Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (4) (2024), 1039 – 1059
DOI : 10.15672/hujms.1189391

Research Article

On left P-Ehresmann and right Ehresmann semigroups: λ-semidirect products and Zappa-Szép products

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Abstract

We show that the λ-semidirect product $M \rtimes^{\lambda} W$ of a left P-Ehresmann semigroup M and a left restriction semigroup W is a left P-Ehresmann semigroup. We explore the behavior of generalized Green's relations on $M \rtimes^{\lambda} W$, and investigate some properties of $M \rtimes^{\lambda} W$. Then the Zappa-Szép product of a right Ehresmann semigroup and its distinguished semilattice is studied. An example of Zappa-Szép product in the context of right Ehresmann semigroups is also given.

Mathematics Subject Classification (2020). 20M99

Keywords. generalized Green's relations, left P-Ehresmann semigroup, right Ehresmann semigroup, Zappa-Szép product

1. Introduction

Left restriction semigroups are non-regular semigroups and are generalizations of inverse semigroups. These semigroups arise very naturally from partial transformation monoids in the same way that inverse semigroups arise from symmetric inverse monoids. For any set X, the partial transformation monoid $\mathcal{PT}_X$ becomes left restriction semigroup under the unary operation $\beta \mapsto I_{\text{dom}\,\beta}$. Left restriction semigroups are precisely the $(2, 1)$-subalgebras of some $\mathcal{PT}_X$. These semigroups were termed as weakly left E-ample semigroups—the (former) York terminology. Since the 1960s, left restriction semigroups transpired with various names and from diverse points of view in literature. For the first time in 1973, these semigroups arose in their own right in Trokhimenko's paper [26]. Also, they were studied in the setting of *type SL2 γ-semigroups* in [1, 2]. Left restriction semigroups were also studied as the idempotent connected Ehresmann semigroups in [23]. Later, they occurred in [19] as *(left) twisted C-semigroups*. In [24], they were studied as *guarded semigroups* which appeared from the restriction categories in [6]. Covering results for restriction semigroups can be found in [7, 10, 12, 13]. New development in the theory of restriction semigroups implies that these semigroups are a subject of fruitful research. For the history and more exposition of (left) restriction semigroups, we suggest [14, 17].

Semidirect products and their generalization Zapp-Szép products are now believed to be momentous tools, which were employed to decompose semigroups. For semidirect product,

Email address: amath1141@gmail.com
Received: 15.10.2022; Accepted: 10.09.2023

see, e.g., [16]. In [27], Zappa introduced Zappa-Szép products which were developed in the setting of groups in [25]. Kunze [21] and Brin [5] applied them to more general structures. The author of [21] studied the characterization of $\mathcal{R}$, $\mathcal{L}$, called the Green's relations, on Zappa-Szép products of semigroups which involve actions of two semigroups on each other. Zenab [28] studied the behavior of generalized Green's relations $\mathcal{R}^*$, $\mathcal{L}^*$, $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ on Zappa-Szép products of semigroups and monoids. Moreover, for a left restriction semigroup T with semilattice (a semigroup of idempotents in which every two elements commute) of projections E, the Zappa-Szép product $E \bowtie T$ is obtained (see [28, Lemma 3.1]). Also, $E \bowtie T$ is not left restriction but there exists a left restriction subsemigroup of $E \bowtie T$.

Now we render a succinct background of λ-semidirect products. If A_1 and A_2 are inverse semigroups, then the semidirect product of A_1 and A_2 may not be an inverse semigroup in general. In [4], by altering the definition of foregoing semidirect product, Billhardt proved that the λ-semidirect product of A_1 and A_2 is an inverse semigroup. Then Billhardt extended his own result to the case where first component, a semilattice, and second component, a left ample semigroup [3]. The λ-semidirect product of semilattice S and weakly left ample semigroup T (a left restriction semigroup T with set of idempotents of T as the semilattice of projections) was calculated in [11]. Recently, in [16], Gould and Zenab considered the case in which both components are left restriction semigroups—if R_1 and R_2 are left restriction semigroups, then the λ-semidirect product of R_1 and R_2 is a left restriction semigroup.

Now we explain how to structure the rest of the paper. The next section is furnished with basic definitions and related facts. In Section 3, we generalize the result of Gould and Zenab [16] by proving that the λ-semidirect product $M \rtimes^{\lambda} W$ of a left P-Ehresmann semigroup M and a left restriction semigroup W is a left P-Ehresmann semigroup. We also characterize generalized Green's relations $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ on $M \rtimes^{\lambda} W$, and discuss some properties of $M \rtimes^{\lambda} W$. In Section 4, motivated by [28, Lemma 3.1], the Zappa-Szép product $Y \bowtie E$ is obtained, where Y is a right Ehresmann semigroup, and E is the distinguished semilattice of Y. Then by defining the unary operation (as defined on Y) on our foregoing Zappa-Szép product, we seek a subset of the image of unary operation, which is the distinguished semilattice of a right Ehresmann subsemigroup G of $Y \bowtie E$. There exists an isomorphism between Y and G, which is also an order-isomorphism. Then from the order-theoretic aspect, some properties of G are discovered. After that, we study the Zappa-Szép product $Y \bowtie E$, where Y is a right Ehresmann semigroup with zero 0, and E is the distinguished semilattice of Y such that $0 \in E$. In this case, the set G' possesses properties from the order-theoretic perspective, which are different from that of G. Lastly, we include an example of Zappa-Szép product in the context of right Ehresmann semigroups.

2. Preliminaries

We incorporate some helpful notions and related facts. For rudimentary notions related to semigroup theory, and Green's relations $\mathcal{R}$, $\mathcal{L}$, see the monograph [18].

The author of [23] introduced the *generalized Green's relations*, i.e., $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ on a semigroup S, where E is a subset of $E(S)$ the set of idempotents of S. For any $v, w \in S$, $\widetilde{\mathcal{R}}_E$ can be defined as:

$$v \ \widetilde{\mathcal{R}}_E \ w \iff \Big[(\ \forall\ f \in E\ )\ fv = v \Leftrightarrow fw = w \Big].$$

The relation $\widetilde{\mathcal{L}}_E$ is defined dually. The relation $\widetilde{\mathcal{R}}_E$ ($\widetilde{\mathcal{L}}_E$) is an equivalence relation. Green's relation $\mathcal{R}$ ($\mathcal{L}$) is left (right) compatible. On the contrary, $\widetilde{\mathcal{R}}_E$ ($\widetilde{\mathcal{L}}_E$) needs not be left (right) compatible. Note that $\mathcal{R} \subseteq \widetilde{\mathcal{R}}_E$ ($\mathcal{L} \subseteq \widetilde{\mathcal{L}}_E$).

Let $v \in S$ and $f \in E$. Let $v \ \widetilde{\mathcal{R}}_E \ f$. Then

$$ff = f \Rightarrow fv = v. \tag{2.1}$$

Moreover,

$$v \, \widetilde{\mathcal{R}}_E \, f \iff fv = v \text{ and } \forall \, h \in E \, [hv = v \Rightarrow hf = f]. \tag{2.2}$$

If E is a semilattice, then the idempotent of E is unique in the $\widetilde{\mathcal{R}}_E$-class of v, and by (2.2), f is the minimum element of $\mathsf{Ll}_v(E)$ the set of all left identities of v belonging to E, with respect to the usual partial order on semilattice E. For more details regarding $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$, see, e.g., [15, 28].

Throughout this paper, we always denote the unique idempotents of E in the $\widetilde{\mathcal{R}}_E$-class and $\widetilde{\mathcal{L}}_E$-class of an element v by $v^\dagger$ and $v^\star$ respectively.

Now we give a brief account of right Ehresmann semigroups from the generalized Green's relations and varietal perspectives, where right Ehresmann semigroups are non-regular generalizations of inverse semigroups. For more details, we prescribe [15].

Definition 2.1 ([15])**.** Suppose that Y is a semigroup and let $E \subseteq E(Y)$. Then Y is a *right Ehresmann semigroup* if:

(E1) E is a semilattice in Y;
(E2) every $\widetilde{\mathcal{L}}_E$-class contains a (unique) element of E;
(E3) $\widetilde{\mathcal{L}}_E$ is a right congruence.

In the above definition, the property of Y that each $\widetilde{\mathcal{L}}_E$-class has a unique idempotent of E, yields a unary operation $y \mapsto y^\star$ on Y. Therefore, every right Ehresmann semigroup may be regarded as a unary semigroup.

The following result is the characterization of right Ehresmann semigroups from the varietal standpoint.

Lemma 2.2 ([15])**.** *A semigroup Y is a right Ehresmann semigroup if and only if the unary operation '$\star$' on Y satisfies the identities:*

$$yy^\star = y, \tag{2.3}$$
$$(y^\star z^\star)^\star = y^\star z^\star, \tag{2.4}$$
$$y^\star z^\star = z^\star y^\star, \tag{2.5}$$
$$(yz)^\star z^\star = (yz)^\star, \tag{2.6}$$
$$(yz)^\star = (y^\star z)^\star \tag{2.7}$$

and

$$E = \{y^\star : y \in Y\}.$$

In Lemma 2.2, the image of the unary operation $\star$ is the set E. The set E is called the distinguished semilattice of right Ehresmann semigroup Y. For all $w_1^\star \in E$, $w_1^\star = (w_1 w_1^\star)^\star = (w_1^\star w_1^\star)^\star = (w_1^\star)^\star$. A partial order $\leq_\ell$ on a right Ehresmann semigroup Y is defined by the rule that for all $x, z \in Y$,

$$x \leq_\ell z \iff x = zx^\star.$$

Any inverse semigroup T is a right Ehresmann semigroup by defining a unary operation $\star$ on T by $t^\star = t^{-1}t$.

Let A and B be two right Ehresmann semigroups with distinguished semilattices E and F respectively. We recall (from [22]) that a *right Ehresmann semigroup morphism* from A to B is a semigroup homomorphism $\alpha : A \to B$ such that for all $x \in A$, $\alpha(x^\star) = (\alpha(x))^\star$. If α is also bijective, then right Ehresmann semigroup morphism from A to B is an *isomorphism*.

Definition 2.3 ([22])**.** A *right Ehresmann semigroup with zero* 0 is a unary semigroup $(S, \cdot, ^\star)$, where $(S, \cdot)$ is a semigroup with zero, * is a unary operation satisfying (2.3)−(2.7).

In the above definition, for all $w \in S$, $0 \leq_\ell w$.

Lemma 2.4 ([22]). *Let S be a right Ehresmann semigroup with zero 0 and the distinguished semilattice E. If $0 \in E$, then for $a \in S$,*

$$a = 0 \Longleftrightarrow a^{\star} = 0.$$

Next we need to recall left restriction semigroups and related facts. For convenience, see, e.g., [14, 28].

