Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (4) (2023), 983 – 994
DOI : 10.15672/hujms.1189672

Research Article

On the low Lagrangian formulation of Vlasov-Poisson equations

Derya Çoksak Er

Yeditepe University, Faculty of Arts and Sciences, Mathematics Department, Istanbul, Turkey

Abstract

In this work, two problems related with the Low Lagrangian formulation of the Vlasov-Poisson equations are solved. The first problem is related to the space on which the Low Lagrangian is defined. It is shown that the Low Lagrangian is defined on the tangent bundle of the densities of configuration space. The second problem is related to the assumptions which are called Low constraints. It is shown that Low constraints amount to the fact that the Low Lagrangian is invariant under a group action.

Mathematics Subject Classification (2020). 70G65, 70G75, 70H30

Keywords. low Lagrangian, Vlasov-Poisson equations, group actions on Lagrangians

1. Poisson-Vlasov equations

The evolution of collisionles plasma is modeled by the Poisson-Vlasov equations in [13], which is in Russian, and [14] in English. In 1958, Low gave an action principle for self consistent Poisson-Vlasov equations of a collisionless plasma. This Lagrangian, which is called the Low Lagrangian, is defined on the group of canonical diffeomorphisms of particle phase space. However, usually one expects a Lagrangian to be defined on the tangent bundle of some manifold rather than the manifold itself. This discrepancy suggests that Low Lagrangian description of plasma dynamics requires further explanation.

The Poisson-Vlasov equations are,

$$\begin{aligned} \nabla_q^2\phi_f(\mathbf{q}) + e\int f(\mathbf{q},\mathbf{p})d^3p &= 0 \\ \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\nabla_q f - e\nabla_q\phi_f.\nabla_p f &= 0 \end{aligned} \tag{1.1}$$

Here $f(\mathbf{q},\mathbf{p})$ is the density of plasma and $\phi_f(\mathbf{q})$ is the electrostatic potential [14]. Let $Q \subset \mathbf{R}^3$ be the configuration space of plasma particles, and let T^*Q be the phase space. Particle motion is a curve on the canonical diffeomorphisms of T^*Q. Equivalently, each configuration of plasma is in $Diff_{can}(T^*Q)$, where $Diff_{can}(T^*Q)$ is the group of diffeomorphisms which preserve the canonical symplectic structure on T^*Q. For general information we refer to [1–7] and [9–12] for earlier works. The characteristic curve of the Vlasov

Email address: derya.coksak@yeditepe.edu.tr
Received: 15.10.2022; Accepted: 09.12.2022

equation is defined by

$$\begin{aligned}
\frac{d\mathbf{q}}{dt} &= \frac{\mathbf{p}}{m} \\
\frac{d\mathbf{p}}{dt} &= -e\nabla_q \phi_f \qquad (1.2)\\
\frac{df(\mathbf{z})}{dt} &= 0
\end{aligned}$$

where $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ are the Eulerian coordinates. The first two equations are Newton's equations

$$m\ddot{\mathbf{q}} + e\nabla_q \phi_f = 0 \qquad (1.3)$$

and the last equation

$$\frac{df(\mathbf{z})}{dt} = 0 \qquad (1.4)$$

implies that the plasma density remains unchanged in time, i.e.

$$f(\mathbf{z}) = f_0\,(Z) \qquad (1.5)$$

Now, integrating the plasma density with compact support (1.5) over T^*Q along the characteristic curve gives the Low constraint

$$\int_{\varphi_t(T^*Q)} f(\mathbf{z})\; d^6\mathbf{z} - \int_{T^*Q} f_0(Z)\; d^6 Z = 0 \qquad (1.6)$$

expressing the conservation of density in the Eulerian coordinates. The plasma density and the electrostatic potential are related by the Poisson equation

$$\nabla_q^2 \phi_f(\mathbf{q}) + e \int f(\mathbf{q}, \mathbf{p}) d^3\mathbf{p} = 0 \qquad (1.7)$$

with $\mathbf{p} = m\dot{\mathbf{q}}$ being the conjugate momenta, and another constraint introduced by Low where $\varphi_t \in Diff_{can}(T^*Q)$ is an arbitrary motion of particles.

2. Low Lagrangian formulation

A Lagrangian formulation of this problem is given by Low [8]. Let $f_0(Z)$ be the plasma density at a reference configuration $Z \in T^*Q$ and $\phi_f(\mathbf{q})$ be the electrostatic potential in the Eulerian coordinates $\mathbf{z} = (\mathbf{q}, \mathbf{p})$. Then, the Low Lagrangian density in time is given by

$$L_{Low}[\mathbf{q}, \dot{\mathbf{q}}, \phi_f] = \int_{T^*Q} f_0(Z)[\frac{1}{2} m |\dot{\xi}\,(Z)\,|^2 - e\phi_f(\xi\,(Z)\,, t)] d^6 Z + \frac{1}{2} \int_Q |\nabla_q \phi_f(\mathbf{q})|^2 d^3\mathbf{q} \qquad (2.1)$$

which is the single particle Lagrangian where m is the particle mass and e is the particle charge, and $\mathbf{q} = \xi\,(Z)$.

