B-Scroll Weıngarten Surface

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Abstract

In this paper, firstly, anchor curve, which is helix curve, of the B- scroll surface was shown as a Weingarten surface in this study. Secondly, shape operator, fundamental forms, asymptotic curve, Gaussian curvature, mean curvature and Christoffel symbols of the B- scroll Weingarten surface were calculated. Lastly, it was indicated that B- scroll Weingarten surface has not parallel surface.

Key Words: B- scroll, Weingarten surface, Asymptotic curve, Gaussian curvature, Mean curvature, Christoffel symbols, Anchor curve, Striction curve

Mathematics Subject Classification (2010): 53A04, 53A05.

Özet

Bu çalışmada, ilk olarak dayanak eğrisi helis olarak alındığında oluşan B- scroll yüzeyin bir Weingarten yüzey olduğu gösterildi. Daha sonra, oluşan bu yüzeyin şekil operatörü, temel formları, asimptotik eğrisi, Gaus eğriliği, ortalama eğrilik ve Christoffel sembolleri hesaplanmıştır. Son olarak, B- scroll Weingarten yüzeyin bir paralel yüzey olmadığı gösterilmiştir.

Anahtar Kelimeler: B- scroll, Weingarten yüzey, Asymptotic eğri, Gaussian eğrilik, Ortalama eğrilik, Christoffel sembolleri, Striction eğrisi

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1. INTRODUCTION

Three-dimensional space is often used in mathematics without being formally defined. Looking at the corner of a room, one can picture the familiar process by which rectangular coordinate axes are introduced and three numbers are measured to describe the position of each point. A precise definition that realizes this intuitive picture may be obtained by this device instead of saying that three numbers describe the position of a point. Examples of surfaces abound in everyday life: Balloons, tubes, cans, the surface of our planet earth are all physical models of surfaces. In order to study the geometry of these objects, one needs coordinates to make calculations. Of course, all these surfaces can be thought of as embedded in Euclidean space $E^3 -$ which means 3 coordinates. But just as a (space) curve needs only 1 coordinate, the very definition of a surface is that it can be described using just two coordinates: We are used to describe points on the surface of the earth by the two geographical coordinates: longitude and latitude and One can produce cylindrical cans by rolling a plane (2-dimensional) piece of metal. The surface is said to be “ruled” if it is generated by moving a straight line continuously in Euclidean space (van–Brunt & Grant 1996).

Ruled surfaces are one of the simplest objects in geometric modeling. The results is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight. The basis notions about ruled surfaces in are given in (Hacısalihoğlu 1994). It is well-known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. B-scrolls are non-developable ruled surfaces. B-scroll over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves, (Graves 1979). Kılıçoğlu defined and worked on involute curves and involute B-scroll of any curve in E^3, (Kılıçoğlu 2011). When β is considered as the involute of the α curve, Frenet vectors, curvature and torsion of β are given, respectively depending on the angle, φ which is between W Darboux vector and B binormal vector of α curve. In this case Gaussian and mean curvatures, I. and II. Fundamental forms of B-scroll generated by involute curve have been calculated. Finally the involute B-scrolls generated by helix curve have drawn application, (Şenyurt 2014). A surface in E^3 is called a Weingarten surface if there is some relation between its two principal curvature K and H are not independent of one another, that is, if there is a smooth function Φ of two variables such that $\Phi(H,K)=0$. Weingarten surface is a classical topic in differential geometry, as introduced by Weingarten, ((Weingarten 1861), (Weingarten 1863),). Applications of Weingarten surfaces on computer aided design and shape investigation can seen in (van–Brunt & Grant 1996). E. Beltrami and U. Dini proved that a helicoidal ruled surface is the only non-developable Weingarten ruled surface in E^3, ((Beltrami 1865-1866), (Dini 1865-1866)). This result is later reproved by W. Kühnel, (Kühnel 1994). It has been obtained that parallel surfaces of a ruled Weingarten surface are also Weingarten surfaces. It has been shown that the parallel surfaces of a developable ruled surface are ruled Weingarten surfaces, (Ziya Savcı 2011). Sipus studied ruled Weingarten surfaces in Galilean space, (Sipus 2008). Weingarten surfaces are surfaces of revolution, tubes around curves where one principal curvature is constant, helicodial surfaces, surfaces of constant Gaussian curvature and surfaces of constant mean curvature, (Kühnel & Steller 2005).