Definition 2.5 ([28]). A *left restriction semigroup* is a unary semigroup $(S, \cdot, ^{\dagger})$ which satisfies the following identities:

$$v_1^{\dagger} v_1 = v_1, \tag{2.8}$$

$$v_1^{\dagger} w_1^{\dagger} = w_1^{\dagger} v_1^{\dagger}, \tag{2.9}$$

$$(v_1^{\dagger} w_1)^{\dagger} = v_1^{\dagger} w_1^{\dagger}, \tag{2.10}$$

$$v_1 w_1^{\dagger} = (v_1 w_1)^{\dagger} v_1. \tag{2.11}$$

If we put $E = S^{\dagger} = \{w_1^{\dagger} \mid w_1 \in S\}$, then one can check that E is a semilattice.

Lemma 2.6. [28, Lemma 1.3 (iii)] *For any $x, z \in S$, $(xz^{\dagger})^{\dagger} = (xz)^{\dagger}$.*

In [29], by applying Lemma 2.6, it is proved that for all $w_1^{\dagger} \in E$, $(w_1^{\dagger})^{\dagger} = w_1^{\dagger}$. Each element of E is called a projection of S. The set E is known as the *semilattice of projections* of S. A partial order $\leq$ on S is defined by the rule that for all $v_1, w_1 \in S$,

$$v_1 \leq w_1 \Longleftrightarrow v_1 = v_1^{\dagger} w_1.$$

An alternative characterization for left restriction semigroups is given by Lemma 2.7.

Lemma 2.7. [28, Lemma 1.4] *Suppose that $(S, \cdot, ^{\dagger})$ is a unary semigroup. Then S is a left restriction semigroup with semilattice of projections E if and only if*

(i) *$E \subseteq E(S)$ is a semilattice;*
(ii) *every $\widetilde{\mathcal{R}}_E$-class contains a (unique) element of E;*
(iii) *$\widetilde{\mathcal{R}}_E$ is a left congruence;*
(iv) *the left ample condition holds, i.e., for any $w_1 \in S$, $g \in E$, $w_1 g = (w_1 g)^{\dagger} w_1$.*

Jones invented (left, right) P-Ehresmann semigroups from the view of variety, which extend (left, right) Ehresmann semigroups. The left P-Ehresmann semigroups, their right-sided and two-sided versions are extracted from reducts of regular $*$-semigroups. For facts relevant to (left, right) P-Ehresmann semigroups, the reader is referred to [20].

Definition 2.8 ([20]). Let $(Z, \cdot, ^{\dagger})$ be a unary semigroup. If Z satisfies the identities

$$w_1^{\dagger} w_1 = w_1, \tag{2.12}$$

$$(w_1 x_1)^{\dagger} = (w_1 x_1^{\dagger})^{\dagger}, \tag{2.13}$$

$$(w_1^{\dagger} x_1^{\dagger})^{\dagger} = w_1^{\dagger} x_1^{\dagger} w_1^{\dagger}, \tag{2.14}$$

$$w_1^{\dagger} w_1^{\dagger} = w_1^{\dagger}, \tag{2.15}$$

then Z is called a *left P-Ehresmann semigroup*.

Put $P_z = \{w_1^{\dagger} : w_1 \in Z\}$. The set P_z is the *set of projections* of Z.

Remark 2.9. The (left, right) Ehresmann semigroups are generalizations of (left, right) restriction semigroups from both the generalized Green's relations and varietal perspectives. The (left, right) P-Ehresmann semigroups are generalizations of (left, right) restriction semigroups from the varietal viewpoint.

We now remind the following indispensable notions.

Definition 2.10 ([29]). Let A be a semigroup. Then A *acts on the left of a set* Z if there is a map $A \times Z \to Z, (a, z) \mapsto a \cdot z$ such that for any $z \in Z$ and for any $a, a' \in A$, we have $(aa') \cdot z = a \cdot (a' \cdot z)$.

Dually, A *acts on the right of* Z if there is a map $Z \times A \to Z, (z, a) \mapsto z^a$ such that for any $z \in Z$ and for any $a, a' \in A$, we have $z^{aa'} = (z^a)^{a'}$.

Definition 2.11 ([29]). Let A and Z be two semigroups. Suppose that A acts on the left of Z. If for any $y, z \in Z$ and for any $a \in A$, we have $a \cdot (yz) = (a \cdot y)(a \cdot z)$, then A *acts on* Z *by endomorphisms.*

Definition 2.11 is equivalent to say that there exists a map $\xi : A \to \mathrm{End}\, Z$ which is a homomorphism, where $\mathrm{End}\, Z$ is the endomorphism monoid of Z. We denote $(\xi(a))(z)$ by $a \cdot z$.

Definition 2.12 ([28]). Let M and Z be two semigroups. Let

$$Z \times M \to M, (x, m) \mapsto x \cdot m \text{ and } Z \times M \to Z, (x, m) \mapsto x^m$$

be two maps such that for every $m, m' \in M$, $x, x' \in Z$,

(Z1) $xx' \cdot m = x \cdot (x' \cdot m)$;
(Z2) $x \cdot (mm') = (x \cdot m)(x^m \cdot m')$;
(Z3) $(x^m)^{m'} = x^{mm'}$;
(Z4) $(xx')^m = x^{x' \cdot m} x'^m$.

Define a binary operation $\circ$ on $M \times Z$ by

$$(m, x) \circ (m', x') = (m(x \cdot m'), x^{m'} x').$$

Then by (Z1)-(Z4), $\circ$ is associative. Hence, $(M \times Z, \circ)$ is a semigroup, called the *(external) Zappa-Szép product* of M and Z, denoted by $M \bowtie Z$.

In the above definition, by (Z1), Z acts on M from the left, and by (Z3), M acts on Z from the right.

Definition 2.13 ([29]). Suppose that A is a left restriction semigroup, and U is a semigroup. Suppose that A is acting on the left of U by endomorphisms. Let $U \rtimes^{\lambda} A$ denote the set $\{(x, a) \in U \times A : a^{\dagger} \cdot x = x\}$. Define a binary composition on $U \rtimes^{\lambda} A$ by

$$(x, a)(y, b) = (((ab)^{\dagger} \cdot x)(a \cdot y), ab). \tag{2.16}$$

Then $U \rtimes^{\lambda} A$ is a semigroup and is called the *λ-semidirect product* of U by A.

In [29, Lemma 6.1.6], it is proved that $U \rtimes^{\lambda} A$ is a semigroup.

Next we also need to quote the following useful terminologies. We refer the reader to [8, 9].

Let $(Q, \leq)$ and $(Q', \leq)$ be two partial ordered sets (posets). A map $\psi : Q \to Q'$ is said to be an *order-isomorphism* if ψ is surjective, and $x \leq y$ in Q if and only if $\psi(x) \leq \psi(y)$ in Q'. Let $X \subseteq Q$. Then X is a *down-set* if $a \in X$, $d \in Q$ and $d \leq a$ imply that $d \in X$. Dually, X is an *up-set* if $a \in X$, $u \in Q$ and $a \leq u$ imply that $u \in X$. Note that the empty set $\emptyset$ is a down-set and up-set as well. Let $(\overline{Q}, \leq)$ be a poset containing the least element 0. If $0 \neq w \in \overline{Q}$ is such that w is a minimal element of $\overline{Q} \setminus \{0\}$, then w is an *atom*. The poset $\overline{Q}$ is called *atomic* if every non-zero element dominates an atom (i.e., there is an atom less than or equal to it).

3. Generalized Green's relations on λ-semidirect product

We commence this section by proving that the λ-semidirect product of a left P-Ehresmann semigroup and a left restriction semigroup is a left P-Ehresmann semigroup.

Let M be a left P-Ehresmann semigroup. If a semigroup W is acting on the left of M such that for all m_1, ${m_1}' \in M$ and $w_1 \in W$, we have

$$w_1 \cdot (m_1 m_1') = (w_1 \cdot m_1)(w_1 \cdot m_1') \tag{3.1}$$

and

$$w_1 \cdot m_1^{\dagger} = (w_1 \cdot m_1)^{\dagger}, \tag{3.2}$$

i.e., the morphism $W \to \mathrm{End}\, M$ is into the monoid of endomorphisms of M *as a unary semigroup*, then we say that W is acing on M by *left P-Ehresmann endomorphisms.*

Proposition 3.1. *Let M be a left P-Ehresmann semigroup. Suppose that W is a left restriction semigroup which is acting on the left of M by left P-Ehresmann endomorphisms. Then*

$$M \rtimes^{\lambda} W = \{(m_1, w_1) \in M \times W : w_1^{\dagger} \cdot m_1 = m_1\}$$

is a left P-Ehresmann semigroup under the binary operation

$$(l_1, v_1)(m_1, w_1) = (((v_1 w_1)^{\dagger} \cdot l_1)(v_1 \cdot m_1), v_1 w_1), \tag{3.3}$$

where

$$(m_1, w_1)^{\dagger} = (m_1^{\dagger}, w_1^{\dagger}). \tag{3.4}$$

Moreover, $P_{M \rtimes^{\lambda} W} = \{(m_1^{\dagger}, w_1^{\dagger}) : w_1^{\dagger} \cdot m_1^{\dagger} = m_1^{\dagger}\}$ is the set of projections of $M \rtimes^{\lambda} W$.

Proof. Since W is acting on M by left P-Ehresmann endomorphisms, for all $w_1 \in W$, $m_1 \in M$, we have $w_1 \cdot m_1^{\dagger} = (w_1 \cdot m_1)^{\dagger}$.

For every $(m_1, w_1) \in M \rtimes^{\lambda} W$,

$$w_1^{\dagger} \cdot m_1^{\dagger} = (w_1^{\dagger} \cdot m_1)^{\dagger} = m_1^{\dagger}. \tag{3.5}$$

Consequently, we have $P_{M \rtimes^{\lambda} W} \subseteq M \rtimes^{\lambda} W$.

By definition 2.13, $M \rtimes^{\lambda} W$ is a semigroup under the binary operation, defined in (3.3), and is called the λ-semidirect product of M by W.

