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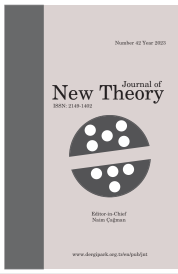

Multiplicity of Scator Roots and the Square Roots in $\mathbb{S}^{1+2}$

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Abstract — This paper presents the roots of elliptic scator numbers in $\mathbb{S}^{1+n}$, which includes both the fundamental 2π symmetry and the π-pair symmetry for $n \geq 2$. Here, the scator set $\mathbb{S}^{1+n}$ is a subset of $\mathbb{R}^{1+n}$ with the scator product and the multiplicative representation. These roots are expressed in terms of both additive (rectangular) and multiplicative (polar) variables. Additionally, the paper provides a comprehensive description of square roots in $\mathbb{S}^{1+2}$, which includes a geometrical representation in three-dimensional space that provides a clear visualization of the concept and makes it easier to understand and interpret. Finally, the paper handles whether the aspects should be further investigated.

Keywords *Roots, non-distributive algebras, hypercomplex numbers*

Mathematics Subject Classification (2020) 30G35, 20M14

1. Introduction

Due to the extra number of dimensions, hypercomplex number systems generally have a larger set of roots than those obtained in the complex plane. However, the number of roots varies considerably depending on the algebraic system [1]. The existence of roots and the obtention of their actual value are two different problems, just as in polynomial real algebra. Formulae to find the roots in diverse systems is subject to active research [2].

Scator algebra is an extension of complex algebra to higher dimensions where the real axis is unique, but there can be an arbitrary number n of hyperimaginary units. In the scator context, the scalar component corresponds to the real part, and each of the n director components corresponds to the imaginary part of a different complex set. In this sense, there are n copies of the complex set embedded in a $1+n$ dimensional scator algebra, just as in Clifford algebras. A geometric representation of scator elements is possible in Argand type diagrams with the appropriate increase of extra imaginary axes. In accordance with Froebenius Theorem and accord with other algebraic systems, not all group properties can be satisfied for scators for $n \geq 2$. Scator algebra is endowed with addition and product operations and a main second-order involution. However, a peculiarity of the scator system is that the product is generally not distributive over addition. The scator product definition gives rise to two branches, elliptic and hyperbolic [3], that are, to some extent, related to Clifford algebras and higher dimensional versions of complex and perplex algebras [4]. This communication is devoted to the description of roots of elliptic scators, also referred to as imaginary or cuspheric scators.

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A general description of the roots of elliptic scators relies on two main theorems that give rise to the Victoria equations in the multiplicative and additive representations. The former establishes that an exponent $\frac{1}{q}$ distributes over the scator component factors. The latter translates this result to the additive representation retaining the multiplicative (angle) variables. The Victoria equation in the additive representation may be viewed as a higher dimensional version of the de Moivre theorem. In [5], these theorems were presented, and several cases were expounded with particular emphasis in the roots of unity. An asset of scator roots is that their number is always finite, contrasting with some infinite solutions obtained in Clifford algebras [6].

In the present communication, the multiplicity of roots is treated in general in the multiplicative (polar) and additive (rectangular) representations in Sections 2 and 3. Particular attention is given to the π-pair symmetry overseen in the seminal publication [5]. The reader may choose to skip the two initial sections in the first approach, where arbitrary $\mathbb{S}^{1+n}$ dimensions and q roots are undertaken. In the remaining manuscript, square roots in $\mathbb{S}^{1+2}$ are treated in detail. In Section 4, square roots in the additive representation are expounded using multiplicative angle variables and additive rectangular variables. In Section 5, the geometric representation of scators and their construction via the addition or product of their components is described. In Subsection 5.1, the geometric visualization of the square root in $\mathbb{S}^{1+2}$ is presented. Conclusions are drawn in the last section.

2. Scators Roots

Scator elements in the multiplicative representation are written as a product of exponentials

$$\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \in \mathbb{S}^{1+n}$$

where the multiplicative scalar φ_0 and the multiplicative director coefficients φ_j, for j from 1 to $n \in \mathbb{N}$, are real quantities and $\check{\mathbf{e}}_j \notin \mathbb{R}$. The scator set $\mathbb{S}^{1+n}$ is a subset of $\mathbb{R}^{1+n}$ where the scator product and the multiplicative representation exist. The product of two scators is evaluated by performing the multiplicative scalars product and the addition of the multiplicative director coefficients with the same director unit,

$$\overset{o}{\alpha}\overset{o}{\beta} = \left(\alpha_0 \prod_{j=1}^{n} \exp\left(\alpha_j \check{\mathbf{e}}_j\right)\right)\left(\beta_0 \prod_{j=1}^{n} \exp\left(\beta_j \check{\mathbf{e}}_j\right)\right) = \alpha_0\beta_0 \prod_{j=1}^{n} \exp\left[\left(\alpha_j + \beta_j\right)\check{\mathbf{e}}_j\right]$$

The components having the same director $\check{\mathbf{e}}_j$ satisfy the addition theorem for exponents. In contrast, components with different director units $\check{\mathbf{e}}_l$ and $\check{\mathbf{e}}_m$ $(l \neq m)$ do not, i.e., $\exp\left(\alpha_l \check{\mathbf{e}}_l\right)\exp\left(\beta_m \check{\mathbf{e}}_m\right) \neq \exp\left(\alpha_l \check{\mathbf{e}}_l + \beta_m \check{\mathbf{e}}_m\right)$. An expression for the exponential of a scator with 1+2 components has been derived in [7]. The conjugate of the scator $\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j}$ is obtained by taking the negative of the director components $\overset{o}{\varphi}^* \equiv \varphi_0 \prod_{j=1}^{n} e^{-\varphi_j \check{\mathbf{e}}_j}$. The magnitude of a scator $\overset{o}{\varphi}$ is $\|\overset{o}{\varphi}\| = \sqrt{\overset{o}{\varphi}\overset{o}{\varphi}^*} = \varphi_0$, the multiplicative scalar thus represents the scator magnitude. The multiplicative inverse $\overset{o}{\varphi}^{-1} = \overset{o}{\varphi}^* \|\overset{o}{\varphi}\|^{-2}$ exists, if the scator magnitude is not zero. The additive representation of scator elements is

$$\overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j$$

where the additive scalar component f_0 and the additive director components f_j, for j from 1 to $n \in \mathbb{N}$, are real quantities and $\check{\mathbf{e}}_j \notin \mathbb{R}$. The scator set $\mathbb{S}^{1+n}$ requires that the additive scalar component must be different from zero, if two or more additive director components are not zero,

$$\mathbb{S}^{1+n} = \left\{\overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j : f_0 \neq 0 \text{ if } \exists\ f_j f_l \neq 0, \text{ for any } j \neq l\right\}$$

The multiplicative and additive representations are related by

$$\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} = \varphi_0 \prod_{k=1}^{n} \cos(\varphi_k) + \varphi_0 \sum_{j=1}^{n} \prod_{k \neq j}^{n} \cos(\varphi_k) \sin(\varphi_j)\, \check{\mathbf{e}}_j = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j$$

If $f_0 \neq 0$, then the magnitude in terms of additive variables is given by

$$\|\overset{o}{\varphi}\| = |f_0| \prod_{j=1}^{n} \sqrt{1 + \frac{f_j^2}{f_0^2}} \tag{1}$$

and if $f_0 = 0$, then $\|\overset{o}{\varphi}\| = |f_j|$. A constant magnitude generates the cusphere isometric surface. Other relevant properties of elliptic scator algebra are summarized in [7].

In $\mathbb{S}^{1+1}$, the multiplicity of roots is due to the trigonometric functions 2π periodicity. Scators with a single director component are isomorphic to the set of complex numbers, i.e., $\mathbb{S}^{1+1} \cong \mathbb{C}$. Thus, the q roots familiar from complex algebra are reproduced, for each $\check{\mathbf{e}}_j$, if all the other director components vanish. In $\mathbb{S}^{1+2}$ or higher dimensions ($\mathbb{S}^{1+n}, n \geq 2$), the 2π trigonometric functions periodicity can be applied to each of the n φ_j's. Then, there are q roots per each of the n hypercomplex director directions. According to this reasoning, Corollary 1 in [5] stated incorrectly: "There are at most q^n different roots for a scator $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$ raised to the power $\frac{1}{q}$". In scator algebra, when two or more hyper-imaginary units are present, the arguments of two multiplicative components can be simultaneously modified by π. Their product leaving the element invariant. This symmetry increases the multiplicity of the roots. These assertions are formulated in the following propositions.