2 . PRELIMINARIES

The Euclidean 3-space E^3 be inner product given by $\langle , \rangle = x_1^2 + x_2^3 + x_3^2$ where $(x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame. For an arbitrary curve $\alpha \in E^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by (Do Carmo 1976)

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \tag{2.1}$$

A regular parametrization of a subset $M \subset \mathbb{R}^3$ is a one-to-one function

$$\varphi : U \to M \subset \mathbb{R}^3 \textit{ so that } \varphi_u \wedge \varphi_v \neq 0$$

for some open set $U \subset \mathbb{R}^2$. A connected subset $M \subset \mathbb{R}^3$ is called a **surface** if each point has a neighborhood that is regularly parametrized. There are two unit vectors orthogonal to the tangent plane $T_p M$. Given a regular parametrization φ we know that $\varphi_u \wedge \varphi_v$ is a nonzero vector orthogonal to the plane spanned by φ_u and φ_v; we obtain the corresponding unit vector by taking

$$N = \frac{\varphi_u \wedge \varphi_v}{\| \varphi_u \wedge \varphi_v \|}.$$

This is called the unit normal of the parametrized surface, (O'Neill 1997).

Definition 1.1 If p is a point of M, then for each tangent vector X to at p, let

$$S_p(X) = D_X N,$$

where N is a unit normal vector field on a neighborhood of p in M. S_p is called the shape operator of M at p derived from N, (O'Neill 1997).

It is defined the first fundamental form, the second fundamental form and third **fundamental form**, as follow, respectively. If $X,Y \in \chi(M)$, we have

$$\begin{aligned} I(X,Y) &= \langle X,Y\rangle = Edu^2 + 2Fdudv + Gdv^2, \\ II(X,Y) &= \langle S(X),Y\rangle = \ell du^2 + 2mdudv + ndv^2, \\ III(X,Y) &= \langle S(X),S(Y)\rangle = edu^2 + 2fdudv + gdv^2. \end{aligned} \tag{2.2}$$

Working in a parametrization, we have the natural basis $\{\varphi_u,\varphi_v\}$, and so it is defined

$$\begin{aligned} &E = \langle\varphi_u,\varphi_u\rangle, F = \langle\varphi_u,\varphi_v\rangle, G = \langle\varphi_v,\varphi_v\rangle, \\ &\ell = -\langle\varphi_{uu},N\rangle, m = -\langle\varphi_{uv},N\rangle, n = -\langle\varphi_{vv},N\rangle, \\ &e = \langle N_u,N_u\rangle, f = \langle N_u,N_v\rangle, g = \langle N_v,N_v\rangle. \end{aligned} \tag{2.3}$$

where $E,F,G,\ \ell,m,n\ and\ e,f,g$ are the coefficients of the first, second and third fundamental form, respectively, (Shifrin 2010).

Definition 1.2 Let X_p be a unit vector tangent to $M \subset \mathrm{P}^3$ at a point p. Then the number $k(X_p) = \langle S(X_p),X_p\rangle$ is called the normal curvature of M in the X_p direction, (O'Neill 1997).

Definition 1.3 A regular curve α in $M \subset \mathrm{P}^3$ is an asymptotic curve provided its velocity α' always points in an asymptotic direction. Thus α asymptotic curve if and only if

$$k(\alpha') = \langle S(\alpha'),\alpha'\rangle = 0,$$ (O'Neill 1997)..

Definition 1.4 Gaussian and mean curvatures of the M surface is

$$K = detS_p = \frac{\ell n - m^2}{EG-F^2}\ and\ H = izS_p = \frac{\ell G - 2mF + nE}{EG-F^2}. \tag{2.4}$$

where $E,F,G\ and\ \ell,m,n$ are the coefficients of the first and second fundamental form, S_p is the shape operator of M at a point p, ((Sabuncuoğlu 2001), (Hacısalihoğlu 1994)).