Now we show that $M \rtimes^{\lambda} W$ satisfies (2.12)–(2.15). For (l_1, v_1), $(m_1, w_1) \in M \rtimes^{\lambda} W$, we have

$$\begin{aligned}
(l_1, v_1)^{\dagger}(l_1, v_1) &= (l_1^{\dagger}, v_1^{\dagger})(l_1, v_1) \\
&= (((v_1^{\dagger} v_1)^{\dagger} \cdot l_1^{\dagger})(v_1^{\dagger} \cdot l_1), v_1^{\dagger} v_1) \\
&= ((v_1^{\dagger} \cdot l_1^{\dagger})(v_1^{\dagger} \cdot l_1), v_1) && \text{(using (2.8))} \\
&= ((v_1^{\dagger} \cdot l_1^{\dagger}) l_1, v_1) && (v_1^{\dagger} \cdot l_1 = l_1) \\
&= (l_1^{\dagger} l_1, v_1) && \text{(using (3.5))} \\
&= (l_1, v_1) && \text{(using (2.12))}.
\end{aligned}$$

Now we obtain

$$\begin{aligned}
((l_1,v_1)(m_1,w_1)^\dagger)^\dagger &= ((l_1,v_1)(m_1^\dagger,w_1^\dagger))^\dagger \\
&= \Big(((v_1w_1^\dagger)^\dagger\cdot l_1)(v_1\cdot m_1^\dagger), v_1w_1^\dagger\Big)^\dagger \\
&= \Big(((v_1w_1^\dagger)^\dagger\cdot l_1)(v_1\cdot m_1)^\dagger, v_1w_1^\dagger\Big)^\dagger \\
&= \Big((((v_1w_1^\dagger)^\dagger\cdot l_1)(v_1\cdot m_1)^\dagger)^\dagger, (v_1w_1^\dagger)^\dagger\Big) && \text{(using (3.4))} \\
&= \Big((((v_1w_1)^\dagger\cdot l_1)(v_1\cdot m_1)^\dagger)^\dagger, (v_1w_1)^\dagger\Big) && \text{(By Lemma 2.6)} \\
&= \Big((((v_1w_1)^\dagger\cdot l_1)(v_1\cdot m_1))^\dagger, (v_1w_1)^\dagger\Big) && \text{(using (2.13))} \\
&= \Big(((v_1w_1)^\dagger\cdot l_1)(v_1\cdot m_1), v_1w_1\Big)^\dagger \\
&= ((l_1,v_1)(m_1,w_1))^\dagger.
\end{aligned}$$

Next we see that

$$\begin{aligned}
&((l_1,v_1)^\dagger(m_1,w_1)^\dagger)^\dagger = ((l_1^\dagger,v_1^\dagger)(m_1^\dagger,w_1^\dagger))^\dagger \\
&= \Big(((v_1^\dagger w_1^\dagger)^\dagger\cdot l_1^\dagger)(v_1^\dagger\cdot m_1^\dagger), v_1^\dagger w_1^\dagger\Big)^\dagger \\
&= \Big(((v_1^\dagger w_1^\dagger)^\dagger\cdot l_1)^\dagger(v_1^\dagger\cdot m_1)^\dagger, v_1^\dagger w_1^\dagger\Big)^\dagger && \text{(using (3.2))} \\
&= \Big(\big(((v_1^\dagger w_1^\dagger)^\dagger\cdot l_1)^\dagger(v_1^\dagger\cdot m_1)^\dagger\big)^\dagger, (v_1^\dagger w_1^\dagger)^\dagger\Big) && \text{(using (3.4))} \\
&= \Big(((v_1^\dagger w_1^\dagger)^\dagger\cdot l_1)^\dagger(v_1^\dagger\cdot m_1)^\dagger((v_1^\dagger w_1^\dagger)^\dagger\cdot l_1)^\dagger, (v_1^\dagger w_1^\dagger)^\dagger\Big) && \text{(using (2.14))} \\
&= \Big(((v_1^\dagger w_1^\dagger)^\dagger\cdot l_1^\dagger)(v_1^\dagger\cdot m_1^\dagger)((v_1^\dagger w_1^\dagger)^\dagger\cdot l_1^\dagger), (v_1^\dagger w_1^\dagger)^\dagger\Big) \\
&= \Big((v_1^\dagger w_1^\dagger v_1^\dagger\cdot l_1^\dagger)(v_1^\dagger\cdot m_1^\dagger)(v_1^\dagger w_1^\dagger v_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) && \text{(using (2.14) in } W) \\
&= \Big((v_1^\dagger w_1^\dagger\cdot(v_1^\dagger\cdot l_1^\dagger))(v_1^\dagger\cdot m_1^\dagger)(v_1^\dagger w_1^\dagger\cdot(v_1^\dagger\cdot l_1^\dagger)), v_1^\dagger w_1^\dagger v_1^\dagger\Big) \\
&= \Big((v_1^\dagger w_1^\dagger\cdot l_1^\dagger)(v_1^\dagger\cdot m_1^\dagger)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) && \text{(using (3.5))} \\
&= \Big((v_1^\dagger w_1^\dagger\cdot l_1^\dagger)(v_1^\dagger\cdot(w_1^\dagger\cdot m_1^\dagger))(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) && \text{(using (3.5))} \\
&= \Big((v_1^\dagger w_1^\dagger\cdot l_1^\dagger)(v_1^\dagger w_1^\dagger\cdot m_1^\dagger)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) \\
&= \Big((v_1^\dagger w_1^\dagger\cdot l_1^\dagger m_1^\dagger)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) && \text{(using (3.1))} \\
&= \Big(((v_1^\dagger w_1^\dagger)^\dagger(v_1^\dagger w_1^\dagger)\cdot l_1^\dagger m_1^\dagger)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) && \text{(using (2.8))} \\
&= \Big(\big((v_1^\dagger w_1^\dagger)^\dagger\cdot((v_1^\dagger w_1^\dagger)\cdot l_1^\dagger m_1^\dagger)\big)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) \\
&= \Big(\big((v_1^\dagger w_1^\dagger)^\dagger\cdot((v_1^\dagger w_1^\dagger\cdot l_1^\dagger)(v_1^\dagger w_1^\dagger\cdot m_1^\dagger))\big)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) && \text{(using (3.1))} \\
&= \Big(\big((v_1^\dagger w_1^\dagger)^\dagger\cdot((v_1^\dagger w_1^\dagger\cdot l_1^\dagger)(v_1^\dagger\cdot(w_1^\dagger\cdot m_1^\dagger)))\big)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) \\
&= \Big(\big((v_1^\dagger w_1^\dagger)^\dagger\cdot((v_1^\dagger w_1^\dagger\cdot l_1^\dagger)(v_1^\dagger\cdot m_1^\dagger))\big)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) && \text{(using (3.5))} \\
&= \Big(\big((v_1^\dagger w_1^\dagger)^\dagger\cdot((v_1^\dagger w_1^\dagger\cdot(v_1^\dagger\cdot l_1^\dagger))(v_1^\dagger\cdot m_1^\dagger))\big)(v_1^\dagger w_1^\dagger\cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) && \text{(using (3.5))}
\end{aligned}$$

$$= \Big(\Big(((v_1^\dagger w_1^\dagger)^\dagger)^\dagger \cdot ((v_1^\dagger w_1^\dagger v_1^\dagger \cdot l_1^\dagger)(v_1^\dagger \cdot m_1^\dagger))\Big)(v_1^\dagger w_1^\dagger \cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big)$$
$$= \Big(\Big((v_1^\dagger w_1^\dagger v_1^\dagger)^\dagger \cdot \Big(((v_1^\dagger w_1^\dagger)^\dagger \cdot l_1^\dagger)(v_1^\dagger \cdot m_1^\dagger)\Big)\Big)(v_1^\dagger w_1^\dagger \cdot l_1^\dagger), v_1^\dagger w_1^\dagger v_1^\dagger\Big) \quad \text{(using (2.14) in } W)$$
$$= \Big(((v_1^\dagger w_1^\dagger)^\dagger \cdot l_1^\dagger)(v_1^\dagger \cdot m_1^\dagger), v_1^\dagger w_1^\dagger\Big)(l_1^\dagger, v_1^\dagger)$$
$$= (l_1^\dagger, v_1^\dagger)(m_1^\dagger, w_1^\dagger)(l_1^\dagger, v_1^\dagger)$$
$$= (l_1, v_1)^\dagger (m_1, w_1)^\dagger (l_1, v_1)^\dagger.$$

Finally, we have

$$\begin{aligned}(l_1, v_1)^\dagger (l_1, v_1)^\dagger &= (l_1^\dagger, v_1^\dagger)(l_1^\dagger, v_1^\dagger)\\ &= (((v_1^\dagger v_1^\dagger)^\dagger \cdot l_1^\dagger)(v_1^\dagger \cdot l_1^\dagger), v_1^\dagger v_1^\dagger)\\ &= ((v_1^\dagger \cdot l_1^\dagger)(v_1^\dagger \cdot l_1^\dagger), v_1^\dagger)\\ &= (l_1^\dagger l_1^\dagger, v_1^\dagger) && \text{(using (3.5))}\\ &= (l_1, v_1)^\dagger.\end{aligned}$$

Therefore, $M \rtimes^\lambda W$ is a left P-Ehresmann semigroup with $(m_1, w_1)^\dagger = (m_1^\dagger, w_1^\dagger)$. It is straightforward to show that for all $(m_1, w_1)^\dagger \in P_{M \rtimes^\lambda W}$, $((m_1, w_1)^\dagger)^\dagger = (m_1, w_1)^\dagger$. Hence, $P_{M \rtimes^\lambda W}$ is the set of projections of $M \rtimes^\lambda W$. □

Our next aim is to ascertain the behavior of generalized Green's relations $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ on the λ-semidirect product of a left P-Ehresmann semigroup by a left restriction semigroup.

Proposition 3.2. *Let $M \rtimes^\lambda W$ be as in Proposition 3.1. Put $\mathcal{P}=M \rtimes^\lambda W$. Suppose that*

$$E(\mathcal{P}) = \{(m_1, w_1) \in \mathcal{P} : (m_1, w_1)(m_1, w_1) = (m_1, w_1)\}.$$

(a) *Suppose that for all $t \in W$, we have*

$$t \cdot e = e, \tag{3.6}$$

where e is a right identity of M. Let $\overline{E}_e=\{(e, h) : h \in E \subseteq E(W)\}$. Then we have

$$\overline{E}_e \subseteq E(\mathcal{P})$$

and

$$(l_1, v_1)\ \widetilde{\mathcal{L}}_{\overline{E}_e}\ (m_1, w_1) \text{ in } \mathcal{P} \iff v_1\ \widetilde{\mathcal{L}}_E\ w_1 \text{ in } W.$$

(b) *Suppose that for all $p \in M$, we have*

$$f \cdot p = p, \tag{3.7}$$

where f is a left identity of W. Let $\overline{F}_f=\{(g, f) : g \in F \subseteq E(M)\} \subseteq \mathcal{P}$. If in addition, for any $t \in W$,

$$t \cdot g = g, \tag{3.8}$$

where $g \in F$, then we have

$$\overline{F}_f \subseteq E(\mathcal{P})$$

and

$$(l_1, v_1)\ \widetilde{\mathcal{R}}_{\overline{F}_f}\ (m_1, w_1) \text{ in } \mathcal{P} \iff l_1\ \widetilde{\mathcal{R}}_F\ m_1 \text{ in } M.$$

Proof. (a) It can be easily checked that $\overline{E}_e \subseteq \mathcal{P}$. For every $(e, h) \in \overline{E}_e$, we have

$$\begin{aligned}(e, h)(e, h) &= (((hh)^\dagger \cdot e)(h \cdot e), hh)\\ &= (ee, hh) && \text{(using (3.6))}\\ &= (e, h) && (e \text{ is a right identity and } hh = h).\end{aligned}$$

Hence, $\overline{E}_e \subseteq E(\mathcal{P})$.