Definition 2.1. The π-pair transformation symmetry requires the simultaneous displacement by π of the argument of two multiplicative director components of a scator element. Given $\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \in \mathbb{S}^{1+n}$, a π-pair transformation $\overset{o}{\varphi} \to \overset{o}{\varphi}{}'$ is

$$\overset{o}{\varphi}{}' = \varphi_0 \prod_{j \neq l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} e^{(\varphi_l \pm \pi)\check{\mathbf{e}}_l} e^{(\varphi_m \pm \pi)\check{\mathbf{e}}_m}$$

for any l, m pair from 1 to n.

Proposition 2.2. Elliptic scators are invariant under π-pair transformations.

Proof.
For the components $\check{\mathbf{e}}_l$ and $\check{\mathbf{e}}_m$ of a scator $\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j}$,

$$\overset{o}{\varphi} = \varphi_0 \prod_{j \neq l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} e^{\varphi_l \check{\mathbf{e}}_l} e^{\varphi_m \check{\mathbf{e}}_m} \in \mathbb{S}^{1+n}$$

Perform a π-pair displacement of the components $\check{\mathbf{e}}_l$ and $\check{\mathbf{e}}_m$

$$\overset{o}{\varphi}{}' = \varphi_0 \prod_{j \neq l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} e^{(\varphi_l \pm \pi)\check{\mathbf{e}}_l} e^{(\varphi_m \pm \pi)\check{\mathbf{e}}_m} = \varphi_0 \prod_{j \neq l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} e^{\varphi_l \check{\mathbf{e}}_l} e^{\varphi_m \check{\mathbf{e}}_m} e^{\pm\pi \check{\mathbf{e}}_l} e^{\pm\pi \check{\mathbf{e}}_m}$$

then

$$\overset{o}{\varphi}{}' = \overset{o}{\varphi} e^{\pm\pi \check{\mathbf{e}}_l} e^{\pm\pi \check{\mathbf{e}}_m} = \overset{o}{\varphi}(-1)(-1) = \overset{o}{\varphi}$$

This π-pair displacement can be carried over an arbitrary pair of components. Therefore,

$$\overset{o}{\varphi}{}' = \varphi_0 \prod_{j=1}^{n} e^{(\varphi_j + \sigma_j \pi)\check{\mathbf{e}}_j} = \overset{o}{\varphi}$$

for $\sigma_j = 1$ applied in pairs. If $\sigma_j = 0$, then the j^{th} component is unaltered. $\square$

Corollary 2.3. The π-pair displacement of components k and l and subsequent π-pair displacement of k and m is equal to the π-pair displacement of l and m.

PROOF.
The k and l π-pair displacement is

$$\varphi_0 \prod_{j\neq k,l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \exp\left(\left(\varphi_k+\pi\right)\check{\mathbf{e}}_k\right)\exp\left(\left(\varphi_l+\pi\right)\check{\mathbf{e}}_l\right)\exp\left(\varphi_m\check{\mathbf{e}}_m\right)$$

and the subsequent k and m π-pair displacement is

$$\varphi_0 \prod_{j\neq k,l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \exp\left(\left(\varphi_k+\pi+\pi\right)\check{\mathbf{e}}_k\right)\exp\left(\left(\varphi_l+\pi\right)\check{\mathbf{e}}_l\right)\exp\left(\left(\varphi_m+\pi\right)\check{\mathbf{e}}_m\right)$$

Due to the 2π symmetry this scator is equal to

$$\varphi_0 \prod_{j\neq k,l,m}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \exp\left(\varphi_k\check{\mathbf{e}}_k\right)\exp\left(\left(\varphi_l+\pi\right)\check{\mathbf{e}}_l\right)\exp\left(\left(\varphi_m+\pi\right)\check{\mathbf{e}}_m\right)$$

that is the l and m π-pair displacement. □

In the multiplicative representation, Theorem 1 in [5] established the roots of a scator due to the 2π trigonometric periodicity. This theorem can now be extended to include the roots arising from the π-pair symmetry.

Theorem 2.4. In the multiplicative representation, for a scator $\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \in \mathbb{S}^{1+n}$ raised to the power $\frac{1}{q}$ such that $q \in \mathbb{Z}$, the exponent $\frac{1}{q}$ distributes over the scator component factors

$$\overset{o}{\varphi}^{\frac{1}{q}} = \left(\varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j}\right)^{\frac{1}{q}} = \varphi_0^{\frac{1}{q}} \prod_{j=1}^{n} e^{\frac{1}{q}\left(\varphi_j+2\pi r_j+\sigma_j\pi\right)\check{\mathbf{e}}_j} \tag{2}$$

where $r_j \in \mathbb{Z}$, from 0 to $q-1$ and σ_j is 0 or 1, the sum of all σ_j is even, for j from 1 to n.

PROOF.
Let $\overset{o}{\varphi} = \overset{o}{\zeta}^{q}$. From the distributivity of an integer exponent over the scator factors, stated in Theorem 3 [7],

$$\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} \exp\left(\left(\varphi_j+2\pi r_j+\sigma_j\pi\right)\check{\mathbf{e}}_j\right) = \left(\zeta_0 \prod_{j=1}^{n} \exp\left(\zeta_j\check{\mathbf{e}}_j\right)\right)^{q} = \zeta_0^{q} \prod_{j=1}^{n} \exp\left(q\zeta_j\check{\mathbf{e}}_j\right)$$

equating components, $\varphi_0 = \zeta_0^q$ and $\varphi_j + 2\pi r_j + \sigma_j\pi = q\zeta_j$, where $r_j \in \mathbb{Z}$ takes values from 0 to $q-1$ and $\sigma_j = 0$ or 1 in pairs, for each subindex j. The $2\pi r_j$ addend in the argument makes explicit the fundamental symmetry of the exponential function with unit directors that satisfy $\check{\mathbf{e}}_j\check{\mathbf{e}}_j = -1$. Whereas the even sum of σ_j exhibits the π-pair symmetry of components couples. Evaluate the above equation to the power $\frac{1}{q}$,

$$\overset{o}{\varphi}^{\frac{1}{q}} = \left(\varphi_0 \prod_{j=1}^{n} \exp\left(\varphi_j\check{\mathbf{e}}_j\right)\right)^{\frac{1}{q}} = \zeta_0 \prod_{j=1}^{n} \exp\left(\zeta_j\check{\mathbf{e}}_j\right)$$

Substitute $\zeta_j = \frac{\varphi_j+2\pi r_j+\sigma_j\pi}{q}$ and $\zeta_0 = \varphi_0^{\frac{1}{q}}$, Equation 2,

$$\overset{o}{\varphi}^{\frac{1}{q}} = \left(\varphi_0 \prod_{j=1}^{n} \exp\left(\varphi_j\check{\mathbf{e}}_j\right)\right)^{\frac{1}{q}} = \varphi_0^{\frac{1}{q}} \prod_{j=1}^{n} \exp\left(\frac{1}{q}\left(\varphi_j+2\pi r_j+\sigma_j\pi\right)\check{\mathbf{e}}_j\right)$$

is obtained. □

Corollary 2.5. A scator $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$ to the power $\frac{1}{q}$ has at most $2pq^n$ different roots, where p is the number of different π-pair possibilities.

Proof.
For $r_j \in \mathbb{Z}$, from 0 to $q-1$, there are q possible arguments for each of the n director components. Therefore, there are q^n possible permutations. For each of them, there is a number p of π-pair possibilities, where every π-pair has two possible values. Thus, there are $2p\,q^n$ possible configurations. □

In many cases, the number of different roots is less than $2pq^n$, either because some root values are repeated or involve only a single director component. Restricted to $\mathbb{S}^{1+2}$, $\overset{o}{\varphi} = \varphi_0 e^{\varphi_x \check{\mathbf{e}}_x} e^{\varphi_y \check{\mathbf{e}}_y}$, where in low dimensions x, y, z lower case roman letters are used instead of numbering the subindices. The multiplicative Victoria Equation 2 is then

$$\overset{o}{\varphi}^{\frac{1}{q}} = \left(\varphi_0 e^{\varphi_x \check{\mathbf{e}}_x} e^{\varphi_y \check{\mathbf{e}}_y}\right)^{\frac{1}{q}} = \varphi_0^{\frac{1}{q}} e^{\frac{1}{q}(\varphi_x + 2\pi r_x + \sigma\pi)\check{\mathbf{e}}_x} e^{\frac{1}{q}(\varphi_y + 2\pi r_y + \sigma\pi)\check{\mathbf{e}}_y} \tag{3}$$

where $r_x, r_y \in \mathbb{Z}$, from 0 to $q-1$ and $\sigma = 0, 1$. For $\mathbb{S}^{1+2}$, there is only one π-pair possibility, both components with either 0 or π phase shift. Thus, there are at most $2q^2$ roots in $\mathbb{S}^{1+2}$.