Definition 1.5 M is $(n-1)$-space riemann manifold, $\Gamma_{ij}^k : M \to \mathrm{P}\ (i,j,k=1,\ldots,n)$ the function is called Christoffel symbols, (Shifrin 2010).

$$\Gamma_{11}^{1} = \frac{E_u G - 2F_u F + EvF}{2(EG - F^2)}, i = j = k = 1$$

$$\Gamma_{11}^{2} = \frac{2F_u E - E_V E - E_u F}{2(EG - F^2)}, i = j = 1, k = 2$$

$$\begin{aligned} \Gamma_{12}^{1} &= \frac{GE_v - FG_u}{2(EG - F^2)}, i = k = 1, j = 2 \\ \Gamma_{12}^{2} &= \frac{EG_u - FE_v}{2(EG - F^2)}, i = 1, j = k = 2 \\ \Gamma_{22}^{1} &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, i = j = 2, k = 1 \\ \Gamma_{22}^{2} &= \frac{G_v E - 2FF_v + FG_u}{2(EG - F^2)}, i = j = k = 2. \end{aligned} \tag{2.5}$$

These values are the provision on the M surface of the Christoffel symbols, (Gray 2006).

A differentiable one-parameter family of (straight) lines $\alpha(u), X(u)$ is a correspondence that assigns to each $u \in I$ a point $\alpha(u) \in \mathrm{P}^3$ and a vector $X(u) \in \mathrm{P}^3, X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ depend differentiably on u. For each $u \in I$, the line L which passes through $\alpha(u)$ and is parallel to $X(u)$ is called the line of the family at u. Given a one-parameter family of lines $\alpha(u), X(u)$ the parametrized surface

$$\varphi(u,v) = \alpha(u) + vX(u), u \in I, v \in \mathrm{P} \tag{2.6}$$

is called the **ruled surface** generated by the family $\alpha(u), X(u)$. The lines L, are called the rulings, and the curve $\alpha(u)$ is called a anchor of the surface φ, (Do Carmo 1976).

Definition 1.6 Let striction curve and generator vector of $\varphi(u,v)$ ruled surface be $\overline{\alpha}(s)$ and $X(s)$, respectively. In this situation, parametric equations of striction curve is

$$\overline{\alpha}(s) = \alpha(s) - \frac{\langle X'(s), \alpha'(s)\rangle}{\|X'(s)\|^2} X(s). \tag{2.7}$$

Theorem 1.1 S is the matrix corresponding to the shape opeerator of B-scroll surface, the matrix is ,(Kılıçoğlu 2006).

$$S=\begin{pmatrix} \dfrac{\kappa-v\tau'+v^2\kappa\tau^2}{(1+v^2\tau^2)^{\frac{3}{2}}} & \dfrac{-\tau}{1+v^2\tau^2} \\ \\ \dfrac{-\tau}{1+v^2\tau^2} & 0 \end{pmatrix} \tag{2.8}$$

Theorem 1.2 Gaussian and mean curvatures of the B-scroll surface is

$$K=\frac{-\tau^2}{(1+v^2\tau^2)^2} \text{ and } H=\frac{v^2\tau^2\kappa+\kappa-v\tau'}{(1+v^2\tau^2)^{\frac{3}{2}}} \tag{2.9}$$

((Kılıçoğlu 2006) , (Şenyurt 2014)).

Let M_1, M_2 be two surface oriented by U and let N be a unit normal vector field. If there is a mapping $f: M_1 \to M_2$ to be

$$f(P)=P+rN, r\in \mathrm{P}, P\in M \tag{2.10}$$

then M_2 is called the **parallel surface** of M_1, ((Kılıçoğlu 2006), (Hacısalihoğlu 1994), (O'Neill 1997)).

Definition 1.7 Surface M to be the Weingarten surfaces, having a nontrivial functional relation between their Gaussian K and mean curvature H, (Sipus 2008)

$$\Phi(H,K)=0.$$

Theorem 1.3 Let $M\subset \mathrm{E}^3$ be a surface. For surfaces of class M, Weingarten surfaces are exactly those surfaces whose gradients of K and H are linearly dependent which gives , (Sipus 2008).

$$K_uH_v-K_vH_u=0 \tag{2.11}$$

Theorem 1.4 The parallel surface of a ruled Weingarten surface are also Weingarten surfaces. The parallel surface of a developable ruled surface are ruled Weingarten surfaces, (Ziya Savcı 2011).