Suppose that for all $(l_1, v_1), (m_1, w_1) \in \mathcal{P}$, $(l_1, v_1)\ \widetilde{\mathcal{L}}_{\overline{E}_e}\ (m_1, w_1)$. Suppose that for all $h \in E$, $v_1 h = v_1$. Then

$$\begin{aligned}(l_1, v_1)(e, h) &= (((v_1 h)^\dagger \cdot l_1)(v_1 \cdot e), v_1 h) \\ &= ((v_1^\dagger \cdot l_1)(v_1 \cdot e), v_1) \\ &= ((v_1^\dagger \cdot l_1)e, v_1) && \text{(using (3.6))} \\ &= (l_1 e, v_1) && ((l_1, v_1) \in \mathcal{P}) \\ &= (l_1, v_1).\end{aligned}$$

Since $(l_1, v_1)\ \widetilde{\mathcal{L}}_{\overline{E}_e}\ (m_1, w_1)$, we obtain

$$\begin{aligned}(m_1, w_1)(e, h) &= (m_1, w_1) \\ (((w_1 h)^\dagger \cdot m_1)(w_1 \cdot e), w_1 h) &= (m_1, w_1).\end{aligned}$$

Then we have $w_1 h = w_1$. Therefore, we have $v_1 h = v_1 \Rightarrow w_1 h = w_1$. Similarly, we obtain $w_1 h = w_1 \Rightarrow v_1 h = v_1$. Hence, $v_1\ \widetilde{\mathcal{L}}_E\ w_1$ in W.

Conversely, suppose that for all $v_1, w_1 \in W$, $v_1\ \widetilde{\mathcal{L}}_E\ w_1$. Suppose that for all $(e, h) \in \overline{E}_e$,

$$\begin{aligned}(l_1, v_1)(e, h) &= (l_1, v_1) \\ (((v_1 h)^\dagger \cdot l_1)(v_1 \cdot e), v_1 h) &= (l_1, v_1).\end{aligned}$$

Then we have $v_1 h = v_1$. Since $v_1\ \widetilde{\mathcal{L}}_E\ w_1$, we deduce $w_1 h = w_1$. Also, we have

$$\begin{aligned}(m_1, w_1)(e, h) &= (((w_1 h)^\dagger \cdot m_1)(w_1 \cdot e), w_1 h) \\ &= ((w_1^\dagger \cdot m_1)(w_1 \cdot e), w_1) \\ &= ((w_1^\dagger \cdot m_1)e, w_1) && \text{(using (3.6))} \\ &= (m_1 e, w_1) && ((m_1, w_1) \in \mathcal{P}) \\ &= (m_1, w_1).\end{aligned}$$

Therefore, we have $(l_1, v_1)(e, h) = (l_1, v_1) \Rightarrow (m_1, w_1)(e, h) = (m_1, w_1)$. Similarly, we obtain $(m_1, w_1)(e, h) = (m_1, w_1) \Rightarrow (l_1, v_1)(e, h) = (l_1, v_1)$. Hence, $(l_1, v_1)\ \widetilde{\mathcal{L}}_{\overline{E}_e}\ (m_1, w_1)$.

(b) For every $(g, f) \in \overline{F}_f$, we have

$$\begin{aligned}(g, f)(g, f) &= (((ff)^\dagger \cdot g)(f \cdot g), ff) \\ &= (gg, ff) && \text{(using (3.8))} \\ &= (g, f) && (gg = g \text{ and } f \text{ is left identity}).\end{aligned}$$

This implies that $\overline{F}_f \subseteq E(\mathcal{P})$.

Suppose that for all $(l_1, v_1), (m_1, w_1) \in \mathcal{P}$, $(l_1, v_1)\ \widetilde{\mathcal{R}}_{\overline{F}_f}\ (m_1, w_1)$. Suppose that for all $g \in F$, $g l_1 = l_1$. Then

$$\begin{aligned}(g, f)(l_1, v_1) &= (((f v_1)^\dagger \cdot g)(f \cdot l_1), f v_1) \\ &= (g l_1, f v_1) && \text{(using (3.8) and (3.7))} \\ &= (l_1, v_1).\end{aligned}$$

Since $(l_1, v_1)\ \widetilde{\mathcal{R}}_{\overline{F}_f}\ (m_1, w_1)$, we obtain

$$\begin{aligned}(g, f)(m_1, w_1) &= (m_1, w_1)\\ (((fw_1)^\dagger \cdot g)(f \cdot m_1), fw_1) &= (m_1, w_1)\\ (gm_1, fw_1) &= (m_1, w_1).\end{aligned}$$

Then we have $gm_1 = m_1$. Therefore, we have $gl_1 = l_1 \Rightarrow gm_1 = m_1$. Similarly, $gm_1 = m_1 \Rightarrow gl_1 = l_1$. Hence, $l_1\ \widetilde{\mathcal{R}}_F\ m_1$ in M.

Conversely, suppose that for all $l_1, m_1 \in M$, $l_1\ \widetilde{\mathcal{R}}_F\ m_1$. Suppose that for all $(g, f) \in \overline{F}_f$,

$$\begin{aligned}(g, f)(l_1, v_1) &= (l_1, v_1)\\ (((fv_1)^\dagger \cdot g)(f \cdot l_1), fv_1) &= (l_1, v_1)\\ (gl_1, fv_1) &= (l_1, v_1).\end{aligned}$$

Then we have $gl_1 = l_1$. Since $l_1\ \widetilde{\mathcal{R}}_F\ m_1$, we have $gm_1 = m_1$. Also, we have

$$\begin{aligned}(g, f)(m_1, w_1) &= (((fw_1)^\dagger \cdot g)(f \cdot m_1), fw_1)\\ &= (gm_1, fw_1) && \text{(using (3.8) and (3.7))}\\ &= (m_1, w_1).\end{aligned}$$

Therefore, we have $(g, f)(l_1, v_1) = (l_1, v_1) \Rightarrow (g, f)(m_1, w_1) = (m_1, w_1)$. Similarly, $(g, f)(m_1, w_1) = (m_1, w_1) \Rightarrow (g, f)(l_1, v_1) = (l_1, v_1)$. Hence, $(l_1, v_1)\ \widetilde{\mathcal{R}}_{\overline{F}_f}\ (m_1, w_1)$. □

Now we study properties of the λ-semidirect product of a left P-Ehresmann semigroup and a left restriction semigroup.

Proposition 3.3. *Let $\mathcal{P}$ be as in Proposition 3.2. Then*

(a) $E(\mathcal{P}) = \{(n_1, w_1) \in \mathcal{P} : w_1 \cdot n_1$ *is a right identity of* n_1 *and* $w_1w_1 = w_1\}$;

(b) *Suppose that for all $t \in W$, $t \cdot e = e$, where e is a right identity of M. Let $\overline{E}_e$ be as in Proposition 3.2. If $(l_1, v_1)\ \widetilde{\mathcal{R}}_{\overline{E}_e}\ (e, h)\ \widetilde{\mathcal{L}}_{\overline{E}_e}\ (l_1, v_1)$ in $\mathcal{P}$, then h is a two-sided identity of v_1;*

(c) *Suppose that for all $p \in M$, $f \cdot p = p$, where f is a left identity of W. Let $\overline{F}_f$ be as in Proposition 3.2. If $(l_1, v_1)\ \widetilde{\mathcal{R}}_{\overline{F}_f}\ (g, f)\ \widetilde{\mathcal{L}}_{\overline{F}_f}\ (l_1, v_1)$ in $\mathcal{P}$, then f is a two-sided identity of v_1;*

(d) *Let E be the semilattice of projections of W. Suppose that for all $t \in W$, $t \cdot e = e$, where e is a right identity of M. Let $\overline{E}_e = \{(e, h) : h \in E\}$. Then*

$$(l_1, v_1)\ \widetilde{\mathcal{R}}_{\overline{E}_e}\ (e, h) \text{ in } \mathcal{P} \Rightarrow v_1^\dagger \le h \text{ in } W.$$

Proof. (a) If $(n_1, w_1) \in E(\mathcal{P})$, then

$$\begin{aligned}(n_1, w_1)(n_1, w_1) = (n_1, w_1) &\Rightarrow (((w_1w_1)^\dagger \cdot n_1)(w_1 \cdot n_1), w_1w_1) = (n_1, w_1)\\ &\Rightarrow ((w_1w_1)^\dagger \cdot n_1)(w_1 \cdot n_1) = n_1 \text{ and } w_1w_1 = w_1\\ &\Rightarrow (w_1^\dagger \cdot n_1)(w_1 \cdot n_1) = n_1\\ &\Rightarrow n_1(w_1 \cdot n_1) = n_1 && (w_1^\dagger \cdot n_1 = n_1).\end{aligned}$$

Let $(n_1, w_1) \in \mathcal{P}$ be such that $n_1(w_1 \cdot n_1) = n_1$ and $w_1w_1 = w_1$. Since $w_1^\dagger \cdot n_1 = n_1$, it follows that

$$\begin{aligned}(w_1^\dagger \cdot n_1)(w_1 \cdot n_1) = n_1 &\Rightarrow ((w_1w_1)^\dagger \cdot n_1)(w_1 \cdot n_1) = n_1\\ &\Rightarrow (((w_1w_1)^\dagger \cdot n_1)(w_1 \cdot n_1), w_1w_1) = (n_1, w_1)\\ &\Rightarrow (n_1, w_1)(n_1, w_1) = (n_1, w_1)\\ &\Rightarrow (n_1, w_1) \in E(\mathcal{P}).\end{aligned}$$

(b) By Proposition 3.2 (a), $\overline{E}_e \subseteq E(\mathcal{P})$. Since $(l_1, v_1)\ \widetilde{\mathcal{R}}_{\overline{E}_e}\ (e,h)$ in $\mathcal{P}$ and $(e,h)(e,h) = (e,h)$, it follows that $(e,h)(l_1,v_1) = (l_1,v_1)$. Then we have $(((hv_1)^{\dagger} \cdot e)(h \cdot l_1), hv_1) = (l_1, v_1)$. Then $hv_1 = v_1$. Since $(e,h)\ \widetilde{\mathcal{L}}_{\overline{E}_e}\ (l_1,v_1)$ in $\mathcal{P}$, we have $(l_1,v_1)(e,h) = (l_1,v_1)$. Then we have $(((v_1h)^{\dagger} \cdot l_1)(v_1 \cdot e), v_1h) = (l_1,v_1)$. Then $v_1h = v_1$. Hence, h is a two-sided identity of v_1.

(c) The proof is the same as that of (b).

(d) By Proposition 3.2 (a), $\overline{E}_e \subseteq E(\mathcal{P})$. The proof of $hv_1 = v_1$ is the same as that of (b). Since $v_1^{\dagger}$ is the unique idempotent in the $\widetilde{\mathcal{R}}_E$-class of v_1 and is the minimum element of $\mathsf{LI}_{v_1}(E)$ the set of all left identities of v_1 belonging to E, we deduce $v_1^{\dagger} \leq h$. □

4. Zappa-Szép products

In this section, in general, we study the Zappa-Szép product of a right Ehresmann semigroup and its distinguished semilattice.