Low variational principal is a generalization of the Hamilton's principle of classical mechanics of particles through the use of Lagrangian displacement variable ξ to describe a continuum [4].

The solution space of particle trajectories is obtained by solving the Low variational principle and the densities are solutions of the Vlasov equation. Given a trajectory one solves linear equations for the potential ϕ_f and then an integro-differential equation for the density f which is also linear for known potential. Conversely, given a density one solves the Poisson equation for ϕ_f and Newton's equations for the trajectories. Both procedures involve linear partial differential equations and hence there appear arbitrary functions in the general solution. The electrostatic constraint given by the Poisson equation seems to eliminate this arbitrariness in constructing solutions from different representations of dynamics thereby ensuring the consistency. Thus, we are led to consider the Poisson equation as a constraint ensuring the consistency between motions of plasma particles and evolution of the plasma density [15].

The variation of Low Lagrangian gives

$$\begin{aligned}\delta L_{Low}[\mathbf{q},\dot{\mathbf{q}},\phi_f] &= \underbrace{-\int f_0(Z)(\ddot{\mathbf{q}}+e\nabla_{\mathbf{q}}\phi_f)\cdot\delta\mathbf{q}d^6Z}_{Newton's\ Eq.}\\ &+ \underbrace{e\Big[\int f(z)\delta\phi_f d^6z-\int f_0(Z)\delta\phi_f d^6Z\Big]}_{Low.Constraint}\\ &- \underbrace{\int\Big(\nabla_q^2\phi_f+\int ef(z)d^3p\Big)\delta\phi_f d^3q}_{Poisson\ Eq.}\\ &+ \underbrace{\int f_0(Z)\frac{d}{dt}(\dot{q}\delta q)d^6Z}_{non-effective\ term}+\underbrace{\int\nabla_q\cdot(\nabla_q\phi_f\delta\phi)d^3q}_{Divergence\ Term}\end{aligned} \tag{2.2}$$

The variation of the Low Lagrangian with respect to the fields $\mathbf{q}$ and ϕ_f gives

$$\begin{aligned}\tfrac{\delta L_{Low}}{\delta q} &= -\left(\ddot{\mathbf{q}}+e\nabla_q\phi_f\right)\\ \tfrac{\delta L_{Low}}{\delta\phi_f} &= \underbrace{-\int\Big(\nabla_q^2\phi_f+\int ef(z)d^3p\Big)d^3q}_{Poisson\ Eq.}+\underbrace{e\Big[\int f(z)d^6z-\int f_0(Z)d^6Z\Big]}_{Vlasov\ Eq.}\end{aligned} \tag{2.3}$$

The variation with respect to $\mathbf{q}$ gives Newton's equations while the variation with respect to ϕ_f gives Poisson-Vlasov equations. Therefore in order to obtain the Poisson equation from the Low Lagrangian one has to assume the Vlasov equation, or vice versa.

Although the computation is quite obvious and straightforward, there are two problems in the Low Lagrangian formulation:

(1) First of all, one expects a Lagrangian to be defined as a function on the tangent bundle of a manifold, and the variation is computed with respect to the coordinates of the base manifold. In the case of the Low Lagrangian, coordinates of our base manifold should be $\mathbf{q}$ and ϕ_f, and $[\mathbf{q},\phi_f,\dot{\mathbf{q}}]$ should be the coordinates of its tangent bundle, which is not. So we have a problem with the space on which L_{Low} is defined.

(2) The variation of the L_{Low} with respect to ϕ_f leads to the Poisson equation under the assumption of the Low constraint which is nothing but the global conservation of density over T^*Q, while the Vlasov equation states the local conservation of density. Hence, in order to obtain the Low constraint one should assume the Vlasov equation which one expects to get as a result of the variation process.

Now, it seems that these two major problems are somehow related to each other. We first try to solve the first problem, and then by using some properties of this solution, we try to explain the second.

3. The first problem: The space on which the Low Lagrangian is defined

First we try to define the base manifold with coordinates $\mathbf{q}$ and ϕ_f. Since ϕ_f is a function on Q, it could be identified with a zero-form on Q

$$\phi_f \in F(Q) \approx \Lambda^0 T^*Q \tag{3.1}$$

With the help of the Hodge star map the space of densities D can be defined to be

$$\begin{aligned} D &= Q\times\Lambda^0T^*Q &&\approx Q\times\Lambda^3T^*Q\\ & (q,\rho) && (q,\rho d^3q)\end{aligned} \tag{3.2}$$

