + \frac{\kappa_n \lambda_1}{C} Z. \tag{21}$$

Moreover, we know already that

$$Z^f = \frac{\lambda_1 \kappa_1 \left(1 + \lambda_3 \kappa_2\right)}{A} T + \frac{\lambda_2 \kappa_2 \left(1 + \lambda_3 \kappa_1\right)}{A} X + \frac{\left(1 + \lambda_3 \kappa_1\right)\left(1 + \lambda_3 \kappa_2\right)}{A} Z. \tag{22}$$

For orthonormal frame $\left\{T^f, X^f, Z^f\right\}$, if we consider that $X^f = Z^f \times T^f$, we get

$$\begin{aligned} X^f = & \left[\frac{\kappa_n \lambda_1 \lambda_2 \left(\kappa_2 + \lambda_3 K\right)}{AC}\right] T \\ & + \left[\frac{\left(1 + \lambda_3 H + \lambda_3^2 K\right)\left(1 - \kappa_n \lambda_3\right) - \kappa_n \lambda_1^2 \left(\kappa_1 + \lambda_3 K\right)}{AC}\right] X \\ & - \left[\frac{\lambda_2 \left(\kappa_1 + \lambda_3 K\right)\left(1 - \kappa_n \lambda_3\right)}{AC}\right] Z \end{aligned} \tag{23}$$

where $K = \kappa_1 \kappa_2$, $H = \kappa_1 + \kappa_2$ are Gauss curvature and mean curvature of the surface M, respectively.

Now, we calculate the geodesic curvature κ_g^f, the normal curvature κ_n^f, the geodesic torsion t_r^f of the curve $\beta(u)$. We will use to calculate these curvatures following equations [11]

$$\kappa_g^f = \frac{1}{C^2}\left\langle \beta^{''}(u), X^f \right\rangle \tag{24}$$

$$\kappa_n^f = \frac{1}{C^2}\left\langle \beta^{''}(u), Z^f \right\rangle \tag{25}$$

$$t_r^f = -\frac{1}{C}\left\langle \left(Z^f\right)^{'}, X^f \right\rangle. \tag{26}$$

Firstly we find vector $\beta^{''}(u)$. If we take derivative of eq.20 according to u and use equations 4, we obtain

$$\beta^{''}(u) = -\left(\lambda_3\kappa_n^{'} + \lambda_1\kappa_n^2\right)T + \left(\lambda_1\kappa_n^{'} + \kappa_n\left(1-\kappa_n\lambda_3\right)\right)Z. \tag{27}$$

Furthermore we find vector $\left(Z^f\right)^{'}$. If we take derivative of eq.22 according to u and use equations 4, we obtain

$$\left(Z^f\right)^{'} = \frac{1}{A}\left\{\begin{array}{c} \left(\lambda_1\left(\kappa_1^{'} + \lambda_3 K^{'}\right) - A^{-1}B\lambda_1\left(\kappa_1+\lambda_3 K\right) - \kappa_n\left(1+\lambda_3 H + \lambda_3^2 K\right)\right)T \\ + \left(\lambda_2\left(\kappa_2^{'} + \lambda_3 K^{'}\right) - A^{-1}B\lambda_2\left(\kappa_2+\lambda_3 K\right)\right)X \\ + \left(\lambda_3 H^{'} + \lambda_3^2 K^{'} - A^{-1}B\left(1+\lambda_3 H+\lambda_3^2 K\right) + \kappa_n\lambda_1\left(\kappa_1+\lambda_3 K\right)\right)Z \end{array}\right\} \tag{28}$$

where

$$\begin{aligned} B &= A^{'} = \frac{1}{A}[\lambda_1^2\kappa_1\kappa_1^{'} + \lambda_2^2\kappa_2\kappa_2^{'} + \lambda_3\left(\lambda_1^2\kappa_1^{'} + \lambda_2^2\kappa_2^{'}\right)K + \lambda_3\left(\lambda_1^2\kappa_1+\lambda_2^2\kappa_2\right)K^{'} \\ &\quad + \left(\lambda_1^2+\lambda_2^2\right)\lambda_3^2 K K^{'} + \left(1+\lambda_3 H+\lambda_3^2 K\right)\left(\lambda_3 H^{'} + \lambda_3^2 K^{'}\right)]. \end{aligned}$$

So, if we substitute equations 23 and 27 into eq.24, we obtain

$$\begin{aligned} \kappa_g^f = -\frac{1}{AC^3}\{ \quad & \kappa_n\lambda_2\left(\kappa_2+\lambda_3 K\right)\left(\left(1-\kappa_n\lambda_3\right)^2 + \lambda_1\lambda_3\kappa_n^{'}\right) \\ & +\lambda_1^2\lambda_2\kappa_n^3\left(\kappa_2+\lambda_3 K\right) \\ & +\kappa_n^{'}\lambda_1\lambda_2\left(\kappa_2+\lambda_3 K\right)\left(1-\kappa_n\lambda_3\right)\}. \end{aligned} \tag{29}$$