4.1. Zappa-Szép product of a right Ehresmann semigroup and its distinguished semilattice

We gain the Zappa-Szép product $Y \bowtie E$, where Y is a right Ehresmann semigroup, and E is the distinguished semilattice of Y. We define the unary operation (as defined on Y) on our foregoing Zappa-Szép product, and find a subset of the image of unary operation, which is the distinguished semilattice of a right Ehresmann subsemigroup G of $Y \bowtie E$. We also show that G is isomorphic and order-isomorphic to Y. Then we study some properties of G from the order-theoretic viewpoint.

Lemma 4.1. *Let Y be a right Ehresmann semigroup with distinguished semilattice E. Define the maps $E \times Y \to Y$ and $E \times Y \to E$ by $(a_1^{\star}, w_1) \mapsto a_1^{\star} \cdot w_1 = a_1^{\star} w_1$ and $(a_1^{\star}, w_1) \mapsto (a_1^{\star})^{w_1} = (a_1 w_1)^{\star}$ respectively. Then we have $Y \bowtie E$ the Zappa-Szép product of Y and E.*

Proof. In order to check that the given maps satisfy the Zappa-Szép product axioms, let $a_1^{\star}, c_1^{\star} \in E$ and $w_1, x_1 \in Y$.

(Z1)

$$\begin{aligned} a_1^{\star} \cdot (c_1^{\star} \cdot w_1) &= a_1^{\star} \cdot (c_1^{\star} w_1) \\ &= a_1^{\star}(c_1^{\star} w_1) \\ &= (a_1^{\star} c_1^{\star}) w_1 \\ &= (a_1^{\star} c_1^{\star})^{\star} w_1 && \text{(using (2.4))} \\ &= (a_1^{\star} c_1^{\star})^{\star} \cdot w_1 \\ &= (a_1^{\star} c_1^{\star}) \cdot w_1. \end{aligned}$$

Hence, (Z1) holds.

(Z2)

$$\begin{aligned} (a_1^{\star} \cdot w_1)((a_1^{\star})^{w_1} \cdot x_1) &= (a_1^{\star} w_1)((a_1 w_1)^{\star} \cdot x_1) \\ &= (a_1^{\star} w_1)((a_1^{\star} w_1)^{\star} \cdot x_1) && \text{(using (2.7))} \\ &= (a_1^{\star} w_1)((a_1^{\star} w_1)^{\star} x_1) \\ &= ((a_1^{\star} w_1)(a_1^{\star} w_1)^{\star}) x_1 \\ &= (a_1^{\star} w_1) x_1 && \text{(using (2.3))} \\ &= a_1^{\star}(w_1 x_1) \\ &= a_1^{\star} \cdot (w_1 x_1). \end{aligned}$$

Hence, (Z2) holds.

(Z3)

$$\begin{aligned}((a_1^\star)^{w_1})^{x_1} &= ((a_1w_1)^\star)^{x_1}\\ &= ((a_1w_1)x_1)^\star\\ &= (a_1(w_1x_1))^\star\\ &= (a_1^\star)^{w_1x_1}.\end{aligned}$$

Therefore, (Z3) holds.

(Z4)

$$\begin{aligned}(a_1^\star)^{c_1^\star\cdot w_1}(c_1^\star)^{w_1} &= (a_1^\star)^{c_1^\star w_1}(c_1^\star)^{w_1}\\ &= (a_1(c_1^\star w_1))^\star(c_1w_1)^\star\\ &= (a_1^\star(c_1^\star w_1))^\star(c_1^\star w_1)^\star && \text{(using (2.7))}\\ &= (a_1^\star(c_1^\star w_1))^\star && \text{(using (2.6))}\\ &= ((a_1^\star c_1^\star)w_1)^\star\\ &= ((a_1^\star c_1^\star)^\star)^{w_1}\\ &= (a_1^\star c_1^\star)^{w_1} && \text{(using (2.4))}.\end{aligned}$$

Thus, (Z4) holds. Then $Y \times E$ is a semigroup under the binary operation $\circ$, defined by

$$(w_1, a_1^\star)\circ(x_1, c_1^\star) = (w_1(a_1^\star\cdot x_1), (a_1^\star)^{x_1}c_1^\star). \tag{4.1}$$

Hence, we have the semigroup $(Y \times E, \circ)$, called the Zappa-Szép product of Y and E, denoted by $Y \bowtie E$. ☐

Lemma 4.2. *Let $Y \bowtie E$ be the Zappa-Szép product of Y and E, formed in Lemma 4.1. Let the unary operation * (as defined on Y) be defined on $Y \bowtie E$ by*

$$(w_1, a_1^\star)^\star = (w_1^\star, a_1^\star). \tag{4.2}$$

Then the unary semigroup $(Y \times E, \circ, ^\star)$=$(Y \bowtie E, ^\star)$ satisfies:

(i) $((w_1, a_1^\star)^\star\circ(x_1, c_1^\star)^\star)^\star = (w_1, a_1^\star)^\star\circ(x_1, c_1^\star)^\star$;
(ii) $((w_1, a_1^\star)^\star\circ(x_1, c_1^\star))^\star = ((w_1, a_1^\star)\circ(x_1, c_1^\star))^\star$;
(iii) $((w_1, a_1^\star)\circ(x_1, c_1^\star)^\star)^\star = (w_1, a_1^\star)^\star\circ(x_1, c_1^\star)^\star$;
(iv) $(w_1, w_1^\star)^\star\circ(w_1, w_1^\star)^\star = (w_1, w_1^\star)^\star$;
(v) $(w_1, a_1^\star)^\star\circ(x_1, a_1^\star)^\star = (x_1, a_1^\star)^\star\circ(w_1, a_1^\star)^\star$ *whenever* $a_1^\star \leq_\ell w_1^\star$ *and* $a_1^\star \leq_\ell x_1^\star$.

Proof. One can verify that * is well-defined. Now we hasten to prove (i)-(v).

First, we intend to prove that for all $(w_1, a_1^\star), (x_1, c_1^\star) \in Y \bowtie E$, (i)-(iii) are satisfied.

(i)

$$
\begin{aligned}
((w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star)^\star &= ((w_1^\star, a_1^\star) \circ (x_1^\star, c_1^\star))^\star && \text{(using (4.2))}\\
&= (w_1^\star(a_1^\star \cdot x_1^\star), (a_1^\star)^{x_1^\star} c_1^\star)^\star\\
&= (w_1^\star(a_1^\star x_1^\star), (a_1 x_1^\star)^\star c_1^\star)^\star\\
&= (w_1^\star(a_1^\star x_1^\star), ((a_1 x_1^\star)^\star c_1^\star)^\star)^\star && \text{(using (2.4))}\\
&= ((w_1^\star(a_1^\star x_1^\star))^\star, ((a_1 x_1^\star)^\star c_1^\star)^\star) && \text{(using (4.2))}\\
&= (((w_1^\star a_1^\star) x_1^\star)^\star, ((a_1 x_1^\star)^\star c_1^\star)^\star)\\
&= (((w_1^\star a_1^\star)^\star x_1^\star)^\star, ((a_1 x_1^\star)^\star c_1^\star)^\star) && \text{(using (2.7))}\\
&= ((w_1^\star a_1^\star)^\star x_1^\star, (a_1 x_1^\star)^\star c_1^\star) && \text{(using (2.4))}\\
&= ((w_1^\star a_1^\star) x_1^\star, (a_1 x_1^\star)^\star c_1^\star) && \text{(using (2.4))}\\
&= (w_1^\star(a_1^\star x_1^\star), (a_1 x_1^\star)^\star c_1^\star)\\
&= (w_1^\star(a_1^\star \cdot x_1^\star), (a_1^\star)^{x_1^\star} c_1^\star)\\
&= (w_1^\star, a_1^\star) \circ (x_1^\star, c_1^\star)\\
&= (w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
((w_1, a_1^\star)^\star \circ (x_1, c_1^\star))^\star &= ((w_1^\star, a_1^\star) \circ (x_1, c_1^\star))^\star\\
&= (w_1^\star(a_1^\star \cdot x_1), (a_1^\star)^{x_1} c_1^\star)^\star\\
&= (w_1^\star(a_1^\star x_1), (a_1 x_1)^\star c_1^\star)^\star\\
&= (w_1^\star(a_1^\star x_1), ((a_1 x_1)^\star c_1^\star)^\star)^\star && \text{(using (2.4))}\\
&= ((w_1^\star(a_1^\star x_1))^\star, ((a_1 x_1)^\star c_1^\star)^\star) && \text{(using (4.2))}\\
&= ((w_1(a_1^\star x_1))^\star, ((a_1 x_1)^\star c_1^\star)^\star) && \text{(using (2.7))}\\
&= (w_1(a_1^\star x_1), ((a_1 x_1)^\star c_1^\star)^\star)^\star && \text{(using (4.2))}\\
&= (w_1(a_1^\star x_1), (a_1 x_1)^\star c_1^\star)^\star\\
&= (w_1(a_1^\star \cdot x_1), (a_1^\star)^{x_1} c_1^\star)^\star\\
&= ((w_1, a_1^\star) \circ (x_1, c_1^\star))^\star.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
((w_1, a_1^\star) \circ (x_1, c_1^\star)^\star)^\star &= ((w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star)^\star && \text{(using (ii))}\\
&= (w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star && \text{(using (i)).}
\end{aligned}
$$

Hence, for all $(w_1, a_1^\star), (x_1, c_1^\star) \in Y \bowtie E$, (i)-(iii) are satisfied.
(iv)

$$
\begin{aligned}
(w_1, w_1^\star)^\star \circ (w_1, w_1^\star)^\star &= (w_1^\star, w_1^\star) \circ (w_1^\star, w_1^\star)\\
&= (w_1^\star(w_1^\star \cdot w_1^\star), (w_1^\star)^{w_1^\star} w_1^\star)\\
&= (w_1^\star, (w_1 w_1^\star)^\star w_1^\star)\\
&= (w_1^\star, w_1^\star w_1^\star) && \text{(using (2.3))}\\
&= (w_1^\star, w_1^\star)\\
&= (w_1, w_1^\star)^\star.
\end{aligned}
$$