Then we have the inclusion

$$\underset{(q,\rho)}{D} \hookrightarrow \underset{(q,d\rho)}{T^*Q} \tag{3.3}$$

and the maps

$$\begin{array}{ccccc} \underset{(q^i,p_i)}{T^*Q} & \rightarrow & \underset{\left(q,\ \pi_Q^*(p)\wedge^*\pi_Q^*(p)\right)}{Q\times\wedge^6 T^*Q} & \rightarrow & \underset{\left(q,\ \int_{T_q^*Q}\pi_Q^*(p)\wedge^*\pi_Q^*(p)\right)}{D} \end{array} \tag{3.4}$$

or explicitly,

$$\begin{array}{ccccc} T^*Q & \rightarrow & Den(T^*Q) & \rightarrow & D \\ (q^i,p_i) & \longmapsto & (q, f(q,p)d^3p\wedge d^3q) & \longmapsto & \left(q, \left(\int_{T_q^*Q} f(q,p)d^3p\right)d^3q\right) \end{array} \tag{3.5}$$

where $Den(T^*Q)$ is the space of densities, i.e. the set of multiples of volume form on T^*Q. Since D is a subbundle of T^*Q

$$\begin{array}{ccccc} TD & \hookrightarrow & T\left(T^*Q\right) & \approx & T^*\left(TQ\right) \\ (q,\rho,\dot{q},\dot{\rho}) & & (q,d\rho,\dot{q},d\dot{\rho}) & & (q,\dot{q},d\rho,d\dot{\rho}) \end{array} \tag{3.6}$$

the metric defined on the fibers of T^*Q given by

$$\|(q,p,\dot{q},\dot{p})\|^2 = \int_{T^*Q} f\left(Q,P\right)\|\dot{q}\|^2\, d^3Qd^3P + \int_{T^*Q} \pi_Q^*\left(p\right)\wedge^*\pi_Q^*\left(p\right) \tag{3.7}$$

Theorem 3.1. *The Low Lagrangian, L_{Low}, is a function on TD, i.e. it is given by*

$$L_{Low} : TD \rightarrow \mathbb{R}$$

4. The second problem: The assumption of Low constraint

In this subsection starting from the definition of the usual action of $Diff\,(Q)$ on Q, we are going to define natural actions of $Diff\,(Q)$ on T^*Q, $F\,(Q)$ and finally on D.

4.1. The action of $Diff\,(Q)$ on Q

Define maps Ψ_g for the actions of $Diff(Q)$ on Q as to satisfy

$$\begin{array}{ccccc} \Psi & : & Q & \longrightarrow & Q \\ & & q & \mapsto & \Psi(q) \end{array}$$

Let Ψ_t be a curve in $Diff(Q)$ with generator

$$X\left(q\right) = \frac{d}{dt}\,|_{t=0}\ \Psi_t\left(q\right), \quad X\in\mathfrak{X}(Q)$$

The flow Ψ_t on Q is the maximal integral curve

$$q(t) = \Psi_t(q_0)$$

Differentiating this at $t=0$ we find the fundamental vector field

$$X_Q(q) = \frac{d}{dt}\,|_{t=0}\ \Psi_t(q) = \left(\Psi_q\right)_*\circ X\left(q_0\right)$$

$$X_Q = \Psi_q^*\circ X\circ\Psi_{-t}$$

on Q for the action of $Diff(Q)$

4.2. The action of $Diff(Q)$ on T^*Q

These are defined by the cotangent lifts of Ψ

$$\Phi : T^*Q \longrightarrow T^*Q, \quad \Phi(q,p) = \left(\Psi(q), \left(\Psi^{-1}\right)^* p\right)$$

The fundamental vector fields on T^*Q generating these actions are

$$X_{T^*Q}(q,p) = \frac{d}{dt}\mid_{t=0} \Phi_{t(}(q,p)) = \left(\Phi_{(q,p)}\right)_* \circ X$$

where

$$(q(t), p(t)) = \Phi_t(Q_0, P_0)$$

is the curve on T^*Q induced by the flow Ψ_t of X.

$$X(q) = X^i(q)\partial q^i$$

The coordinate expression for the generator is the complete lift to the cotangent bundle of the vector field X, which we simply denote as the cotangent lift:

$$X_{T^*Q}(q,p) = X^i(q)\partial q^i - p_j \frac{\partial X^j}{\partial q^i}\partial p_i$$

of the generator on Q.