3. B-SCROLL WEINGARTEN SURFACE

In this subsection, It was shown to be a Weingarten surface of the B-scroll surface and we found that some of its most important geometric properties belong to the this surface. Later, the relation between striction curve and anchor curve was given. Lastly, we said that B-scroll Weingarten surface has not parallel surface.

Theorem 3.1 B-scroll surface is Weingarten surface $\Rightarrow$ κ and τ values are constant.

Proof. Parametric equation of the B-scroll surface is

$$\varphi(u,v) = \alpha(u) + vB(u). \tag{3.1}$$

From the equation (2.9), Gaussian and mean curvatures of the B-scroll surface are

$$K = \frac{-\tau^2}{(1+v^2\tau^2)^2} \text{ and } H = \frac{v^2\tau^2\kappa + \kappa - v\tau'}{(1+v^2\tau^2)^{\frac{3}{2}}}, \tag{3.2}$$

respectively. The (2.11) equation is needed for Weingarten surface to be a B-scroll surface. In the (3.2) equation, the derivatives of K, H with respect to u and v are

$$K_u = \frac{-2\tau\tau'(1-v^2\tau^2)}{(1+v^2\tau^2)^3}, K_v = \frac{4v\tau^4}{(1+v^2\tau^2)^3},$$

$$H_u = \frac{[v^2(2\tau\tau'\kappa + \kappa'\tau^2) + \kappa' - v\tau''](1+v^2\tau^2) - 3\tau\tau'v^2(v^2\tau^2\kappa + \kappa - v\tau')}{(1+v^2\tau^2)^{\frac{5}{2}}},$$

$$H_v = \frac{(2v\tau^2\kappa - \tau')(1+v^2\tau^2) - 3v\tau^2(v^2\tau^2\kappa + \kappa - v\tau')}{(1+v^2\tau^2)^{\frac{5}{2}}}.$$

If these results are written instead of the (2.11), equation is

$$K_uH_v - K_vH_u = \frac{\begin{array}{l} -2\tau\tau'(1-v^2\tau^2)[(2v\tau^2\kappa - \tau')(1+v^2\tau^2) - 3v\tau^2(v^2\tau^2\kappa + \kappa \\ -v\tau')] - 4v\tau^4[(v^2(2\tau\tau'\kappa + \kappa'\tau^2) + \kappa' - v\tau'')(1+v^2\tau^2) \\ -3\tau\tau'v^2(v^2\tau^2\kappa + \kappa - v\tau')] \end{array}}{(1+v^2\tau^2)^{\frac{11}{2}}}.$$

$\kappa' = 0$ and $\tau' = 0$ are provided for Weingarten surface to be a B-scroll surface. Then the κ and τ must be a constant size. So, anchor curve of the B-scroll surface must be helical curve.

Theorem 3.2 The shape operator of B-scroll Weingarten surface, S is

$$S = \begin{pmatrix} \dfrac{\kappa}{\sqrt{1+v^2\tau^2}} & \dfrac{-\tau}{1+v^2\tau^2} \\ & \\ \dfrac{-\tau}{1+v^2\tau^2} & 0 \end{pmatrix} \tag{3.3}$$

Proof. $\tau' = 0$ is for Weingarten surface to be a B-scroll surface. Given the equation (2.8), the proof is complete.

Theorem 3.3 Gaussian and mean curvatures of the B-scroll Weingarten surface is

$$K = \frac{-\tau^2}{(1+v^2\tau^2)^2} \; and \; H = \frac{\kappa}{\sqrt{1+v^2\tau^2}} \cdot \tag{3.4}$$

Proof. From the equations (2.4) and (3.3), Gaussian and mean curvatures of the B-scroll Weingarten surface are

$$K = detS = \frac{-\tau^2}{(1+v^2\tau^2)^2} \; and \; H = izS = \frac{\kappa}{\sqrt{1+v^2\tau^2}} \cdot$$

Theorem 3.4 Let $Y_p = \dfrac{\varphi_v}{\|\varphi_v\|} \neq 0$ be a tangent vector on B-scroll Weingarten surface. This vector is called the asymptotic direction of the surface and the curve whose tangent vector is Y_p is called as the asymptotic line of this surface.