(v) Suppose that $a_1^\star \leq_\ell w_1^\star$ and $a_1^\star \leq_\ell x_1^\star$. Then

$$a_1^\star = w_1^\star a_1^\star = x_1^\star a_1^\star. \tag{4.3}$$

Now we obtain

$$\begin{aligned}
(w_1, a_1^\star)^\star \circ (x_1, a_1^\star)^\star &= (w_1^\star, a_1^\star) \circ (x_1^\star, a_1^\star) \\
&= (w_1^\star(a_1^\star \cdot x_1^\star), (a_1^\star)^{x_1^\star} a_1^\star) \\
&= (w_1^\star(a_1^\star x_1^\star), (a_1 x_1^\star)^\star a_1^\star) \\
&= (w_1^\star a_1^\star x_1^\star, (a_1^\star x_1^\star)^\star a_1^\star) && \text{(using (2.7))} \\
&= (w_1^\star a_1^\star x_1^\star, a_1^\star x_1^\star a_1^\star) && \text{(using (2.4))} \\
&= (x_1^\star a_1^\star w_1^\star, a_1^\star x_1^\star a_1^\star) && \text{(using (2.5))} \\
&= (x_1^\star a_1^\star w_1^\star, a_1^\star w_1^\star a_1^\star) && \text{(using (4.3))} \\
&= (x_1^\star a_1^\star w_1^\star, (a_1^\star w_1^\star)^\star a_1^\star) && \text{(using (2.4))} \\
&= (x_1^\star(a_1^\star w_1^\star), (a_1 w_1^\star)^\star a_1^\star) && \text{(using (2.7))} \\
&= (x_1^\star(a_1^\star \cdot w_1^\star), (a_1^\star)^{w_1^\star} a_1^\star) \\
&= (x_1^\star, a_1^\star) \circ (w_1^\star, a_1^\star) \\
&= (x_1, a_1^\star)^\star \circ (w_1, a_1^\star)^\star.
\end{aligned}$$

□

Lemma 4.3. *Let $(Y \bowtie E, ^\star)$ be as in Lemma 4.2. Put*

$$F = \{(w_1, a_1^\star)^\star : (w_1, a_1^\star) \in Y \bowtie E\},$$

where F is the image of $Y \bowtie E$ under *. Then F is a unary subsemigroup of $(Y \bowtie E, ^\star)$.

Proof. Let $(w_1, a_1^\star)^\star, (x_1, c_1^\star)^\star \in F$. Then

$$\begin{aligned}
(w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star &= (w_1^\star, a_1^\star) \circ (x_1^\star, c_1^\star) \\
&= (w_1^\star(a_1^\star \cdot x_1^\star), (a_1^\star)^{x_1^\star} c_1^\star) \\
&= (w_1^\star(a_1^\star x_1^\star), (a_1 x_1^\star)^\star c_1^\star) \\
&= (w_1^\star(a_1^\star x_1^\star), (a_1^\star x_1^\star)^\star c_1^\star) && \text{(using (2.7))} \\
&= (w_1^\star(a_1^\star x_1^\star)^\star, ((a_1^\star x_1^\star)^\star c_1^\star)^\star) && \text{(using (2.4))} \\
&= ((w_1^\star(a_1^\star x_1^\star)^\star)^\star, ((a_1^\star x_1^\star)^\star c_1^\star)^\star) && \text{(using (2.4))} \\
&= (w_1^\star(a_1^\star x_1^\star)^\star, ((a_1^\star x_1^\star)^\star c_1^\star)^\star)^\star && \text{(using (4.2)).}
\end{aligned}$$

Since $(w_1^\star(a_1^\star x_1^\star)^\star, ((a_1^\star x_1^\star)^\star c_1^\star)^\star) \in Y \bowtie E$, we deduce $(w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star \in F$. Hence, F is a subsemigroup of $Y \bowtie E$.

Next for any $(w_1, a_1^\star)^\star \in F$,

$$((w_1, a_1^\star)^\star)^\star = (w_1^\star, a_1^\star)^\star = ((w_1^\star)^\star, a_1^\star) = (w_1^\star, a_1^\star) = (w_1, a_1^\star)^\star \in F.$$

Thus, F is a unary subsemigroup of $(Y \bowtie E, ^\star)$. □

Theorem 4.4. *Let $F = \{(w_1, a_1^\star)^\star : (w_1, a_1^\star) \in Y \bowtie E\}$ be the unary subsemigroup of $(Y \bowtie E, ^\star)$, constructed in Lemma 4.3. Let $G = \{(w_1, w_1^\star) : w_1 \in Y\}$ and $D_G = \{(w_1, w_1^\star)^\star : (w_1, w_1^\star) \in G\} \subseteq F$. Then $(G, \circ, ^\star)$ is a right Ehresmann subsemigroup of $(Y \bowtie E, ^\star)$ and D_G is the distinguished semilattice of $(G, \circ, ^\star)$. Moreover, there exists an isomorphism between Y and G, which is also an order-isomorphism.*

Proof. Our first objective is to show that the given set G is a unary subsemigroup of $(Y \bowtie E,^\star)$. Let $(w_1, w_1^\star), (x_1, x_1^\star) \in G$. Then $(w_1, w_1^\star) \circ (x_1, x_1^\star) = (w_1(w_1^\star \cdot x_1), (w_1^\star)^{x_1} x_1^\star) = (w_1(w_1^\star x_1), (w_1 x_1)^\star x_1^\star)$. By (2.3), $(w_1, w_1^\star) \circ (x_1, x_1^\star) = (w_1 x_1, (w_1 x_1)^\star x_1^\star)$. By (2.6),

$$(w_1, w_1^\star) \circ (x_1, x_1^\star) = (w_1 x_1, (w_1 x_1)^\star). \tag{4.4}$$

Since $w_1 x_1 \in Y$, it follows that $(w_1, w_1^\star) \circ (x_1, x_1^\star) = (w_1 x_1, (w_1 x_1)^\star) \in G$. Therefore, G is a subsemigroup of $Y \bowtie E$.

Now we show that G is closed under *. Let $(w_1, w_1^\star) \in G$. Then by (4.2), $(w_1, w_1^\star)^\star = (w_1^\star, w_1^\star)$. Since $w_1^\star \in Y$, we deduce $(w_1, w_1^\star)^\star = (w_1^\star, w_1^\star) \in G$. Hence, $(G, \circ,^\star)$ is a unary subsemigroup of $(Y \bowtie E,^\star)$.

Second, we prove that the unary semigroup G satisfies the identities (2.3)−(2.7). For all $(w_1, w_1^\star), (x_1, x_1^\star) \in G$, the identities (2.4) and (2.7) hold in G with the aid of Lemma 4.2 (i) and (ii) respectively.

For proving the identities (2.3), (2.5) and (2.6), let $(w_1, w_1^\star), (x_1, x_1^\star) \in G$. Then we have

$$\begin{aligned}
(w_1, w_1^\star) \circ (w_1, w_1^\star)^\star &= (w_1, w_1^\star) \circ (w_1^\star, w_1^\star) \\
&= (w_1, w_1^\star) \circ (w_1^\star, (w_1^\star)^\star) \\
&= (w_1 w_1^\star, (w_1 w_1^\star)^\star) && \text{(using (4.4))} \\
&= (w_1, w_1^\star).
\end{aligned}$$

Now we obtain

$$\begin{aligned}
(w_1, w_1^\star)^\star \circ (x_1, x_1^\star)^\star &= (w_1^\star, w_1^\star) \circ (x_1^\star, x_1^\star) \\
&= (w_1^\star x_1^\star, (w_1^\star x_1^\star)^\star) && \text{(using (4.4))} \\
&= (w_1^\star x_1^\star, w_1^\star x_1^\star) \\
&= (x_1^\star w_1^\star, x_1^\star w_1^\star) && \text{(using (2.5))} \\
&= (x_1, x_1^\star)^\star \circ (w_1, w_1^\star)^\star.
\end{aligned}$$

Next we see that

$$\begin{aligned}
((w_1, w_1^\star) \circ (x_1, x_1^\star))^\star \circ (x_1, x_1^\star)^\star &= (w_1 x_1, (w_1 x_1)^\star)^\star \circ (x_1, x_1^\star)^\star && \text{(using (4.4))} \\
&= ((w_1 x_1)^\star, (w_1 x_1)^\star) \circ (x_1^\star, x_1^\star) \\
&= ((w_1 x_1)^\star x_1^\star, ((w_1 x_1)^\star x_1^\star)^\star) && \text{(using (4.4))} \\
&= ((w_1 x_1)^\star, ((w_1 x_1)^\star)^\star) && \text{(using (2.6))} \\
&= ((w_1 x_1)^\star, (w_1 x_1)^\star) \\
&= (w_1 x_1, (w_1 x_1)^\star)^\star && \text{(using (4.4))} \\
&= ((w_1, w_1^\star) \circ (x_1, x_1^\star))^\star.
\end{aligned}$$

Therefore, for any $(w_1, w_1^\star), (x_1, x_1^\star) \in G$, (2.3), (2.5) and (2.6) are satisfied. Thus, $(G, \circ,^\star)$ is a right Ehresmann subsemigroup of $(Y \bowtie E,^\star)$. In this case, the set $D_G = \{(w_1, w_1^\star)^\star : (w_1, w_1^\star) \in G\}$ is the distinguished semilattice of G.

Next we define a map $\alpha : Y \to G$ by

$$\alpha(w_1) = (w_1, w_1^\star). \tag{4.5}$$

We show that α is an isomorphism. It is not tricky to check that α is well-defined. One can verify that for all $w_1, x_1 \in Y$, $\alpha(w_1 x_1) = \alpha(w_1) \circ \alpha(x_1)$. For all $w_1 \in Y$, $\alpha(w_1^\star) = (w_1^\star, (w_1^\star)^\star) = (w_1^\star, w_1^\star) = (w_1, w_1^\star)^\star = (\alpha(w_1))^\star$. So α is a right Ehresmann semigroup morphism. Also, α is surjective and one-one. Hence, Y and G are isomorphic.

Lemma 4.5. *Assume that $w_1, x_1 \in Y$ and $(w_1, w_1^\star), (x_1, x_1^\star) \in G$. Then $w_1 \leq_\ell x_1$ in Y implies $(w_1, w_1^\star) \leq_\ell (x_1, x_1^\star)$ in G, and $(w_1, w_1^\star) \leq_\ell (x_1, x_1^\star)$ in G implies $w_1 \leq_\ell x_1$ and $w_1^\star \leq_\ell x_1^\star$ in Y.*

Proof. Assume that $w_1 \leq_\ell x_1$ in Y. Then we have

$$w_1 = x_1 w_1^\star. \tag{4.6}$$

In G, we get

$$\begin{aligned}(x_1, x_1^\star) \circ (w_1, w_1^\star)^\star &= (x_1, x_1^\star) \circ (w_1^\star, w_1^\star)\\ &= (x_1(x_1^\star \cdot w_1^\star), (x_1^\star)^{w_1^\star} w_1^\star)\\ &= (x_1 x_1^\star w_1^\star, (x_1 w_1^\star)^\star w_1^\star)\\ &= (x_1 w_1^\star, (x_1 w_1^\star)^\star w_1^\star)\\ &= (w_1, w_1^\star w_1^\star) \qquad \text{(using (4.6))}\\ &= (w_1, w_1^\star).\end{aligned}$$

So $(w_1, w_1^\star) \leq_\ell (x_1, x_1^\star)$.