4.3. The action of $F(Q)$ on T^*Q

The action of $F(Q)$ on T^*Q is defined as the fiber translation given by $t : F(Q) \times T^*Q \to T^*Q$ given by $t_\rho(q,p) = (q, p - d\rho(q))$. The generator is computed to be

$$\begin{aligned} Y &= \frac{d}{ds} t_{\rho_t}(q,p)\mid_{t=0} = \frac{d}{ds}(q, p - \nabla\rho_s)\mid_{s=0} \\ &= -\frac{\partial\rho}{\partial q_1}\frac{\partial}{\partial p_1} - \frac{\partial\rho}{\partial q_2}\frac{\partial}{\partial p_2} - \dots - \frac{\partial\rho}{\partial q_n}\frac{\partial}{\partial p_n} \\ &= -\frac{\partial\rho}{\partial q^i}\frac{\partial}{\partial p_i} \end{aligned}$$

4.4. The action of $Diff(Q)$ on $F(Q)$

We have now actions of an infinite dimensional group on an infinite dimensional space. This is defined by the push-forward of functions by the action $\Psi^{R,L}_{g(t)}$ on Q, namely, for $\rho \in F(Q)$, we let

$$\sigma_g(\rho) = \rho \circ \Psi_{g^{-1}} = (\Psi_{g^{-1}})^*(\rho).$$

To verify that this defines an action, we will show that

$$\sigma_{g_1 g_2} = \sigma_{g_2} \circ \sigma_{g_1} \tag{4.1}$$

We compute

$$\begin{aligned} \sigma_{g_2} \circ \sigma_{g_1}(\rho) &= \sigma_{g_2} \circ (\rho \circ \Psi_{g_1^{-1}}) = \rho \circ \Psi_{g_1^{-1}} \circ \Psi_{g_2^{-1}} \\ &= \rho \circ \Psi_{g_2^{-1} g_1^{-1}} = \rho \circ \Psi_{(g_1 g_2)^{-1}} = \sigma_{g_1 g_2} \end{aligned}$$

The generators of these actions are the vector fields

$$\begin{aligned} X_{F(Q)}(\rho(q)) &= \frac{d}{dt}\,\sigma_{g(t)}(\rho(q))\Big|_{t=0} = \frac{d}{dt}\,(\rho\circ\Psi_{g^{-1}})(q))\Big|_{t=0} \\ &= T_{\Psi_q(g^{-1}(t))}\rho\circ T_{g^{-1}(t)}\Psi_q\circ \left.\frac{dg^{-1}(t)}{dt}\right|_{t=0} \\ &= -T_q\rho\circ T_{id}\Psi_q\circ X = -T_q\rho\circ X_Q(q) \end{aligned} \tag{4.2}$$

on $F(Q)$. To see that, $X_{F(Q)}$ is really the Lie derivative of X_Q, let $g_t = g(t)$ be a curve on $Diff(Q)$, then the corresponding curve on $F(Q)$ is $\sigma_{g_t}(\rho)$ and the tangent vector to this curve is

$$\frac{d}{dt}\sigma_{g_t}(\rho) = \frac{d}{dt}\left(\Psi_{g_t^{-1}}\right)^*\rho = -\left(\Psi_{g_t^{-1}}\right)^* X_Q(\rho). \tag{4.3}$$

Evaluating at $t=0$, this gives

$$\left.\frac{d}{dt}\sigma_{g_t}(\rho)\right|_{t=0} = -X_Q(\rho) = -d\rho(X_Q) = -\mathcal{L}_{X_Q}(\rho)$$

which is the vector $-T_q\rho\circ X_Q(q)$ generating the action of $Diff(Q)$ on $F(Q)$.

4.5. The action of $Diff\,(Q)$ on D

This density formulation reveals the following action of $Diff\,(Q)$ on D. For $\Psi\in Diff\,(Q)$

$$\begin{array}{ccc} T^*Q & \longrightarrow & D \\ (Q_0,P_0) & \longmapsto & (Q_0,\rho\,(Q_0)\,d^3Q_0) \\ \Psi^*\downarrow & & \downarrow\Psi^* \\ T^*Q & \longrightarrow & D \\ (q,p) & \longmapsto & (q,\rho(q)d^3q) \end{array} \tag{4.4}$$

5. The group S and its Lie algebra

5.1. The group S

The action of $Diff\,(Q)$ on D reveals that

$$\Psi^*\left(\rho(Q_0)d^3Q_0\right) = \rho(\Psi\,(Q_0))d^3\psi\,(Q_0) = \rho\,(\Psi(Q_0))\,J_\Psi\,(Q_0)\,d^3Q_0 \tag{5.1}$$

where $J_\Psi\,(Q_0)$ is the Jacobian of the diffeomorphism Ψ at Q_0.Therefore, one can define the action of $\Psi\in Diff(Q)$ on the densities ρ by

$$\Psi\cdot\rho = J_\Psi(\rho\circ\Psi) \tag{5.2}$$

Since our diffeomorphisms comes from the flow Ψ_t, it is possible to distinguish backward and forward flows, identifying forward flows with orientation preserving diffeomorphisms $Diff_+\,(Q)$, we can assume that J_Ψ is always positive, i.e.

$$J_\Psi = e^\sigma \tag{5.3}$$

for some $\sigma\in F\,(Q)$. Now, we can define the action of $S = (Diff\,(Q)\,,\circ)$ Ⓢ $(F\,(Q)\,,+)$ on densities by

$$(\Psi,\sigma)\cdot\rho = e^\sigma(\rho\circ\Psi) \tag{5.4}$$

The semidirect product structure of the group S comes from the fact that the second component $\sigma = \ln(J_\Psi)$ depends on the first component ψ.