3. Roots in the Additive Representation

Theorem 2 in [5] establishes the equation for the roots of scator numbers with the multiplicity due to the fundamental 2π symmetry of the trigonometric functions. This theorem is extended here in order to encompass the roots arising from the π-pair symmetry.

Theorem 3.1. A scator $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$ raised to the power $\frac{1}{q}$, $q \in \mathbb{Z}$, in the additive representation satisfies the Victoria equation

$$\begin{aligned} \overset{o}{\varphi}^{\frac{1}{q}} &= \left(\varphi_0 \textstyle\prod_{k=1}^n \cos\varphi_k + \sum_{j=1}^n \varphi_0 \prod_{k\neq j}^n \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j\right)^{\frac{1}{q}} \\ &= \overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n} \\ &= \varphi_0^{\frac{1}{q}} \textstyle\prod_{k=1}^n \cos\left(\frac{1}{q}\left(\varphi_k + 2\pi r_k + \sigma_k \pi\right)\right) \\ &\quad + \textstyle\sum_{j=1}^n \varphi_0^{\frac{1}{q}} \prod_{k\neq j}^n \cos\left(\frac{1}{q}\left(\varphi_k + 2\pi r_k + \sigma_k\pi\right)\right) \sin\left(\frac{1}{q}\left(\varphi_j + 2\pi r_j + \sigma_j \pi\right)\right) \check{\mathbf{e}}_j \end{aligned} \tag{4}$$

for $r_j \in \mathbb{Z}$, from 0 to $q-1$, and $\sigma_k, \sigma_j = 0$ or 1, where $\sum_{k=1}^n \sigma_k$ is even for j, k from 1 to n. Provided that the q products of $\overset{o}{\zeta}{}^{q}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$ and its n components are associative for a given set of r_j's and σ-pairs , $\overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$ is a root of $\overset{o}{\varphi} = \overset{o}{\zeta}{}^{q}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$.

Proof.
The sum of σ_k is even if the value of 1 is always assigned in pairs. For each pair $\sigma_l = \sigma_m = 1$, $l \neq m$ from 1 to n, the φ_l and φ_m arguments of the trigonometric functions are displaced by π. The scator $\overset{o}{\varphi} = \varphi_0 \prod_{k=1}^n \cos\varphi_k + \sum_{j=1}^n \varphi_0 \prod_{k\neq j}^n \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j$ is left unchanged by this transformation since $\cos(\varphi_l + \pi)\cos(\varphi_m + \pi) = \cos\varphi_l \cos\varphi_m$, $\cos(\varphi_m + \pi)\sin(\varphi_l + \pi) = \cos\varphi_m \sin\varphi_l$ and $\cos(\varphi_l + \pi)\sin(\varphi_m + \pi) = \cos\varphi_l \sin\varphi_m$. For odd n director dimension, the remaining unpaired φ_j should not be displaced to leave $\overset{o}{\varphi}$ invariant. This π-pair symmetry is carried through to the RHS of Equation 4. Write the scator $\overset{o}{\varphi} = \overset{o}{\zeta}{}^{q}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$ in the additive representation with multiplicative variables

$$\varphi_0 \prod_{k=1}^n \cos\varphi_k + \sum_{j=1}^n \varphi_0 \prod_{k\neq j}^n \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j = \left(\zeta_0 \prod_{k=1}^n \cos\zeta_k + \sum_{j=1}^n \zeta_0 \prod_{k\neq j}^n \cos\zeta_k \sin\zeta_j \check{\mathbf{e}}_j\right)^q \tag{5}$$

From Theorem 4 in [7], that generalizes De Moivre formula to $\mathbb{S}^{1+n}$ scator space, provided that the product of the factors and its components are associative,

$$\varphi_0 \prod_{k=1}^{n} \cos\varphi_k + \sum_{j=1}^{n} \varphi_0 \prod_{k\neq j}^{n} \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j = \zeta_0^q \prod_{k=1}^{n} \cos(q\zeta_k) + \sum_{j=1}^{n} \zeta_0^q \prod_{k\neq j}^{n} \cos(q\zeta_k)\sin(q\zeta_j)\,\check{\mathbf{e}}_j \tag{6}$$

Equating the additive scalar components

$$\varphi_0 \prod_{k=1}^{n} \cos(\varphi_k + 2\pi r_k + \sigma_k\pi) = \zeta_0^q \prod_{k=1}^{n} \cos(q\zeta_k) \tag{7}$$

whereas for each j director component

$$\varphi_0 \prod_{k\neq j}^{n} \cos(\varphi_k + 2\pi r_k + \sigma_k\pi)\sin(\varphi_j + 2\pi r_j + \sigma_j\pi)\,\check{\mathbf{e}}_j = \zeta_0^q \prod_{k\neq j}^{n} \cos(q\zeta_k)\sin(q\zeta_j)\,\check{\mathbf{e}}_j$$

where the fundamental 2π symmetry of the trigonometric functions as well as the π-pair symmetry are written explicitly, each $r_j \in \mathbb{Z}$ goes from 0 to $q-1$ and $\sigma_l = \sigma_m = 0, 1$ are set in pairs with equal values, any unpaired σ is set equal to zero. If all $\check{\mathbf{e}}_j$ coefficients are zero except one, say the $\check{\mathbf{e}}_l$ coefficient, $\varphi_0 \sin(\varphi_l + 2\pi r_l + \sigma_l\pi) = \zeta_0^q \sin(q\zeta_l)$ and the relationship between angles is straightforward. In this case, σ_l is unpaired and equal to zero. If two or more $\check{\mathbf{e}}_j$ coefficients are different from zero, $\cos(\varphi_j + 2\pi r_j + \sigma_j\pi) \neq 0$ and $\cos(q\zeta_j) \neq 0$ for all j, since $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$. The products can then be completed for all k and each of the $\check{\mathbf{e}}_j$ equations become

$$\varphi_0 \prod_{k=1}^{n} \cos(\varphi_k + 2\pi r_k + \sigma_k\pi)\tan(\varphi_j + 2\pi r_j + \sigma_j\pi)\,\check{\mathbf{e}}_j = \zeta_0^q \prod_{k=1}^{n} \cos(q\zeta_k)\tan(q\zeta_j)\,\check{\mathbf{e}}_j$$

With the use of Equation 7,

$$\tan(q\zeta_j) = \tan(\varphi_j + 2\pi r_j + \sigma_j\pi) \Rightarrow \zeta_j = \frac{1}{q}(\varphi_j + 2\pi r_j + \sigma_j\pi) \tag{8}$$

for all j from 1 to n. Replace the angles $\zeta_k = \frac{1}{q}(\varphi_k + 2\pi r_k + \sigma_j\pi)$ in Equation 7, to find $\zeta_0 = \varphi_0^{\frac{1}{q}}$. Evaluate Equation 5 to the power $\frac{1}{q}$,

$$\begin{aligned}
\overset{o}{\varphi}^{\frac{1}{q}} &= \left(\varphi_0 \prod_{k=1}^{n} \cos\varphi_k + \sum_{j=1}^{n} \varphi_0 \prod_{k\neq j}^{n} \cos\varphi_k \sin\varphi_j \check{\mathbf{e}}_j\right)^{\frac{1}{q}} \\
&= \overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n} \\
&= \zeta_0 \prod_{k=1}^{n} \cos\zeta_k + \sum_{j=1}^{n} \zeta_0 \prod_{k\neq j}^{n} \cos\zeta_k \sin\zeta_j \check{\mathbf{e}}_j
\end{aligned}$$

Rewrite the ζ_j variables in terms of φ_j from Equation 8 to obtain,

$$\begin{aligned}
\overset{o}{\varphi}^{\frac{1}{q}} = \overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n} = \Bigg(&\varphi_0^{\frac{1}{q}} \prod_{k=1}^{n} \cos\left(\frac{1}{q}(\varphi_k + 2\pi r_k + \sigma_k\pi)\right) \\
&+ \sum_{j=1}^{n} \varphi_0^{\frac{1}{q}} \prod_{k\neq j}^{n} \cos\left(\frac{1}{q}(\varphi_k + 2\pi r_k + \sigma_k\pi)\right)\sin\left(\frac{1}{q}(\varphi_j + 2\pi r_j + \sigma_j\pi)\right)\check{\mathbf{e}}_j\Bigg)
\end{aligned}$$