Proof. Let $X_p = \dfrac{\varphi_u}{\|\varphi_u\|}$, $Y_p = \dfrac{\varphi_v}{\|\varphi_v\|}$ be two tangent vector on B-scroll Weingarten surface. By substituting (3.3), definition (2.3), we get

$$S(Y_p) = \frac{-\tau}{1+v^2\tau^2} X_p,$$

$$k(Y_p) = \langle S(Y_p), Y_p \rangle = \langle \frac{-\tau}{\sqrt{1+v^2\tau^2}} X_p, Y_p \rangle = 0.$$

Then Y_p vector is called the asymptotic direction of the surface and the curve whose tangent vector is Y_p is called as the asymptotic line of this surface.

Theorem 3.5 The first fundamental form, the second fundamental form and third fundamental form of B-scroll Weingarten surface are, respectively

$$
\begin{aligned}
I &= (1+v^2\tau^2)du^2 + dv^2, \\
II &= \kappa(1+v^2\tau^2)du^2 - 2\tau du dv, \\
III &= (\kappa(1+v^2\tau^2)+\tau^2)du^2 - 2\kappa\tau du dv + \tau^2 dv^2.
\end{aligned} \tag{3.5}
$$

Proof. From the (2.3), E, F, G, ℓ, m, n *and* e, f, g the coefficients of the first fundamental form, the second fundamental form and third fundamental form of B-scroll Weingarten surface are, as follow:

$$
\begin{aligned}
E &= \langle \varphi_u, \varphi_u \rangle = 1+v^2\tau^2, \\
F &= \langle \varphi_u, \varphi_v \rangle = 0, \\
G &= \langle \varphi_v, \varphi_v \rangle = 1, \\
\\
\ell &= \langle \varphi_u, N_u \rangle = \kappa(1+v^2\tau^2), \\
m &= \langle \varphi_u, N_v \rangle = -2\tau, \\
n &= \langle \varphi_v, N_v \rangle = 0, \\
\\
e &= \langle N_u, N_u \rangle = \kappa(1+v^2\tau^2)+\tau^2, \\
f &= \langle N_u, N_v \rangle = -2\kappa\tau, \\
g &= \langle N_v, N_v \rangle = \tau^2.
\end{aligned} \tag{3.6}
$$

By substituting (2.2), we get

$$
\begin{aligned}
I &= (1+v^2\tau^2)du^2 + dv^2, \\
II &= \kappa(1+v^2\tau^2)du^2 - 2\tau du dv, \\
III &= (\kappa(1+v^2\tau^2)+\tau^2)du^2 - 2\kappa\tau du dv + \tau^2 dv^2.
\end{aligned}
$$

From the equation (2.5), the provisions on the B-scroll Weingarten surface of the Christoffel symbols are

$$
\Gamma_{11}^1 = \frac{E_u G - 2F_u F + EvF}{2(EG-F^2)} = 0,
$$

$$
\Gamma_{11}^2 = \frac{2F_u E - E_V E - E_u F}{2(EG-F^2)} = -v\tau^2,
$$

$$\Gamma_{12}^{1} = \frac{GE_v - FG_u}{2(EG-F^2)} = \frac{v\tau^2}{1+v^2\tau^2},$$

$$\Gamma_{12}^{2} = \frac{EG_u - FE_v}{2(EG-F^2)} = 0,$$

$$\Gamma_{22}^{1} = \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)} = 0,$$

$$\Gamma_{22}^{2} = \frac{G_v E - 2FF_v + FG_u}{2(EG-F^2)} = 0.$$

Theorem 3.6 Let striction curves of $\varphi(u,v)$ ruled surface be α . Then, the relation between striction curve and anchor curve is, $\alpha(s) = \overline{\alpha}(s)$.