One could visualise the group S by

$$\begin{array}{ccccc} Q & \xrightarrow{\Psi} & Q & \xrightarrow{\sigma} & \mathbb{R} \\ Q_0 & & q = \Psi\,(Q_0) & & \sigma(\Psi\,(Q_0)) \end{array} \tag{5.5}$$

and the group product by

$$\begin{array}{ccccc} Q & \xrightarrow{\Psi_1} & Q & \xrightarrow{\Psi_2} & Q \\ & & \sigma_1 \searrow & & \swarrow \sigma_2 \\ & & & \mathbb{R} & \end{array} \tag{5.6}$$

or componentwise

$$(\Psi_1, \sigma_1)(\Psi_2, \sigma_2) = (\Psi_2 \circ \Psi_1, \sigma_2 + \sigma_1 \circ \Psi_2^{-1}) \tag{5.7}$$

5.2. The Lie algebra $\mathfrak{s}$

The Lie algebra of S, denoted by $\mathfrak{s}$, can be defiend as follows. Choose a curve $(\Psi_t, \sigma_t) \in S$ passing through identity of S which is $(id_{Q_0}, 0)$. We may define left and right actions of S on this curve as

$$(\Psi_t, \sigma_t)(\Theta, \theta) = (\Theta \circ \Psi_t, \theta + \sigma_t \circ \Theta^{-1}) \tag{5.8}$$

is the right action and

$$(\Theta, \theta)(\Psi_t, \sigma_t) = (\Psi_t \circ \Theta, \sigma_t + \theta \circ \Psi_t^{-1}) \tag{5.9}$$

is the left action. Differentiating with respect to t at $t = 0$ gives the right and left actions of S on $\mathfrak{s}$ are

$$\frac{d}{dt}(\Psi_t, \sigma_t)(\Theta, \theta)\mid_{t=0} = \left((\Theta)_* \dot{\Psi}_0, \dot{\sigma}_0 \circ \Theta^{-1}\right) \tag{5.10}$$

and

$$\frac{d}{dt}(\Theta, \theta)(\Psi_t, \sigma_t)\mid_{t=0} = \left(\dot{\Psi}_0 \circ \Theta, \dot{\sigma}_0 - \dot{\Psi}_0(\theta)\right) \tag{5.11}$$

respectively. Therefore the right and left actions of S on $\mathfrak{s}$ are

$$\left(\dot{\Psi}_0, \dot{\sigma}_0\right)(\Theta, \theta) = \left((\Theta)_* \dot{\Psi}_0, \dot{\sigma}_0 \circ \Theta^{-1}\right) \tag{5.12}$$

and

$$(\Theta, \theta)\left(\dot{\Psi}_0, \dot{\sigma}_0\right) = \left(\dot{\Psi}_0 \circ \Theta, \dot{\sigma}_0 - \dot{\Psi}_0(\theta)\right) \tag{5.13}$$

The Adjoint action is given by

$$(\Theta, \theta)\left(\dot{\Psi}_0, \dot{\sigma}_0\right)(\Theta, \theta)^{-1} = ((\Theta^{-1})_* \dot{\Psi}_0 \circ \Theta, \dot{\sigma}_0 \circ \Theta^{-1} - \dot{\Psi}_0(\theta \circ \Theta^{-1})) \tag{5.14}$$

The semidirect product structure of S allows us to write right and left decompositions as

$$(\Psi, \sigma) = (\Psi, 0)(id_q, \sigma) = (id_{Q_0}, \sigma \circ \Psi)(\Psi, 0) \tag{5.15}$$

we will identify the additive subgroups (id_q, ρ) and $(id_{Q_0}, \rho \circ \psi)$ with $(F(Q), +)$. In this notation we identify $\mathfrak{s}$ with the right invariant vector fields on S and use the left action of S on $\mathfrak{s}$.

6. The action of the group S on T^*Q and $T\left(T^*Q\right)$

6.1. Action of S on T^*Q

Now, the action of S on T^*Q is defined by the diagram

$$\begin{array}{ccccc} T^*Q & \overset{\Psi^*}{\longleftarrow} & T^*Q & & \\ \downarrow & & \downarrow & \nwarrow^{d\sigma} & \\ Q & \underset{\longrightarrow}{\Psi} & Q & \underset{\longrightarrow}{\sigma} & \mathbb{R} \end{array} \tag{6.1}$$

or componentwise

$$(\Psi,\sigma)\left(Q_0,P_0\right)=\Big(\Psi(Q_0),(\Psi^*)^{-1}P_0-d\sigma\Big)=(q,p-d\sigma) \tag{6.2}$$