The derivation of Equation 6 from Equation 5 required associativity of the $\overset{o}{\zeta}{}^q_{r_1 r_2 \cdots r_n}$ products. However, due to the multi-valued inversion that followed, it is possible that for certain $\overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n}$ solutions, some of the q products of $\overset{o}{\zeta}{}^q_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n}$ do not satisfy associativity. From [8] Theorem 2.1, associativity is insured if all possible product pairs have a non vanishing additive scalar component. For each $\overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1\sigma_2\cdots\sigma_n}$ to be a root,

$$\begin{aligned}
\overset{o}{\zeta}{}^{q}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n} &= \left[\varphi_0^{\frac{1}{q}} \prod_{k=1}^{n} \cos\left(\tfrac{1}{q}\left(\varphi_k + 2\pi r_k + \sigma_k \pi\right)\right)\right. \\
&\quad \left. + \sum_{j=1}^{n} \varphi_0^{\frac{1}{q}} \prod_{k\neq j}^{n} \cos\left(\tfrac{1}{q}\left(\varphi_k + 2\pi r_k + \sigma_k \pi\right)\right) \sin\left(\tfrac{1}{q}\left(\varphi_j + 2\pi r_j + \sigma_j \pi\right)\right) \check{\mathbf{e}}_j\right]^{q} \\
&= \prod_{j=1}^{n} \left(\cos\left(\tfrac{1}{q}\left(\varphi_k + 2\pi r_k + \sigma_k \pi\right)\right) + \sin\left(\tfrac{1}{q}\left(\varphi_j + 2\pi r_j + \sigma_j \pi\right)\right)\, \check{\mathbf{e}}_j\right)^{q}
\end{aligned} \tag{9}$$

must be associative. Thus, none of the $q \times n$ products should give a scator with zero additive scalar component if two or more director coefficients are different from zero, then $\overset{o}{\zeta}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n}$ satisfies $\overset{o}{\zeta}{}^{q}_{r_1 r_2 \cdots r_n, \sigma_1 \sigma_2 \cdots \sigma_n} = \overset{o}{\varphi}$. □

The scator roots are identical in the multiplicative representation (Theorem 2.4) or the additive representation (Theorem 3.1), unless obstructed by the lack of associativity. Recall that associativity is not an issue in the multiplicative representation. However, in the additive representation, non-associative products can lead to spurious roots. This problem is discussed at length in [5].

4. Square Roots in 1+2 Dimensions

Lemma 4.1. The square roots of $\overset{o}{\varphi} = \varphi_0 \cos\varphi_x \cos\varphi_y + \varphi_0 \cos\varphi_y \sin\varphi_x\, \check{\mathbf{e}}_x + \varphi_0 \cos\varphi_x \sin\varphi_y\, \check{\mathbf{e}}_y$, are

$$\overset{o}{\varphi}{}^{\frac{1}{2}} = \overset{o}{\zeta}_{\pm,0} = \pm\varphi_0^{\frac{1}{2}} \left(\cos\frac{\varphi_x}{2}\cos\frac{\varphi_y}{2} + \cos\frac{\varphi_y}{2}\sin\frac{\varphi_x}{2}\check{\mathbf{e}}_x + \cos\frac{\varphi_x}{2}\sin\frac{\varphi_y}{2}\check{\mathbf{e}}_y\right) \tag{10}$$

and from the π-pair symmetry

$$\overset{o}{\varphi}{}^{\frac{1}{2}} = \overset{o}{\zeta}_{\pm,1} = \pm\varphi_0^{\frac{1}{2}} \left(\sin\frac{\varphi_x}{2}\sin\frac{\varphi_y}{2} - \sin\frac{\varphi_y}{2}\cos\frac{\varphi_x}{2}\check{\mathbf{e}}_x - \sin\frac{\varphi_x}{2}\cos\frac{\varphi_y}{2}\check{\mathbf{e}}_y\right) \tag{11}$$

PROOF.
For $q = 2$ in $\mathbb{S}^{1+2}$, from the Victoria Equation 4 in Theorem 3.1, the roots of $\overset{o}{\varphi}$ are

$$\begin{aligned}
\overset{o}{\varphi}{}^{\frac{1}{2}} &= \varphi_0^{\frac{1}{2}} \cos\left(\tfrac{\varphi_x}{2} + \tfrac{\sigma\pi}{2} + \pi r_x\right)\cos\left(\tfrac{\varphi_y}{2} + \tfrac{\sigma\pi}{2} + \pi r_y\right) + \varphi_0^{\frac{1}{2}} \cos\left(\tfrac{\varphi_y}{2} + \tfrac{\sigma\pi}{2} + \pi r_y\right)\sin\left(\tfrac{\varphi_x}{2} + \tfrac{\sigma\pi}{2} + \pi r_x\right)\check{\mathbf{e}}_x \\
&\quad + \varphi_0^{\frac{1}{2}} \cos\left(\tfrac{\varphi_x}{2} + \tfrac{\sigma\pi}{2} + \pi r_x\right)\sin\left(\tfrac{\varphi_y}{2} + \tfrac{\sigma\pi}{2} + \pi r_y\right)\check{\mathbf{e}}_y \\
&= \overset{o}{\zeta}_{r_x r_y, \sigma}
\end{aligned}$$

for $r_x, r_y, \sigma = 0, 1$. In this particular case, the r_x, r_y different values change the sign of all components, $-$ for $r_x \neq r_y$ and $+$ for $r_x = r_y$. This degeneracy halves the number of roots arising from the 2π symmetry from $q^n = 4$ to 2,

$$\begin{aligned}
\overset{o}{\varphi}{}^{\frac{1}{2}} &= \pm\varphi_0^{\frac{1}{2}} \left[\cos\left(\tfrac{\varphi_x}{2} + \tfrac{\sigma\pi}{2}\right)\cos\left(\tfrac{\varphi_y}{2} + \tfrac{\sigma\pi}{2}\right) + \cos\left(\tfrac{\varphi_y}{2} + \tfrac{\sigma\pi}{2}\right)\sin\left(\tfrac{\varphi_x}{2} + \tfrac{\sigma\pi}{2}\right)\check{\mathbf{e}}_x \right. \\
&\quad \left. + \cos\left(\tfrac{\varphi_x}{2} + \tfrac{\sigma\pi}{2}\right)\sin\left(\tfrac{\varphi_y}{2} + \tfrac{\sigma\pi}{2}\right)\check{\mathbf{e}}_y\right] \\
&= \overset{o}{\zeta}_{\pm,\sigma}
\end{aligned} \tag{12}$$

The multiplicity coming from the π-pair symmetry is $2p = 2$, since there is only one possible pairing. Hence, the total number of possibly different square roots is 4. If $\sigma = 0$ in Equation 12, two of the square roots are given by Equation 10. If $\sigma = 1$, the other two square roots, since $\cos\left(\frac{\varphi_j}{2} + \frac{\pi}{2}\right) = -\sin\frac{\varphi_j}{2}$ and $\sin\left(\frac{\varphi_j}{2} + \frac{\pi}{2}\right) = \cos\frac{\varphi_j}{2}$, are given by Equation 11. □

In the subsets $\mathbb{S}^{1+1}$, where either the $\check{\mathbf{e}}_x$ or $\check{\mathbf{e}}_y$ coefficient is different from zero, there are two roots since these subspaces are isomorphic to the complex plane. In [5], it was wrongly stated that "For elements $\overset{o}{\varphi} \in \mathbb{S}^{1+n} \setminus \mathbb{S}^{1+0}$, there are only two different square roots in the additive representation".

The π-pair symmetry now included, has corrected this mistake. There can be, as shown above, up to four different square roots in $\mathbb{S}^{1+2} \setminus \mathbb{S}^{1+1}$. The number of $2pq^n$ roots in $\mathbb{S}^{1+n}$ for $n > 2$ increase considerably due to the π-pair p different pairing possibilities as well as the n dimensions.