Also, if we substitute equations 22 and 27 into eq.25, we obtain

$$\begin{aligned} \kappa_n^f &= \frac{1}{AC^2}\{\lambda_1\left(\kappa_1+\lambda_3 K\right)\left(-\lambda_3\kappa_n^{'} - \lambda_1\kappa_n^2\right) \\ &\quad + \left(\lambda_1\kappa_n^{'} + \kappa_n\left(1-\kappa_n\lambda_3\right)\right)\left(1+\lambda_3 H+\lambda_3^2 K\right)\}. \end{aligned} \tag{30}$$

And finally, if we substitute equations 23 and 28 into eq.26, we obtain

$$t_r^f = \frac{1}{A^2C^2}\{ \quad \lambda_2\left(\kappa_2+\lambda_3 K\right)\left(1-\kappa_n\lambda_3\right)\left(\kappa_n\lambda_1\left(\kappa_1+\lambda_3 K\right) + \lambda_3 H^{'} + \lambda_3^2 K^{'}\right)$$

$$\begin{aligned}
&-\lambda_1^2\left(\kappa_1^{'}+\lambda_3K^{'}\right)\left(\kappa_n\lambda_2\left(\kappa_2+\lambda_3K\right)\right)\\
&-\lambda_2\left(\kappa_2^{'}+\lambda_3K^{'}\right)\left(1+\lambda_3H+\lambda_3^2K\right)\left(1-\kappa_n\lambda_3\right)\\
&+\lambda_1\lambda_2\kappa_n^2\left(\kappa_2+\lambda_3K\right)\left(1+\lambda_3H+\lambda_3^2K\right)\\
&+\kappa_n\lambda_1^2\lambda_2\left(\kappa_1+\lambda_3K\right)\left(\kappa_2^{'}+\lambda_3K^{'}\right)\}.
\end{aligned} \tag{31}$$

Theorem 1. *Let M be a ruled surface in E^3 and M^f be parallel-like surface of the ruled surface M that formed along directions of E_P lying in plane $Sp\{X,Z\}$, i.e. $\lambda_1=0$. Recall that the curve β on the surface M^f is image curve of the curve α lying on M, then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface $\left(\beta,M^f\right)$ are as follows*

$$\kappa_g^f=-\frac{\lambda_2\kappa_n\left(\kappa_2+\lambda_3K\right)}{AC} \tag{32}$$

$$\kappa_n^f=\frac{\kappa_n\left(1+\lambda_3H+\lambda_3^2K\right)}{AC} \tag{33}$$

$$\begin{aligned}
t_r^f \quad = \quad & -\frac{\lambda_2}{A^2C}\{\left(\kappa_2^{'}+\lambda_3K^{'}\right)\left(1+\lambda_3H+\lambda_3^2K\right)\\
&-\left(\kappa_2+\lambda_3K\right)\left(\lambda_3H^{'}+\lambda_3^2K^{'}\right)\}.
\end{aligned} \tag{34}$$

Proof. If we substitute $\lambda_1=0$ in equations 29, 30, 31, we can easily hold equations 32, 33, 34. □

Corollary 2. *Providing $\lambda_1=0$, the curve β which is image curve of the directrix curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Corollary 3. *Providing $\lambda_1=0$, the curve β which is image curve of the directrix curve α is a geodesic curve if and only if α is a asymptotic curve.*

Theorem 2. *Let M be a ruled surface in E^3 and M^f be parallel-like surface of the ruled surface M that formed along directions of E_P lying in plane $Sp\{T,Z\}$, i.e. $\lambda_2=0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface $\left(\beta,M^f\right)$ are as follows*

$$\kappa_g^f=0 \tag{35}$$

$$\begin{aligned}
\kappa_n^f \quad = \quad & \frac{1}{AC^2}\{\lambda_1\left(\kappa_1+\lambda_3K\right)\left(-\lambda_3\kappa_n^{'}-\lambda_1\kappa_n^2\right)\\
&+\left(1+\lambda_3H+\lambda_3^2K\right)\left(\lambda_1\kappa_n^{'}+\kappa_n\left(1-\kappa_n\lambda_3\right)\right)\}
\end{aligned} \tag{36}$$

$$t_r^f=0. \tag{37}$$

Proof. If we substitute $\lambda_2=0$ in equations 29, 30, 31, we can easily hold equations 35, 36, 37. □

Corollary 4. *Providing $\lambda_2 = 0$, the curve β lying on M^f is both a geodesic curve and a principal line.*