Proposition 6.1. *The action of S on T^*Q preserves the symplectic structure $\omega=dq^i\wedge dp_i$ on T^*Q.*

Proof. Let X denote the infinitesimal generator of the action of S on T^*Q,then,

$$X=X^i\partial_{q^i}-\Big(p_j\partial_{q^i}X^j+\partial_{q^i}\sigma\Big)\partial_{p_i}$$

by taking the Lie derivative of the symplectic two-form along X

$$\mathcal{L}_X(dp_i\wedge dq^i)=d\Big(-X^idp_i-p_j\partial_{q^i}X^jdq^i-\partial_{q^i}\sigma dq^i\Big)=d\Big(-d\Big(X^ip_i+\sigma\Big)\Big)=0 \tag{6.3}$$

Since S preserves the symplectic structure it also preserves the symplectic volume and therefore □

$$\mathcal{L}_X\left(dV\right)=0 \tag{6.4}$$

6.2. Action of S on $T\left(T^*Q\right)$

Analogously, we can define the action of S on TT^*Q by the tangent lift of (6.2)

$$\begin{array}{ccccc} TT^*Q & \xrightarrow{(\Psi^*)_*^{-1}} & TT^*Q & & \\ & & & \nwarrow^{d\dot{\sigma}} & \\ T^*Q & \xrightarrow{(\Psi^*)^{-1}} & T^*Q & & T \\ \downarrow & & \downarrow & \nwarrow^{d\sigma} & \\ Q & \xrightarrow{\Psi} & Q & \xrightarrow{\sigma} & \mathbb{R} \end{array} \tag{6.5}$$

or componentwise

$$(\Psi,\sigma)(Q_0,P_0,\dot{Q}_0,\dot{P}_0)=\Big(\Psi(Q_0),(\Psi^*)^{-1}P_0-d\sigma,\Psi_*(\dot{Q}_0),(\Psi^*)_*^{-1}\dot{P}_0-d\dot{\sigma}\Big) \tag{6.6}$$

where

$$\sigma_t=\sigma(\Psi(Q_t)) \tag{6.7}$$

and

$$d\dot{\sigma}=(d\sigma)_*\left(\dot{Q}_0\right) \tag{6.8}$$

6.3. The infinitesimal generator of the action of the group S on T^*Q

Let Ψ_t be the flow of the equation (1.2) .Then we have

$$(\Psi_t, \sigma)(Q_0, P_0) = (\Psi_t(Q_0), (\Psi_t^*)^{-1}P_0 - d\sigma(\Psi_t(Q_0)) \tag{6.9}$$

In accordance with (6.7) we define

$$\sigma_t(Q_0) = \sigma(\Psi(Q_t)) = \sigma(\Psi_t(Q_0)) \tag{6.10}$$

Then (6.9) becomes

$$(\Psi_t, \sigma)(Q_0, P_0) = \Big(\Psi_t(Q_0), (\Psi_t^*)^{-1}P_0 - d\sigma_t(Q_0)\Big) \tag{6.11}$$

Differentiating (6.9) w.r.t. t at 0 leads to

$$\frac{d}{dt}(\Psi_t, \sigma_t)(Q_0, P_0) \mid_{t=0} = (\dot{\Psi}_0(Q_0), (\dot{\Psi}_0^*)^{-1}P_0 - (d\dot{\sigma}_0)(Q_0)) \tag{6.12}$$

Therefore the action of S on T^*Q by the tangent lift is

$$(\Psi_t, \sigma_t, \dot{\Psi}_t, \dot{\sigma}_t)(Q_0, P_0) = (\Psi_t(Q_0), (\Psi_t^*)^{-1}P_0 - d\sigma_t(Q_0), \dot{\Psi}_t(Q_0), (\dot{\Psi}_t^*)^{-1}P_0 - d\dot{\sigma}_t(Q_0)) \tag{6.13}$$

and the infinitesimal generator determines an element of Lie algebra $\mathfrak{s}$ given by the section of $T\left(T^*Q\right)$

$$(Q_0, P_0, \dot{Q}_0, \dot{P}_0) = (id_{Q_0}, 0, \dot{\Psi}_0, \dot{\sigma}_0)(Q_0, P_0) = (Q_0, P_0, \dot{\Psi}_0(Q_0), (\dot{\Psi}_0^*)^{-1}P_0 - d\dot{\sigma}_0(Q_0)) \tag{6.14}$$

The action of S on $T\left(T^*Q\right)$ is given by

$$\begin{aligned}&(\Psi, \sigma)(Q_0, P_0, \dot{Q}_0, \dot{P}_0)\\ &= \Big(\Psi(Q_0), (\Psi^*)^{-1}P_0 - d\sigma, \Psi_*(\dot{Q}_0), (\Psi^*)^{-1}\dot{P}_0 - d\dot{\sigma}\Big)\\ &= (q, p - d\sigma(q), \dot{q}, \dot{p} - d\dot{\sigma})\end{aligned} \tag{6.15}$$

is consistent with the left action of S on $\mathfrak{s}$

$$(\Psi, \sigma)(id, 0, \dot{\Psi}_0, \dot{\sigma}_0) = (\psi, \sigma, \dot{\Psi}_0 \circ \Psi, \dot{\sigma}_0 - \dot{\Psi}_0(\sigma)) \tag{6.16}$$

The action of S on D and TD could be obtained by restriction of the action on T^*Q to D.