4.1. Square roots with additive variables

Lemma 4.2. The square roots of $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$ are

$$\begin{aligned} \sqrt{\overset{o}{\varphi}} &= \overset{o}{\zeta}_{\pm,0} \\ &= \pm\tfrac{1}{2}\sqrt{\tfrac{1}{|s|}}\left[\sqrt{\left(\sqrt{s^2+x^2}+s\right)\left(\sqrt{s^2+y^2}+s\right)}\right. \\ &\quad +\mathrm{sgn}x\sqrt{\left(\sqrt{s^2+x^2}-s\right)\left(\sqrt{s^2+y^2}+s\right)}\,\check{\mathbf{e}}_x \\ &\quad \left.+\mathrm{sgn}y\sqrt{\left(\sqrt{s^2+x^2}+s\right)\left(\sqrt{s^2+y^2}-s\right)}\,\check{\mathbf{e}}_y\right] \end{aligned} \tag{13}$$

and for the π-pair symmetry

$$\begin{aligned} \sqrt{\overset{o}{\varphi}} &= \overset{o}{\zeta}_{\pm,1} \\ &= \pm\tfrac{1}{2}\sqrt{\tfrac{1}{|s|}}\left[\mathrm{sgn}x\,\mathrm{sgn}y\sqrt{\left(\sqrt{s^2+x^2}-s\right)\left(\sqrt{s^2+y^2}-s\right)}\right. \\ &\quad -\mathrm{sgn}y\sqrt{\left(\sqrt{s^2+x^2}+s\right)\left(\sqrt{s^2+y^2}-s\right)}\check{\mathbf{e}}_x \\ &\quad \left.-\mathrm{sgn}x\sqrt{\left(\sqrt{s^2+x^2}-s\right)\left(\sqrt{s^2+y^2}+s\right)}\check{\mathbf{e}}_y\right] \end{aligned} \tag{14}$$

Proof.
The scator $\overset{o}{\varphi} = \varphi_0 \cos\varphi_x \cos\varphi_y + \varphi_0 \cos\varphi_y \sin\varphi_x\,\check{\mathbf{e}}_x + \varphi_0 \cos\varphi_x \sin\varphi_y\,\check{\mathbf{e}}_y$ in terms of additive variables is $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$. The relationship between multiplicative and additive variables is

$$s = \varphi_0 \cos\varphi_x \cos\varphi_y, \quad x = \varphi_0 \cos\varphi_y \sin\varphi_x, \quad y = \varphi_0 \cos\varphi_x \sin\varphi_y.$$

From the quotient of the directors over the scalar coefficient

$$\frac{x}{s} = \tan\varphi_x, \Rightarrow \cos\varphi_x = \frac{s}{\sqrt{s^2+x^2}} \text{ and } \frac{y}{s} = \tan\varphi_y, \Rightarrow \cos\varphi_y = \frac{s}{\sqrt{s^2+y^2}} \tag{15}$$

In order to write the square roots in terms of the additive variables, rewrite the half angles in terms of angles $\cos\frac{\varphi}{2} = \frac{1}{\sqrt{2}}\sqrt{1+\cos\varphi}$ and $\sin\frac{\varphi}{2} = \frac{1}{\sqrt{2}}\sqrt{1-\cos\varphi}$. These substitutions put together give

$$\cos\left(\frac{\varphi_x}{2}\right)\cos\left(\frac{\varphi_y}{2}\right) = \frac{1}{2}\sqrt{\left(1+\frac{s}{\sqrt{s^2+x^2}}\right)\left(1+\frac{s}{\sqrt{s^2+y^2}}\right)} \tag{16}$$

$$\sin\left(\frac{\varphi_x}{2}\right)\cos\left(\frac{\varphi_y}{2}\right) = \frac{\mathrm{sgn}x}{2}\sqrt{\left(1-\frac{s}{\sqrt{s^2+x^2}}\right)\left(1+\frac{s}{\sqrt{s^2+y^2}}\right)} \tag{17}$$

$$\sin\left(\frac{\varphi_y}{2}\right)\cos\left(\frac{\varphi_x}{2}\right) = \frac{\mathrm{sgn}y}{2}\sqrt{\left(1+\frac{s}{\sqrt{s^2+x^2}}\right)\left(1-\frac{s}{\sqrt{s^2+y^2}}\right)} \tag{18}$$

$$\sin\left(\frac{\varphi_x}{2}\right)\sin\left(\frac{\varphi_y}{2}\right) = \frac{\mathrm{sgn}x\,\mathrm{sgn}y}{2}\sqrt{\left(1-\frac{s}{\sqrt{s^2+x^2}}\right)\left(1-\frac{s}{\sqrt{s^2+y^2}}\right)} \tag{19}$$

where sgn is the sign function. The scator magnitude from Equation 1 is

$$\|\overset{o}{\varphi}\| = \varphi_0 = |s|\sqrt{1+\frac{x^2}{s^2}}\sqrt{1+\frac{y^2}{s^2}} = \frac{1}{|s|}\sqrt{s^2+x^2}\sqrt{s^2+y^2}$$

Evaluate the product of the magnitude's square root

$$\sqrt{\varphi_0} = \sqrt{\frac{1}{|s|}\sqrt{s^2+x^2}\sqrt{s^2+y^2}}$$

times Equations 16-19. Substitution in Equation 10, Lemma 4.1, gives Equation 13. The π-pair multiplicity is obtained from substitution in Equation 11.

If x or y are zero, the usual square root of a complex number is recovered. For example, from Equation 13, if $y=0$, $\sqrt{\overset{o}{\varphi}} = \pm\left(\frac{1}{\sqrt{2}}\sqrt{\sqrt{s^2+x^2}+s}+\frac{\text{sgn}x}{\sqrt{2}}\sqrt{\sqrt{s^2+x^2}-s}\,\check{\mathbf{e}}_x\right)$. The x or y zero limit does not make sense for $\overset{o}{\zeta}_{\pm,1}$, because these novel roots arise from the π-pair symmetry that requires at least two nonvanishing director components. □

Corollary 4.3. The square roots of $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$ lie on the plane

$$(\cos\varphi_x - \cos\varphi_y)\,S + \sin\varphi_x\,X - \sin\varphi_y\,Y = 0 \tag{20}$$

that in additive variables is

$$s\left(\sqrt{s^2+x^2}-\sqrt{s^2+y^2}\right)S - x\sqrt{s^2+y^2}\,X + y\sqrt{s^2+x^2}\,Y = 0 \tag{21}$$

Proof.

Since the roots come in $\pm$ pairs, zero must be on the plane where the roots lie. Let this plane be $a_0S + a_xX + a_yY = 0$. Substitute the positive value of the roots Equations 10 and 11, upon division by $\cos\frac{\varphi_x}{2}\cos\frac{\varphi_y}{2}$, the equations are

$$a_0 + \tan\frac{\varphi_x}{2}a_x + \tan\frac{\varphi_y}{2}a_y = 0 \text{ and } a_0 - \cot\frac{\varphi_x}{2}a_x - \cot\frac{\varphi_y}{2}a_y = 0$$

Write the semiangles in terms of angles $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$ and $\cot\frac{\theta}{2} = \frac{\sin\theta}{1-\cos\theta}$. Isolate $\sin\varphi_y\,a_y$ and add the two equations

$$2a_0 + \left[\frac{(1+\cos\varphi_y)}{1+\cos\varphi_x} - \frac{(1-\cos\varphi_y)}{1-\cos\varphi_x}\right]\sin\varphi_x a_x = 0$$

Upon rearrangement

$$a_x = \frac{-\sin\varphi_x}{(\cos\varphi_y-\cos\varphi_x)}a_0, \quad a_y = \frac{-\sin\varphi_y}{(\cos\varphi_x-\cos\varphi_y)}a_0$$

where a_y follows an analogous procedure. From

$$a_0S - \frac{\sin\varphi_x}{(\cos\varphi_y-\cos\varphi_x)}X - \frac{\sin\varphi_y}{(\cos\varphi_x-\cos\varphi_y)}a_0Y = 0$$

Equation 20 is obtained. A similar procedure starting with Equations 13 and 14 gives Equation 21. □

This result is particular to square roots. Higher order roots no longer lie on a plane as evinced by cube and higher order roots in [5].