Corollary 5. *Providing $\lambda_2 = 0$, the curve β which is image curve of the directrix curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Theorem 3. *Let M be a ruled surface in E^3 and M^f be parallel-like surface of the ruled surface M that formed along directions of E_P lying in plane $Sp\{T, X\}$, i.e. $\lambda_3 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface (β, M^f) are as follows*

$$\kappa_g^f = \frac{1}{AC^3}\{-\lambda_1^2\lambda_2\kappa_2\kappa_n^3 - \lambda_2\kappa_2\left(\lambda_1\kappa_n^{'} + \kappa_n\right)\} \tag{38}$$

$$\kappa_n^f = \frac{1}{AC^2}\{-\lambda_1^2\kappa_1\kappa_n^2 + \lambda_1\kappa_n^{'} + \kappa_n\} \tag{39}$$

$$\begin{aligned} t_r^f \quad = \quad & -\frac{1}{A^3C^2}\{\lambda_1\lambda_2\kappa_n\kappa_2\left(A\lambda_1\kappa_1^{'} - B\lambda_1\kappa_1 - A\kappa_n\right) \\ & + \left(A\kappa_2^{'}\lambda_2 - B\kappa_2\lambda_2\right)\left(1 - \kappa_n\kappa_1\lambda_1^2\right) \\ & -\lambda_2\kappa_2\left(A\kappa_n\kappa_1\lambda_1 - B\right)\}. \end{aligned} \tag{40}$$

Proof. If we substitute $\lambda_3 = 0$ in equations 29, 30, 31, we can easily hold equations 38, 39, 40. □

Corollary 6. *Providing $\lambda_3 = 0$, the curve β which is image curve of the directrix curve α is a geodesic curve if and only if α is an asymptotic curve.*

Corollary 7. *Providing $\lambda_3 = 0$, the curve β which is image curve of the directrix curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Theorem 4. *Let M be a ruled surface in E^3 and M^f be parallel-like surface of the ruled surface M that formed along vector field Z, i.e. $\lambda_1 = \lambda_2 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface (β, M^f) are as follows*

$$\kappa_g^f = 0 \tag{41}$$

$$\kappa_n^f = \frac{\kappa_n}{1 - \kappa_n\lambda_3} \tag{42}$$

$$t_r^f = 0. \tag{43}$$

Proof. If we substitute $\lambda_1 = \lambda_2 = 0$ in equations 29, 30, 31, we can easily hold equations 41, 42, 43. □

Corollary 8. *Providing $\lambda_1 = \lambda_2 = 0$, the curve β lying on M^f is both a geodesic curve and principal line.*

Corollary 9. *Providing $\lambda_1 = \lambda_2 = 0$, the curve β lying on M^f is an asymptotic curve if and only if α lying on M is an asymptotic curve.*

Theorem 5. *Let M be a ruled surface in E^3 and M^f be parallel-like surface of the ruled surface M that formed along vector field X, i.e. $\lambda_1 = \lambda_3 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface $\left(\beta, M^f\right)$ are as follows*

$$\kappa_g^f = \frac{-\lambda_2 \kappa_n \kappa_2}{\sqrt{\lambda_2^2 \kappa_2^2 + 1}} \tag{44}$$

$$\kappa_n^f = \frac{\kappa_n}{\sqrt{\lambda_2^2 \kappa_2^2 + 1}} \tag{45}$$

$$t_r^f = -\frac{\kappa_2^{'} \lambda_2}{\left(\lambda_2^2 \kappa_2^2 + 1\right)}. \tag{46}$$

Proof. If we substitute $\lambda_1 = \lambda_3 = 0$ in equations 29, 30, 31, we can easily hold equations 44, 45, 46. □

Corollary 10. *Providing $\lambda_1 = \lambda_3 = 0$, the curve β which is image curve of the directrix curve α is a geodesic curve if and only if α is an asymptotic curve.*

Corollary 11. *Providing $\lambda_1 = \lambda_3 = 0$, the curve β which is image curve of the directrix curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Theorem 6. *Let M be a ruled surface in E^3 and M^f be parallel-like surface of the ruled surface M that formed along vector field T, i.e. $\lambda_2 = \lambda_3 = 0$. Then curvatures of κ_g^f, κ_n^f, t_r^f for partner of curve-surface $\left(\beta, M^f\right)$ are as follows*

$$\kappa_g^f = 0 \tag{47}$$

$$\kappa_n^f = \frac{1}{AC^2} \left\{ \kappa_n + \lambda_1 \kappa_n^{'} - \lambda_1^2 \kappa_1 \kappa_n^2 \right\} \tag{48}$$

$$t_r^f = 0. \tag{49}$$

Proof. If we substitute $\lambda_2 = \lambda_3 = 0$ in equations 29, 30, 31, we can easily hold equations 47, 48, 49. □

Corollary 12. *Providing $\lambda_2 = \lambda_3 = 0$, the curve β which is image curve of the directrix curve α is both a geodesic curve and a principal line.*

Corollary 13. *Providing $\lambda_2 = \lambda_3 = 0$, the curve β which is image curve of the directrix curve α is an asymptotic curve if and only if α is an asymptotic curve.*

Declaration of Competing Interests The author declares that she has no competing interest.

Acknowledgements The author thanks the referees for their valuable suggestions which made this paper much improved.

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