7. The invariance of Low Lagrangian under the group S

Now, to define the action of S on the Low Lagrangian L_{Low}, we will use the semidirect product structure of S. In order to prove the invariance under the action of (Ψ, σ) we will first investigate the invariance under (id, σ) , then under $(\Psi, 0)$.

$$\begin{aligned}\widetilde{L} &= (id, \sigma) \cdot L\left(q, \dot{q}, \phi_f\left(q\right)\right) = L\left(\left(id, 0, (id)_*, \dot{\sigma}\right) \cdot \left(q, \dot{q}, \phi_f\left(q\right)\right)\right) = L\left(\left(q, \dot{q}, \phi_f\left(q\right) - \dot{\sigma}\right)\right)\\ &= \textstyle\int_{T^*Q} f(z)(\frac{1}{2}m\left|\left|\dot{q}\right|\right|^2 - e\phi_f(q) + e\dot{\sigma})d^6z + \frac{1}{2}\int_Q \left|\nabla_q(\phi_f(q) - \dot{\sigma}(q))\right|^2 d^3q\end{aligned} \tag{7.1}$$

Now, we are going to check the invariance of the Low Lagrangian under the action of S. Therefore we take the variation with respect to $\dot{\sigma}$ rather than σ.Then we have,

$$\begin{aligned} \frac{\delta\widetilde{L}}{\delta\dot{\rho}} &= e\int_{T^*Q} f(z)d^6z + \tfrac{1}{2}\tfrac{\delta}{\delta\dot{\rho}}\int_Q(-2\nabla_q\phi_f(q)\cdot\nabla_q\dot{\sigma})d^3q + \tfrac{1}{2}\tfrac{\delta}{\delta\dot{\rho}}\int_Q\nabla_q\dot{\sigma}\cdot\nabla_q\dot{\sigma}d^3q \\ &= e\int_{T^*Q} f(z)d^6z + \int_Q\nabla_q^2\phi_f(q)d^3q - \underbrace{\int_Q\nabla_q\cdot\left(\left(\nabla_q\phi_f(q)-\nabla_q\dot{\sigma}\right)\right)d^3q}_{Divergence\ Term} - \underbrace{\int_Q\frac{d}{dt}\left(\nabla_q^2\sigma\right)d^3q}_{Ineffective\ Term} \\ &= e\int_{T^*Q} f(z)d^6z + \int_Q\nabla_q^2\phi_f(q)d^3q \end{aligned} \tag{7.2}$$

Therefore

$$\frac{\delta\widetilde{L}}{\delta\dot{\sigma}} = 0 \Longrightarrow \nabla_q^2\phi_f(q) = -e\int_{T^*Q} f(z)d^3p \tag{7.3}$$

which implies that the invariance of Low Lagrangian under the action of $F(Q)$ part amounts to the Poisson equation.

For the invariance under the action of the $Diff(Q)$ part, we use the semidirect product structure and the invariance under the action of $F(Q)$ part (i.e. the Poisson equation) and choose

$$\dot{\sigma} = \phi_f \tag{7.4}$$

Under this assumption Low Lagrangian reduces to

$$L_{Low}(q,\dot{q}) = \frac{1}{2}\int_{T^*Q} f(z)m\,||\dot{q}||^2\,d^6z \tag{7.5}$$

Theorem 7.1. *The Vlasov equation is obtained by the variation of L with respect to $\dot{\Psi}$.*

The action of $(\Psi,0)$ on reduced form of Low Lagrangian is

$$\begin{aligned} \widehat{L} &= (\Psi,0)\cdot L_{Low}\left(Q_0,\dot{Q}_o\right) = L_{Low}\left(\left(\Psi,0,\dot{\Psi},0\right)\cdot\left(Q_0,\dot{Q}_o\right)\right) \\ &= L_{Low}(q,\dot{q},0) = \tfrac{1}{2}\int_{T^*Q} f(z)m\,||\dot{q}||^2\,d^6z \end{aligned} \tag{7.6}$$

Since Ψ is generated by the flow Ψ_t which is $\Psi_0 = id_{Q_o}$ and $\Psi_{t_0} = \Psi$, the Lie derivative of the Lagrangian $\widehat{L}$ along the direction of the flow, X is equal to the variation of the Lagrangian w.r.t.$\dot{\Psi}$.