In the multiplicative representation, the square roots in $\mathbb{S}^{1+2}$ from Equation 3 with $q=2$, are

$$\overset{o}{\varphi}^{\frac{1}{2}} = \left(\varphi_0 e^{\varphi_x\check{\mathbf{e}}_x}e^{\varphi_y\check{\mathbf{e}}_y}\right)^{\frac{1}{2}} = \varphi_0^{\frac{1}{2}}e^{\left(\frac{\varphi_x}{2}+\sigma\frac{\pi}{2}+\pi r_x\right)\check{\mathbf{e}}_x}e^{\left(\frac{\varphi_y}{2}+\sigma\frac{\pi}{2}+\pi r_y\right)\check{\mathbf{e}}_y} \tag{22}$$

for $\sigma = 0,1$ and $r_x = 0,1$ and $r_y = 0,1$. The $e^{\pi r_x\check{\mathbf{e}}_x}$ and $e^{\pi r_y\check{\mathbf{e}}_y}$ factors introduce a minus sign if

$r_x \neq r_y$. The four possibly distinct square roots of a scator $\overset{o}{\varphi} = \varphi_0 e^{\varphi_x \check{\mathbf{e}}_x} e^{\varphi_y \check{\mathbf{e}}_y}$ are

$$\overset{o}{\varphi}^{\frac{1}{2}} = \pm\varphi_0^{\frac{1}{2}} e^{\frac{\varphi_x}{2}\check{\mathbf{e}}_x} e^{\frac{\varphi_y}{2}\check{\mathbf{e}}_y}, \qquad \pm\varphi_0^{\frac{1}{2}} e^{\left(\frac{\varphi_x}{2}+\frac{\pi}{2}\right)\check{\mathbf{e}}_x} e^{\left(\frac{\varphi_y}{2}+\frac{\pi}{2}\right)\check{\mathbf{e}}_y} \tag{23}$$

5. Geometric visualization

The scalar and the two director components of a scator $\overset{o}{\varphi} \in \mathbb{S}^{1+2}$ can be depicted in orthogonal directions in a three dimensional space as shown in Figure 1.

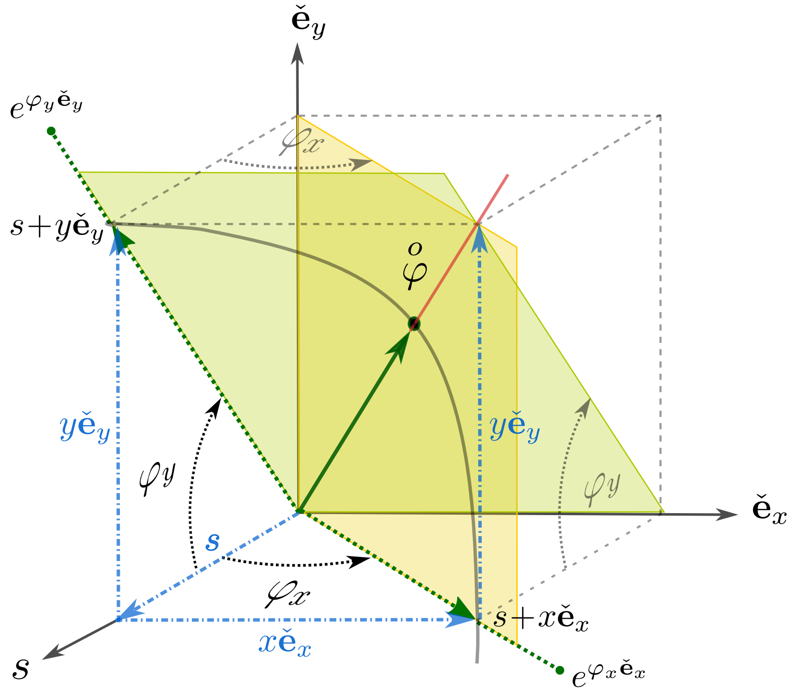

Figure 1. Geometrical representation of the unit magnitude ($\varphi_0 = 1$) scator $\overset{o}{\varphi} = \cos\varphi_x \cos\varphi_y + \cos\varphi_y \sin\varphi_x\, \check{\mathbf{e}}_x + \cos\varphi_x \sin\varphi_y\, \check{\mathbf{e}}_y$. In additive variables $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$.

1. The additive components of a scator $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$ can be represented as directed line segments in the s, $\check{\mathbf{e}}_x$, $\check{\mathbf{e}}_y$ axes respectively.

2. In terms of the multiplicative variables:

(a) φ_0 is the distance given by the scator magnitude of the point $\overset{o}{\varphi}$ to the origin,

(b) φ_x is the angle that the projection of the point $\overset{o}{\varphi}$ onto the s, x plane makes with the positive scalar axis and,

(c) φ_y is the angle of the projection onto the s, y plane with the positive scalar axis.

3. A scator can be constructed from the sum of its components, $\overset{o}{\varphi} = (s) + (x\,\check{\mathbf{e}}_x) + (y\,\check{\mathbf{e}}_y)$. This procedure is visualized with dash-dot blue lines in Figure 1. The tip of the $\overset{o}{\varphi}$ scator does not match the sum of the three components because the scator magnitude is not an Euclidean magnitude.

4. A scator can also be constructed by the sum of two scators with a scalar component, $\overset{o}{\varphi}_{sx} = s + x\,\check{\mathbf{e}}_x = \cos\varphi_y e^{\varphi_x \check{\mathbf{e}}_x}$ and $\overset{o}{\varphi}_{sy} = s + y\,\check{\mathbf{e}}_y = \cos\varphi_x e^{\varphi_y \check{\mathbf{e}}_y}$. $\overset{o}{\varphi}_{sx}$ and $\overset{o}{\varphi}_{sy}$ are depicted by green dashed lines with arrows in Figure 1. However, the additive inverse of the scalar component has to be added to achieve the appropriate result $\overset{o}{\varphi}_{sx} + \overset{o}{\varphi}_{sy} - s = 2s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y - s = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$. Having a scalar and a director component, permits the representation in polar coordinates. Notice that the $\overset{o}{\varphi}_{sx}$ and $\overset{o}{\varphi}_{sy}$ scators have a somewhat smaller magnitude than their counterparts in the multiplicative representation. The scators $e^{\varphi_x \check{\mathbf{e}}_x} = \cos\varphi_x + \sin\varphi_x \check{\mathbf{e}}_x$ and $e^{\varphi_y \check{\mathbf{e}}_y} = \cos\varphi_y + \sin\varphi_y \check{\mathbf{e}}_y$ have been depicted in the figure ending with a dot. The scators $\overset{o}{\varphi}_{sx} = s + x\,\check{\mathbf{e}}_x$ and $e^{\varphi_x \check{\mathbf{e}}_x}$ are collinear, thus are $\overset{o}{\varphi}_{sy} = s + y\,\check{\mathbf{e}}_y$ and $e^{\varphi_y \check{\mathbf{e}}_y}$.

5. Scators can also be constructed from the product of their components,

$$\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y = \frac{1}{s}\overset{o}{\varphi}_{sx}\overset{o}{\varphi}_{sy} = \frac{1}{s}\left(s + x\,\check{\mathbf{e}}_x\right)\left(s + y\,\check{\mathbf{e}}_y\right)$$

provided that product with the multiplicative inverse of the additive scalar component is included. In the multiplicative representation, $\overset{o}{\varphi} = \varphi_0 e^{\varphi_x \check{\mathbf{e}}_x} e^{\varphi_y \check{\mathbf{e}}_y}$ is equal to the product of the multiplicative director components $\overset{o}{\varphi} = \overset{o}{\varphi}_x \overset{o}{\varphi}_y = \left(\sqrt{\varphi_0} e^{\varphi_x \check{\mathbf{e}}_x}\right)\left(\sqrt{\varphi_0} e^{\varphi_y \check{\mathbf{e}}_y}\right)$, where the magnitude has been symmetrically split between components. This product can be seen as a rotation of the $\sqrt{\varphi_0} e^{\varphi_x \check{\mathbf{e}}_x}$ scator by φ_y in the $\check{\mathbf{e}}_y$ direction (plane depicted in semitransparent yellow). It can of course be seen the other way around, a rotation in the $\check{\mathbf{e}}_x$ direction (plane depicted in semitransparent green) of the scator $\sqrt{\varphi_0} e^{\varphi_y \check{\mathbf{e}}_y}$.

This last property of scators is in sharp contrast with vector elements, where the product of components cannot be used to construct a several component vector.