Suppose that $(w_1, w_1^\star) \leq_\ell (x_1, x_1^\star)$ in G. Then we obtain

$$\begin{aligned}(w_1, w_1^\star) &= (x_1, x_1^\star) \circ (w_1, w_1^\star)^\star\\ &= (x_1, x_1^\star) \circ (w_1^\star, w_1^\star)\\ &= (x_1(x_1^\star \cdot w_1^\star), (x_1^\star)^{w_1^\star} w_1^\star)\\ &= (x_1 x_1^\star w_1^\star, (x_1^\star)^{w_1^\star} w_1^\star)\\ &= (x_1 w_1^\star, (x_1^\star)^{w_1^\star} w_1^\star).\end{aligned}$$

Then we have $w_1 = x_1 w_1^\star$. Therefore, $w_1 \leq_\ell x_1$ in Y. Since $w_1 = x_1 w_1^\star$, we have $w_1^\star = (x_1 w_1^\star)^\star$. By (2.7), $w_1^\star = (x_1^\star w_1^\star)^\star$. By (2.4), $w_1^\star = x_1^\star w_1^\star$. Then $w_1^\star = x_1^\star (w_1^\star)^\star$. Thus, $w_1^\star \leq_\ell x_1^\star$ in Y. □

Now we return to prove our Theorem 4.4 by using the above lemma. To prove that Y is order-isomorphic to G, we merely need to prove that

$$w_1 \leq_\ell x_1 \text{ in } Y \Longleftrightarrow \alpha(w_1) \leq_\ell \alpha(x_1) \text{ in } G. \tag{4.7}$$

With the aid of Lemma 4.5, (4.7) is proved. Hence, α is an order-isomorphism. The proof is completed. □

Throughout this article, $\inf A$ *(*$\sup A$*) means the infimum (supremum) of a non-empty set* A*.*

Equipped with the notation, given in Theorem 4.4, we reveal some properties of the set G in the following result.

Proposition 4.6. *Consider* $\alpha : Y \to G$ *the order-isomorphism, proved in the proof of Theorem 4.4. Let* $\emptyset \neq U \subseteq Y$*. Then*

(i) *if* $\sup U$ *exists in* Y*, then* $\sup(\alpha(U))$ *exists in* G *and* $\sup(\alpha(U)) = \alpha(\sup U)$*;*
(ii) *if* $\inf U$ *exists in* Y*, then* $\inf(\alpha(U))$ *exists in* G *and* $\inf(\alpha(U)) = \alpha(\inf U)$*;*
(iii) *if* U *is a down-set in* Y*, then so is* $\alpha(U)$ *in* G*;*
(iv) *if* U *is an up-set in* Y*, then so is* $\alpha(U)$ *in* G*.*

Proof. We only prove (i) and (iii). The proofs of (ii) and (iv) are similar to that of (i) and (iii) respectively.

To prove (i), let $k = \sup U$. Then for all $u \in U$, $u \leq_\ell k$. Then by Lemma 4.5, $\alpha(u) \leq_\ell \alpha(k)$ for all $\alpha(u) \in \alpha(U)$. So $\alpha(k)$ is an upper bound of $\alpha(U)$. Let $c \in G$ be any upper bound of $\alpha(U)$. Since $c = \alpha(s)$ for some $s \in Y$, it follows that $\alpha(u) \leq_\ell \alpha(s)$ for all $\alpha(u) \in \alpha(U)$. Then by Lemma 4.5, we obtain $u \leq_\ell s$ for all $u \in U$. As s is an upper bound of U, we deduce $k \leq_\ell s$. Then $\alpha(k) \leq_\ell \alpha(s) = c$. So $\alpha(k) = \sup(\alpha(U))$, whence $\sup(\alpha(U)) = \alpha(\sup U)$.

To prove (iii), let $t \in \alpha(U)$ and $c \in G$, where $t = \alpha(b)$ for some $b \in U \subseteq Y$, and $c = \alpha(a)$ for some $a \in Y$. Suppose that $c \leq_\ell t$. Since $\alpha(a) \leq_\ell \alpha(b)$ in G. Then we obtain $a \leq_\ell b$

in Y. Since U is a down set such that $b \in U$, we have $a \in U$. Then $\alpha(a) = c \in \alpha(U)$. Therefore, $\alpha(U)$ is a down-set in G. □

4.2. Zappa-Szép product of a right Ehresmann semigroup with zero and its distinguished semilattice

Here we form the Zappa-Szép product $Y \bowtie E$, where Y is a right Ehresmann semigroup with zero 0, and E is the distinguished semilattice of Y such that $0 \in E$. Then we define the unary operation (as defined on Y) on $Y \bowtie E$, and find a subset of the image of unary operation, which is the distinguished semilattice of a right Ehresmann subsemigroup G' of $Y \bowtie E$ with zero. We also show that G' is isomorphic to Y. Then our aim is to discuss some properties of the set G' from the order-theoretic perspective, which are different from that of G, studied in the previous subsection.

Lemma 4.7. *Let Y be a right Ehresmann semigroup with zero 0 and distinguished semilattice E such that $0 \in E$. Define the maps $E \times Y \to Y$ and $E \times Y \to E$ by $(a_1^\star, w_1) \mapsto a_1^\star \cdot w_1 = a_1^\star w_1$ and $(a_1^\star, w_1) \mapsto (a_1^\star)^{w_1} = (a_1 w_1)^\star$ respectively. Then we have $Y \bowtie E$ the Zappa-Szép product of Y and E.*

Proof. We first note that for any $w_1 \in Y$, $0 \cdot w_1 = 0w_1 = 0$, and for any $a_1^\star \in E$, $(a_1^\star)^0 = (a_1 0)^\star = 0^\star = 0$. It can be checked that the given maps satisfy the Zappa-Szép product axioms (Z1)-(Z4). Then $Y \times E$ is a semigroup with zero $(0, 0)$ under the binary operation $\circ$, defined in (4.1). Hence, we have the semigroup $(Y \times E, \circ)$, called the Zappa-Szép product of Y and E, denoted by $Y \bowtie E$. □

Lemma 4.8. *Let $Y \bowtie E$ be the Zappa-Szép product of Y and E, formed in Lemma 4.7. Let the unary operation * (as defined on Y) be defined on $Y \bowtie E$ by*

$$(w_1, a_1^\star)^\star = (w_1^\star, a_1^\star). \tag{4.8}$$

Then the unary semigroup $(Y \times E, \circ, ^\star)$=$(Y \bowtie E, ^\star)$ satisfies:

(i) $((w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star)^\star = (w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star$;
(ii) $((w_1, a_1^\star)^\star \circ (x_1, c_1^\star))^\star = ((w_1, a_1^\star) \circ (x_1, c_1^\star))^\star$;
(iii) $((w_1, a_1^\star) \circ (x_1, c_1^\star)^\star)^\star = (w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star$;
(iv) $(w_1, w_1^\star)^\star \circ (w_1, w_1^\star)^\star = (w_1, w_1^\star)^\star$;
(v) $(w_1, a_1^\star)^\star \circ (x_1, a_1^\star)^\star = (x_1, a_1^\star)^\star \circ (w_1, a_1^\star)^\star$ *whenever* $a_1^\star \leq_\ell w_1^\star$ *and* $a_1^\star \leq_\ell x_1^\star$, *where* $a_1^\star, w_1^\star, x_1^\star \neq 0$.

Proof. By Lemma 2.4, and by (4.8), $(0, 0)^\star = (0, 0^\star)^\star = (0^\star, 0^\star) = (0, 0)$. One can check that * is well-defined.

Now we prove (i)-(v). For all non-zero $(w_1, a_1^\star), (x_1, c_1^\star) \in Y \bowtie E$, the identities (i)-(iii) can be proved in the same way that (i)-(iii) are proved in Lemma 4.2. If $(0, 0) \in \{(w_1, a_1^\star), (x_1, c_1^\star)\}$, then (i)-(iii) can be easily proved. Thus, for all $(w_1, a_1^\star), (x_1, c_1^\star) \in Y \bowtie E$, (i)-(iii) are satisfied.

(iv) We deal with the cases: **(a)** $w_1 \neq 0$, **(b)** $w_1 = 0$.

(a) If $w_1 \neq 0$, then by Lemma 2.4, $w_1^\star \neq 0$. Then (iv) can be proved in the same way that (iv) is proved in Lemma 4.2.

(b) Let $w_1 = 0$. Then by Lemma 2.4, $w_1^\star = 0$. It can be easily verified that $(w_1, w_1^\star)^\star \circ (w_1, w_1^\star)^\star = (w_1, w_1^\star)^\star$.

(v) It can be proved in the same way that (v) is proved in Lemma 4.2. □

Lemma 4.9. *Let $(Y \bowtie E, ^\star)$ be as in Lemma 4.8. Put*

$$F = \{(w_1, a_1^\star)^\star : (w_1, a_1^\star) \in Y \bowtie E\},$$

where F is the image of $Y \bowtie E$ under *. Then F is a unary subsemigroup of $(Y \bowtie E, ^\star)$ with a zero element $(0, 0)$ such that $(0, 0)^\star = (0, 0)$.

Proof. Since $(0,0) \in Y \bowtie E$, we deduce $(0,0)^\star \in F$. Then by (4.8), $(0^\star, 0) \in F$. So $(0,0) \in F$.

Let $(w_1, a_1^\star)^\star, (x_1, c_1^\star)^\star \in F$ be such that $(w_1, a_1^\star)^\star, (x_1, c_1^\star)^\star \neq (0,0)$. Then $(w_1, a_1^\star)^\star \circ (x_1, c_1^\star)^\star \in F$ can be proved in the same way that we proved in Lemma 4.3. It can also be shown that $(w_1, a_1^\star)^\star \circ (0,0) = (0,0) \circ (w_1, a_1^\star)^\star = (0,0) \circ (0,0) = (0,0)$. Hence, F is a subsemigroup of $Y \bowtie E$ with zero $(0,0)$.