$$\frac{\delta\widehat{L}}{\delta\dot{\Psi}} = \frac{1}{2}\int_{T^*Q}(\mathcal{L}_X f(z))\,m\,||\dot{q}||^2\,d^6z + \frac{1}{2}\int_{T^*Q} f(z)m\left(\mathcal{L}_X\,||\dot{q}||^2\right)d^6z \tag{7.7}$$

The assumption of $F(Q)$ invariance amounts to the zero electrostatic potential case, which means that there is only kinetic energy term, $\frac{1}{2}m\,||\dot{q}||^2$, in the particle Lagrangian and particle Hamiltonian, and hence they are the same. Since the particle Hamiltonian is invariant under the flow we have,

$$\mathcal{L}_X\,||\dot{q}||^2 = 0 \tag{7.8}$$

(7.7) amounts to

$$\int_{T^*Q}\mathcal{L}_X f(z)m\,||\dot{q}||^2\,d^6z = 0 \tag{7.9}$$

Now, since

$$\frac{1}{2}m\,||\dot{q}||^2 > 0 \tag{7.10}$$

we get

$$\mathcal{L}_X f(z) = \frac{df(z)}{dt} = 0 \tag{7.11}$$

which is the Vlasov equation.

8. Conclusion

In this work, two problems related with the Low Lagrangian formulation of the Vlasov-Poisson equations are solved. The first problem is related to the space on which the Low Lagrangian is defined. Usually, a Lagrangian is expected to be defined on the tangent bundle of a manifold. However, in the Low Lagrangian case, the space on which this Lagrangian is defined is not so clear. Taking the variation process into consideration, it is shown that Low Lagrangian is defined on the tangent space of the space of densities of the configuration space. Then it is shown that Low Lagrangian can be written as the square of the norm of a tangent vector in this space. The second problem is related to the assumptions which are called Low constraints. To clarify these assumptions further, the group S is introduced by using the action of the diffeomorphism group on the tangent bundle of the space of densities. After computing the infinitesimal generator of the action of S on this space, it is shown that Low constraints amount to the invariance of the Low Lagrangian under this action. Therefore, these two seemingly unrelated problems related to the Low Lagrangian is solved in a unifying approach which formulates the Lagrangian on the tangent space of the space of densities and explains Low constraints as the invariance of the Lagrangian under the action of the group S on the tangent bundle of the same space.

Obviously, once one get the invariance of the Lagrangian under a certain group, the next question will be to investigate the reduction of the Lagrangian. For further work, we are planning to study this reduction and its relation with some other physical phenomena.

Acknowledgment. I would like to thank Prof. Hasan Gümral and Prof. Ender Abadoğlu for their contributions to this work.

References

[1] R. Abraham, J.E Marsden and T. Ratiu, *Manifolds, Tensor Analysis and Applications*, Springer-Verlag, 1988.

[2] H. Cendra, D.D. Holm, M.J.W. Hoyle and J.E. Marsden, *The Maxwell-Vlasov equations in Euler-Poincare form*, J. Math. Phys. **39**, 3138-3157, 1998.

[3] K. Chandrasekhar, *Ellipsoidal Figures of Equilibrium*, Dover, 1977.

[4] H. Gümral, *Geometry of plasma dynamics I. Group of canonical diffeomorphisms*, Journal of Mathematical Physics **51** (8), 501-523, 2010.

[5] Z.R. Iwinski and L.A. Turski, *Canonical theories of systems interacting electromagnetically*, Lett. Appl. Eng. Sci. **4**, 179-191, 1976.

[6] J.H. Jeans, *The stability of spherical nebula*, Philos. Trans. R. Soc. London **199**, 1-53, 1902.

[7] J. Larsson, *An action principle for the Vlasov equation and associated Lie perturbation equations. Part 1. The Vlasov-Poisson system*, Journal of Plasma Physics **48**, 13-35,1992.

[8] F.E. Low, *A Lagrangian formulation of the Boltzmann-Vlasov equation for plasmas*, Proc. R. Soc. London Ser. A **248**, 282-287, 1958.

[9] J.E. Marsden and T. Ratiu, *Introduction to Symmetry and Mechanics*, Springer, 1994.

[10] P.J. Olver, *Applications of Lie groups to Differential Equations*, Springer Science and Business Media LLC, 1986.

[11] H. Poincaré, *Théorie des tourbillions*, Gauthier-Villars, 1890.

[12] H. Poincaré, *Sur la stabilité de l'équilibre des figures piriformes affectées par une masse fluide en rotation*, Philos. Trans. R. Soc. London Ser. A, 333-3 73, 1901.

[13] A.A. Vlasov, *On vibration properties of electron gas*, J. Exp. Theor. Phys. **8** (3), 291-318, 1938.

[14] A.A. Vlasov, *The vibrational properties of an electron gas*, Soviet Physics Uspekhi **10** (6), 721-733, 1968.
[15] H. Ye and P.J. Morrison, *Action principles for the Vlasov equation*, Phys. Fluids B **4**, 771-776, 1992.