Scator products geometrically represent rotations, although the term rotation is a bit of an abuse. The product of an arbitrary scator with a unit scator geometrically represents a rotation that preserves the scator magnitude. However, they are not rotations in the Euclidean sense because the Euclidean metric is not preserved. For this reason, the end point of the scator $\overset{o}{\varphi}$ is not equal to the end point of the three components sum. In contrast, the tip of each of the $\overset{o}{\varphi}_{sx}$ or $\overset{o}{\varphi}_{sy}$ scators coincides with the sum $s + x\check{\mathbf{e}}_x$ or $s + y\check{\mathbf{e}}_x$ respectively, since the scator magnitude in $1 + 1$ dimensions is equal to the Euclidean magnitude.

5.1. Geometric representation of the square root

The scator square roots involve halving the φ_x and φ_y angles and taking the square root of the magnitude φ_0. Some of the square roots also involve adding a π and/or a $\frac{\pi}{2}$ term to the argument. The scator angles and half angles without any other terms are geometrically depicted in Figure 2.

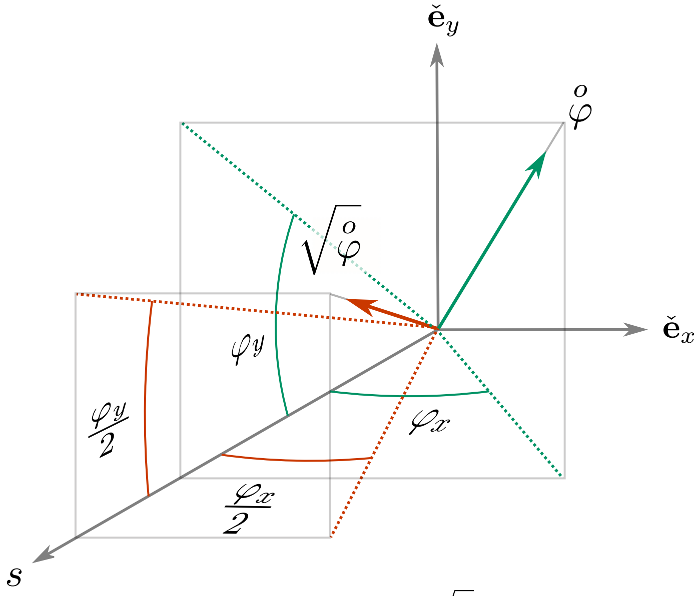

Figure 2. Geometrical representation of a square root $\sqrt{\overset{o}{\varphi}}$ of a scator $\overset{o}{\varphi}$. The φ_x and φ_y angles are halved and the square root of the magnitude is evaluated.

A scator $\overset{o}{\varphi}$ (in green) is projected in the $s, \check{\mathbf{e}}_x$ and $s, \check{\mathbf{e}}_y$ planes (green dotted lines). The angles φ_x and φ_y are the angles that these projections make with respect to the scalar s axis. These angles are halved and represent the projections of the resultant scator $\sqrt{\overset{o}{\varphi}}$ (in red). A unit magnitude scator is assumed, so that the tip of both scators must lie on the unit cusphere surface.

The projection of $\overset{o}{\varphi}$ in the $s, \check{\mathbf{e}}_x$ plane (green dotted line) is the scator $\overset{o}{\varphi}_x = e^{\varphi_x \check{\mathbf{e}}_x}$. It is a unit magnitude hypotenuse with projections $\cos\varphi_x$ and $\sin\varphi_x$ in the s and $\check{\mathbf{e}}_x$ axes, that correspond to the additive representation of this scator $\overset{o}{\varphi}_x = e^{\varphi_x \check{\mathbf{e}}_x} = \cos\varphi_x + \sin\varphi_x\, \check{\mathbf{e}}_x$. The scator magnitude of $\overset{o}{\varphi}_x$, from Equation 1 is $\|\overset{o}{\varphi}\| = |f_0| \sqrt{1 + \frac{f_x^2}{f_0^2}} = \sqrt{f_0^2 + f_x^2}$, equal to the Pythagorean identity. The scator magnitude $\sqrt{s^2 + x^2}$ in $\mathbb{S}^{1+1}$ is identical to the Euclidean magnitude. Thus, a right angle triangle where the tip of the hypotenuse matches the sum of the directed catheti is obtained.

An analogous result is obtained for the projection of $\overset{o}{\varphi}$ in the $s, \check{\mathbf{e}}_y$ plane (green dotted line), $\overset{o}{\varphi}_y = e^{\varphi_y \check{\mathbf{e}}_y} = \cos\varphi_y + \sin\varphi_y\, \check{\mathbf{e}}_y$. Again, a unit magnitude hypotenuse is made up from a right angle triangle, but this time in the $s, \check{\mathbf{e}}_y$ plane.

The product of these two projections $\overset{o}{\varphi} = \overset{o}{\varphi}_x \overset{o}{\varphi}_y = \left(e^{\varphi_x \check{\mathbf{e}}_x}\right)\left(e^{\varphi_y \check{\mathbf{e}}_y}\right)$, construct the $\overset{o}{\varphi}$ scator. Its additive representation is $\overset{o}{\varphi} = \cos\varphi_x \cos\varphi_y + \cos\varphi_y \sin\varphi_x\, \check{\mathbf{e}}_x + \cos\varphi_x \sin\varphi_y\, \check{\mathbf{e}}_y$. Its magnitude, from Equation 1 is

$$\|\overset{o}{\varphi}\| = |s| \sqrt{1 + \frac{x^2}{s^2}} \sqrt{1 + \frac{y^2}{s^2}} = \sqrt{s^2 + x^2 + y^2 + \frac{x^2 y^2}{s^2}}$$

It is no longer the sum of three squares but has the $\frac{x^2 y^2}{s^2}$ term. The magnitude of this scator is one,

$$\|\overset{o}{\varphi}\| = \sqrt{\cos^2\varphi_x \cos^2\varphi_y + \cos^2\varphi_y \sin^2\varphi_x + \cos^2\varphi_x \sin^2\varphi_y + \sin^2\varphi_x \sin^2\varphi_y} = 1$$

where the last term $\frac{x^2 y^2}{s^2} = \frac{\cos^2\varphi_y \sin^2\varphi_x \cos^2\varphi_x \sin^2\varphi_y}{\cos^2\varphi_x \cos^2\varphi_y}$ is crucial to attain this result. The tip of the scator $\overset{o}{\varphi} = s + x\, \check{\mathbf{e}}_x + y\, \check{\mathbf{e}}_y$, cannot match the tip of the directed sum of the three components (That would imply a magnitude $\sqrt{s^2 + x^2 + y^2}$).

The $\overset{o}{\varphi}$ scator root (in red) is now the product of the two projection scators $e^{\frac{\varphi_x}{2} \check{\mathbf{e}}_x}$ and $e^{\frac{\varphi_y}{2} \check{\mathbf{e}}_y}$. It also has unit magnitude and is leaned closer to the s axis in both $\check{\mathbf{e}}_x$ and $\check{\mathbf{e}}_y$ as expected for smaller angles.

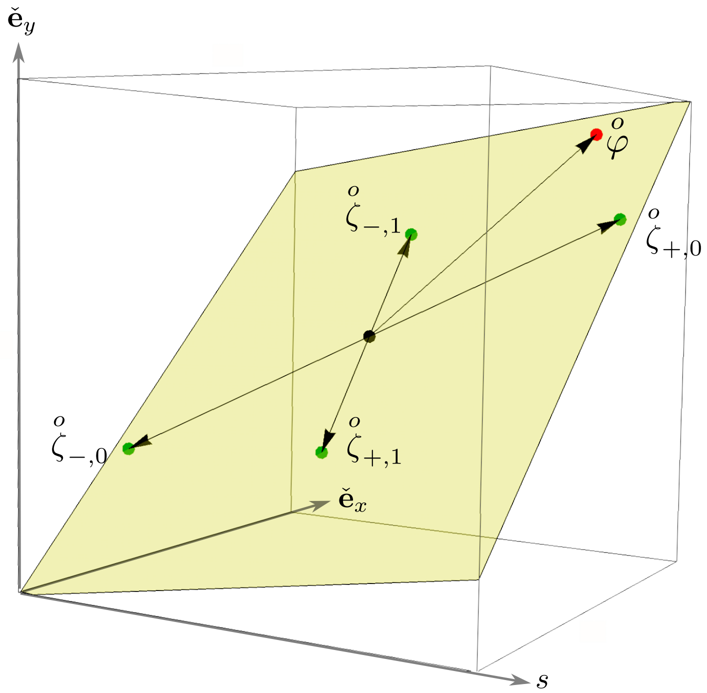

Figure 3. Roots (green points) of $\overset{o}{\varphi} = \cos\frac{\pi}{6}\cos\frac{\pi}{5} + \cos\frac{\pi}{5}\sin\frac{\pi}{6}\, \check{\mathbf{e}}_x + \cos\frac{\pi}{6}\sin\frac{\pi}{5}\, \check{\mathbf{e}}_y$ (red point). The origin is located at the black point. The $\left(2\sqrt{3} - 1 - \sqrt{5}\right) s + 2\,x - 2\sqrt{\frac{5}{2} - \frac{\sqrt{5}}{2}}\, y = 0$ plane is shown in semitransparent yellow. The four roots lie on this plane but not the $\overset{o}{\varphi}$ scator (red).