Next for any non-zero $(v_1, a_1^\star)^\star \in F$, $((v_1, a_1^\star)^\star)^\star = (v_1^\star, a_1^\star)^\star = ((v_1^\star)^\star, a_1^\star) = (v_1^\star, a_1^\star) = (v_1, a_1^\star)^\star \in F$. Also, $(0,0)^\star = (0,0) \in F$. Thus, F is a unary subsemigroup of $(Y \bowtie E, ^\star)$ with zero $(0,0)$ such that $(0,0)^\star = (0,0)$. □

Theorem 4.10. *Let $F = \{(w_1, a_1^\star)^\star : (w_1, a_1^\star) \in Y \bowtie E\}$ be the unary subsemigroup of $(Y \bowtie E, ^\star)$, constructed in Lemma 4.9. Let $G' = \{(w_1, w_1^\star) : w_1 \in Y\}$ and $D_{G'} = \{(w_1, w_1^\star)^\star : (w_1, w_1^\star) \in G'\} \subseteq F$. Then $(G', \circ, ^\star)$ is a right Ehresmann subsemigroup of $(Y \bowtie E, ^\star)$ with zero $(0,0)$ such that $(0,0)^\star = (0,0)$, and $D_{G'}$ is the distinguished semilattice of $(G', \circ, ^\star)$ such that $(0,0) \in D_{G'}$. Moreover, there exists an isomorphism between Y and G'.*

Proof. First, we show that the given set G' is a unary subsemigroup of $(Y \bowtie E, ^\star)$ with zero $(0,0)$ such that $(0,0)^\star = (0,0)$. Since $0 \in Y$, by the definition of G', it follows that $(0,0) = (0, 0^\star) \in G'$. Let $(w_1, w_1^\star), (x_1, x_1^\star) \in G'$ be such that $(w_1, w_1^\star), (x_1, x_1^\star) \neq (0,0)$. Then $(w_1, w_1^\star) \circ (x_1, x_1^\star) \in G'$ can be proved in the same way that we proved in Theorem 4.4. It can also be verified that $(w_1, w_1^\star) \circ (0,0) = (0,0) \circ (w_1, w_1^\star) = (0,0) \circ (0,0) = (0,0)$. Therefore, G' is a subsemigroup of $Y \bowtie E$ with zero $(0,0)$.

Now we show that G' is closed under *. Let $(w_1, w_1^\star) \in G'$ be such that $(w_1, w_1^\star) \neq (0,0)$. Then by (4.8), $(w_1, w_1^\star)^\star = (w_1^\star, w_1^\star)$. Since $w_1^\star \in Y$, we have $(w_1^\star, (w_1^\star)^\star) \in G'$. So $(w_1^\star, w_1^\star) \in G'$. Therefore, $(w_1, w_1^\star)^\star \in G'$. Also, we have $(0,0)^\star = (0,0) \in G'$. Hence, $(G', \circ, ^\star)$ is a unary subsemigroup of $(Y \bowtie E, ^\star)$ containing zero $(0,0)$ such that $(0,0)^\star = (0,0)$.

Second, we prove that the unary semigroup G' satisfies the identities (2.3)$-$(2.7). For all $(w_1, w_1^\star), (x_1, x_1^\star) \in G'$, the identities (2.4) and (2.7) hold in G' with the aid of Lemma 4.8 (i) and (ii) respectively. For any non-zero $(w_1, w_1^\star), (x_1, x_1^\star) \in G'$, (2.3), (2.5) and (2.6) can be proved in the same way that we proved in Theorem 4.4. It can also be checked that if $(0,0) \in \{(w_1, w_1^\star), (x_1, x_1^\star)\}$, then (2.3), (2.5) and (2.6) also hold in G'. Therefore, for all $(w_1, w_1^\star), (x_1, x_1^\star) \in G'$, (2.3), (2.5) and (2.6) are satisfied. Thus, $(G', \circ, ^\star)$ is a right Ehresmann subsemigroup of $(Y \bowtie E, ^\star)$ containing zero $(0,0)$ with $(0,0)^\star = (0,0)$. In this case, the set $D_{G'} = \{(w_1, w_1^\star)^\star : (w_1, w_1^\star) \in G'\}$ is the distinguished semilattice of G'. Since $(0,0) \in G'$, by the definition of $D_{G'}$, $(0,0)^\star = (0,0) \in D_{G'}$.

Next we define a map $\alpha : Y \to G'$ by

$$\alpha(w_1) = (w_1, w_1^\star). \tag{4.9}$$

We show that α is an isomorphism. It is immediate that $\alpha(0) = (0,0)$, and that α is well-defined. One can prove that for all $w_1, x_1 \in Y$, $\alpha(w_1 x_1) = \alpha(w_1) \circ \alpha(x_1)$. It can be checked that for any $w_1 \in Y$, $\alpha(w_1^\star) = (\alpha(w_1))^\star$. So α is a right Ehresmann semigroup morphism. Also, α is surjective and one-one. Hence, Y and G' are isomorphic. The proof is completed. □

Equipped with the notation, given in Theorem 4.10, we prove all the remaining results of this section.

Lemma 4.11. *If $0 \neq k_1 \in Y$, then $(k_1, k_1^\star) \neq (0,0)$ in G' such that $k_1^\star \neq 0$ in Y, and if $(0,0) \neq (k_1, k_1^\star) \in G'$, then $k_1 \neq 0$ (and hence $k_1^\star \neq 0$) in Y.*

Proof. Let $0 \neq k_1 \in Y$. Then by Lemma 2.4, $k_1^\star \neq 0$ in Y. Then we have $(k_1, k_1^\star) \neq (0,0)$ in G'. Conversely, let $(0,0) \neq (k_1, k_1^\star) \in G'$. On the contrary, suppose that $k_1 = 0$ in Y.

Then by Lemma 2.4, $k_1^\star = 0$ in Y. So $(k_1, k_1^\star) = (0,0)$—a contradiction. Therefore, $k_1 \neq 0$ in Y. Then $k_1^\star \neq 0$ in Y. □

Lemma 4.12.

(i) *Let $w_1, x_1 \in Y$ be such that $w_1, x_1 \neq 0$. Then $w_1 \leq_\ell x_1$ in Y implies $(w_1, w_1^\star) \leq_\ell (x_1, x_1^\star)$ in G';*

(ii) *Let $(w_1, w_1^\star), (x_1, x_1^\star) \in G'$ be such that $(w_1, w_1^\star), (x_1, x_1^\star) \neq (0,0)$. Then $(w_1, w_1^\star) \leq_\ell (x_1, x_1^\star)$ in G' implies $w_1 \leq_\ell x_1$ and $w_1^\star \leq_\ell x_1^\star$ in Y.*

Proof. (i) If $w_1, x_1 \in Y$ with $w_1, x_1 \neq 0$, then by Lemma 4.11, $(w_1, w_1^\star), (x_1, x_1^\star) \neq (0,0)$ in G' such that $w_1^\star, x_1^\star \neq 0$ in Y. The proof of the implication is the same as that of implication of Lemma 4.5.

(ii) If $(w_1, w_1^\star), (x_1, x_1^\star) \in G'$ with $(w_1, w_1^\star), (x_1, x_1^\star) \neq (0,0)$, then by Lemma 4.11, $w_1, x_1 \neq 0$ (and hence $w_1^\star, x_1^\star \neq 0$) in Y. The proof of the implication is the same as that of implication of Lemma 4.5. □

Next, from the perspective of order theory, we reveal some properties of the set G'.

Proposition 4.13.

(i) *a_1 is an atom of Y if and only if $(a_1, a_1^\star)$ is an atom of G';*

(ii) *Y is an atomic poset if and only if G' is an atomic poset.*

Proof. (i) Suppose that a_1 is an atom of Y. Then $a_1 \neq 0$. By Lemma 4.11, $(a_1, a_1^\star) \neq (0,0)$ in G'. Suppose that for all $(0,0) \neq (k_1, k_1^\star) \in G'$,

$$(k_1, k_1^\star) \leq_\ell (a_1, a_1^\star).$$

Since $(k_1, k_1^\star) \neq (0,0)$, by Lemma 4.11, we deduce $k_1 \neq 0$ in Y. By Lemma 4.12 (ii), $k_1 \leq_\ell a_1$ in Y. Since a_1 is an atom, it follows that $k_1 = a_1$. Then $k_1^\star = a_1^\star$. So $(k_1, k_1^\star) = (a_1, a_1^\star)$. Thus, $(a_1, a_1^\star)$ is an atom of G'. Conversely, let $(a_1, a_1^\star)$ be an atom of G'. Then $(a_1, a_1^\star) \neq (0,0)$. By Lemma 4.11, $a_1 \neq 0$ in Y. Suppose that for all $0 \neq k_1 \in Y$,

$$k_1 \leq_\ell a_1.$$

Since $k_1 \neq 0$, by Lemma 4.11, we deduce $(k_1, k_1^\star) \neq (0,0)$ in G'. By Lemma 4.12 (i), $(k_1, k_1^\star) \leq_\ell (a_1, a_1^\star)$ in G'. Since $(a_1, a_1^\star)$ is an atom, it follows that $(k_1, k_1^\star) = (a_1, a_1^\star)$. Then $k_1 = a_1$. Accordingly, a_1 is an atom of Y.

(ii) Let Y be an atomic poset. Therefore, for any $0 \neq k_1 \in Y$, there is an atom $a_1 \in Y$ with $a_1 \leq_\ell k_1$. Then by Lemma 4.11, $(0,0) \neq (k_1, k_1^\star) \in G'$. By (i), $(a_1, a_1^\star)$ is an atom of G'. Then by Lemma 4.12 (i), for all $(0,0) \neq (k_1, k_1^\star) \in G'$,

$$(a_1, a_1^\star) \leq_\ell (k_1, k_1^\star).$$

This implies that every non-zero element of G' dominates an atom of G'. Thus, G' is atomic. Conversely, if G' is an atomic poset, then for all $(0,0) \neq (k_1, k_1^\star) \in G'$, there is an atom $(a_1, a_1^\star) \in G'$ with $(a_1, a_1^\star) \leq_\ell (k_1, k_1^\star)$. Then by Lemma 4.11, $0 \neq k_1 \in Y$. By (i), a_1 is an atom of Y. By Lemma 4.12 (ii), for all $0 \neq k_1 \in Y$,

$$a_1 \leq_\ell k_1.$$

Therefore, every non-zero element of Y dominates an atom of Y. Hence, Y is atomic. □

4.3. Example

We end this section by presenting an example of Zappa-Szép product.

Example 4.14. Let V be a non-empty set containing at least two elements. Suppose that A and B are non-empty subsets of V such that $A \cup B = V$ and $A \cap B = \emptyset$. Put

$$M = Z = \{\emptyset, A, B, V\}.$$

Then $(M, \cup, ^\star)$ and $(Z, \cup, ^\star)$ are right Ehresmann semigroups by defining $m^\star = m$ for all $m \in M$, and $z^\star = z$ for all $z \in Z$. Define the maps $Z \times M \to M$ and $Z \times M \to Z$ by $(z, m) \mapsto z \cdot m = m \cup z$ and $(z, m) \mapsto z^m = z \setminus m$ respectively. It is handy to verify (Z1)-(Z4). The binary operation $\circ$ on $M \times Z$ is defined by

$$(m, z) \circ (m', z') = (m(z \cdot m'), z^{m'} z').$$

Then by (Z1)-(Z4), $\circ$ is associative. Hence, we have the semigroup $(M \times Z, \circ)$, called the Zappa-Szép product $M \bowtie Z$.

Example 4.14 can be found in the contexts of inverse semigroups and left restriction semigroups in [16] and [28] respectively, in which $M = Z = P(X)$ (a power set of a set X).

Acknowledgment. I thank to the anonymous referees for their insightful comments and valuable suggestions to enhance merit in this article.

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