Proof. By substituting (2.1) and (2.7), then we can write

$$\overline{\alpha}(s) = \alpha(s) - \frac{\langle B'(s), \alpha'(s)\rangle}{\|B'(s)\|^2} B(s)$$

$$= \alpha(s) - \frac{\langle -\tau(s)N(s), T(s)\rangle}{\|B'(s)\|^2} B(s)$$

$$= \alpha(s)$$

Theorem 3.7 B-scroll Weingarten surface has not parallel surface.
Proof. Let $\varphi^r(u,v)$ be the surface. From the equation (2.10), the surface is

$$\begin{aligned} \varphi^r(u,v) &= \varphi(u,v) + rN(u,v), \\ \varphi^r(u,v) &= \alpha(u) + vB(u) + r(-v\tau(u)T(u) - N(u)), \\ \varphi^r(u,v) &= \alpha(u) - rN(u) + v(-r\tau(u)T(u) + B(u)). \end{aligned} \tag{3.7}$$

$$N^r = \varphi_u{}^r \wedge \varphi_v{}^r = (1+r\kappa)T - (vr\kappa\tau + v\tau)N - r\tau B.$$

N^r is the unit normal of the parametrized $\varphi^r(u,v)$ surface. The unit normal of the parametrized $\varphi(u,v)$ surface was

$$N = \varphi_u \wedge \varphi_u = -v\tau T - N.$$

Thus, N^r and **N** are not linear dependent. Consequently, B-scroll Weingarten surface has not parallel surface.

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REFERENCES

Beltrami, E., Risoluzione di un Problema Relativo alla Teoria delle Superficie Gobbe, Ann. Mat. Pura Appl., 7, 139–150, (1865-1866).

Dini, U., Sulle Superficie Gobbe nelle quali uno dei due Raggi di Curvatura Principale é una Funzione Dell'altro, Ann. Mat. Pura Appl., 7, 205–210, (1865/1866)

Do Carmo, M. P., Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, (1976).

Graves, L.K., Codimension one isometric immersions between Lorentz spaces, Trans. Amer. Math. Soc., 252, 367–392, (1979)

Gray, A., Abbena, E. and Salamon, S., Modern Differential Geometry of Curves and Surfaces with Mathematica, 3rd Edition, Chapman & Hall/CRC, (2006).

Hacısalihoğlu, H.H., Differantial Geometry,Academic Press Inc.Ankara,(1994)

Kılıçoğlu, Ş., On the Involute B–scrolls in E^3, XIII.International Conference Geometry, Varna, Bulgaria, Integrability and Quantization, 3–8 (2011).

Kılıçoğlu, Ş., B–Scrolls in Lorenzt n – Space, Ph.D. Thesis, Ankara University Graduate School of Natural And Applied Sciences Department of Mathematics, (2006).

Kühnel, W., Ruled W–surfaces, Arch. Math., 62, 475–480, (1994).

Kühnel, W. and Steller M., On Closed Weingarten Surfaces, Monatsh. Math., 146, 113–126, (2005)

O'Neill, B., Elemantary Differential Geometry, 2nd ed. Academic Press, New York, (1997).

Sabuncuoğlu, A., Differential Geometry, Nobel publicitaion, (2001).

Shifrin T.,A First Course in Curves and Surfaces,University of Georgia,(2010).

Sipus, Z. M., Ruled Weingarten Surface in Galilean space, Periodica Mathematica Hungarica , 56(2), 213–225, (2008).

Şenyurt, S., On Involute B–Scroll a New View, Ordu Univ. J. Sci. Tech., 4(1), 59–74, (2014).

van–Brunt, B. And Grant, K., Potential applications of Weingarten surfaces in CAGD. I: Weingarten surfaces and surface shape investigation, Comput Aided Geom Des, 13 , 569–582, (1996)

Weingarten, J., über eine Klasse auf einander abwickelbarer Fläachen, J Reine AngewMath, 59, 382–393, (1861).

Weingarten, J., Über eine flächen, derer normalen eine gegebene flächeberühren, Journal für die Reine und Angewandte Mathematik, 62, 61-63, (1863).
Ziya Savcı, Ü., On the Parallel Ruled Weingarten Surfaces in 3–dimensional Euclid Space, PhD Thesis, Osmangazi University, Eskişehir, (2011).