Consider as a numeric example, the scator

$$\overset{o}{\varphi} = \frac{\sqrt{3}}{8}\left(\sqrt{5}+1\right) + \frac{1}{8}\left(\sqrt{5}+1\right)\breve{\mathbf{e}}_x + \frac{1}{4}\sqrt{\frac{3}{2}\left(5-\sqrt{5}\right)}\,\breve{\mathbf{e}}_y$$

where the values have been chosen so that the two different angles are rational (relative primes) functions of π. From Equation 13, two roots are given by

$$\overset{o}{\zeta}_{\pm,0} = \pm\left(\frac{\left(\sqrt{3}+1\right)\sqrt{5+\sqrt{5}}}{8} + \frac{\left(\sqrt{3}-1\right)\sqrt{5+\sqrt{5}}}{8}\breve{\mathbf{e}}_x + \frac{\left(\sqrt{3}+1\right)\left(\sqrt{5}-1\right)}{8\sqrt{2}}\breve{\mathbf{e}}_y\right)$$

and the other two π-pair symmetry roots from Equation 14 are

$$\overset{o}{\zeta}_{\pm,\pi} = \pm\left(\frac{\left(\sqrt{3}-1\right)\left(\sqrt{5}-1\right)}{8\sqrt{2}} - \frac{\left(\sqrt{3}+1\right)\left(\sqrt{5}-1\right)}{8\sqrt{2}}\breve{\mathbf{e}}_x - \frac{\left(\sqrt{3}-1\right)}{8}\sqrt{5+\sqrt{5}}\,\breve{\mathbf{e}}_y\right)$$

This scator in multiplicative variables is

$$\overset{o}{\varphi} = \cos\frac{\pi}{6}\cos\frac{\pi}{5} + \cos\frac{\pi}{5}\sin\frac{\pi}{6}\breve{\mathbf{e}}_x + \cos\frac{\pi}{6}\sin\frac{\pi}{5}\breve{\mathbf{e}}_y$$

Its roots from Equations 10 and 11 are

$$\overset{o}{\varphi}{}^{\frac{1}{2}} = \overset{o}{\zeta}_{\pm,0} = \pm\left(\cos\frac{\pi}{12}\cos\frac{\pi}{10} + \cos\frac{\pi}{10}\sin\frac{\pi}{12}\breve{\mathbf{e}}_x + \cos\frac{\pi}{12}\sin\frac{\pi}{10}\breve{\mathbf{e}}_y\right)$$

and

$$\overset{o}{\varphi}{}^{\frac{1}{2}} = \overset{o}{\zeta}_{\pm,1} = \pm\left(\sin\frac{\pi}{12}\sin\frac{\pi}{10} - \sin\frac{\pi}{10}\cos\frac{\pi}{12}\breve{\mathbf{e}}_x - \sin\frac{\pi}{12}\cos\frac{\pi}{10}\breve{\mathbf{e}}_y\right)$$

These roots are depicted in Figure 3. The equation of the plane where the four roots lie, from Equation 21 is

$$\left(2\sqrt{3}-1-\sqrt{5}\right)s + 2x - 2\sqrt{\frac{5}{2}-\frac{\sqrt{5}}{2}}\,y = 0$$

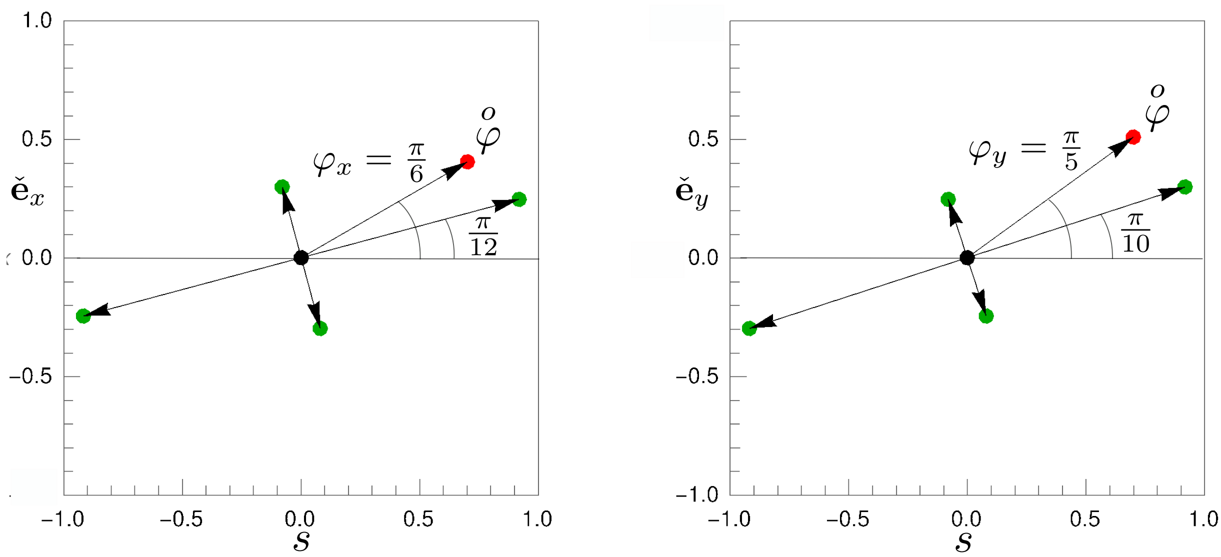

Figure 4. Projections of the roots (green points) of $\overset{o}{\varphi}$ (red point) in the $s, \breve{\mathbf{e}}_x$ (left) and $s, \breve{\mathbf{e}}_y$ (right) planes.

The halving of the angles is not at all evident in the three dimensional plot. However, in Figure 4, where the projections in the $s, \breve{\mathbf{e}}_x$ and $s, \breve{\mathbf{e}}_y$ planes are shown, the angle division is clearly depicted. Furthermore, the other three roots, $\overset{o}{\zeta}_{-,0}$, $\overset{o}{\zeta}_{+,1}$ and $\overset{o}{\zeta}_{-,1}$ are seen as π, $\frac{\pi}{2}$ and $\pi+\frac{\pi}{2}$ rotations respectively of the first root.

6. Conclusion

Scator roots exist in $\mathbb{S}^{1+n}$, and their values in the multiplicative and additive representations are given in closed forms by the Victoria equations in Theorems 2.4 and 3.1. These extended versions of previous theorems exhaust all possible values for the roots of a scator. The multiplicity of the q^{th} root of a scator $\overset{o}{\varphi} \in \mathbb{S}^{1+n}$ is at most $2pq^n$, where p is the number of different π-pair possibilities (Corollary 2.5). The q^{th} root of a scator involves the division of the scator angles by q. The φ_j angles are the multiplicative director components of the scator. They can be represented geometrically as the angle of the projections in the $s, \check{\mathbf{e}}_j$ planes. The square root of a scator in $\mathbb{S}^{1+2}$ has at most four different values that are contained in a plane (Corollary 4.3). Their values are given by Lemmas 4.1 and 4.2. These roots can be nicely depicted in three-dimensional space with the s, $x\,\check{\mathbf{e}}_x$ and $y\,\check{\mathbf{e}}_y$ components drawn in orthogonal axes. The geometric construction of a scator by adding its components is not surprising since vectors and other algebraic structures exhibit this feature. However, the construction of a $1+n$ dimensional scator via the product of its $1+1$ components is quite novel and uncommon in most algebraic structures.

In future studies, the square roots obtained here may be successfully used to find the inverse orbits in the quadratic iteration dynamic scator space. Thus, an algorithm for visualizing the scator fractal Julia sets in 1+2D may be provided. Moreover, this square roots inverse visualization procedure may be implemented. We believe the present results also pave the way for studies on considering higher-order roots and evaluating square roots in higher dimensional scator spaces.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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