



Digraph groups corresponding to digraphs with one more vertex than arcs

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Abstract

Cuno & Williams [1] examined digraph groups $G_{\Gamma}(R)$ in which the undirected graph is triangle free and $|V(\Gamma)| = |A(\Gamma)|$ (the number of generators and realtors are equal). Cuno & Williams showed there are 6 cases to consider, and Cuno & Williams demonstrated that the corresponding group $G_{\Gamma}(R)$ is either infinite or finite cyclic in most cases. In this paper, we investigate the case $|V(\Gamma)| = |A(\Gamma)| - 1$, which is next step, and we prove that there are 35 cases to consider.

Keywords: Digraph Groups, girth, finite, infinite, cyclic.

Köşelerden bir fazla kenarları olan graflara karşılık gelen yönlendirilmiş graf grupları

Öz

Cuno ve Williams yönlendirilmemiş grafında üçgen bulundurmeyen ve $|V(\Gamma)| = |A(\Gamma)|$ (üreteç ve ilişki sayısı eşit) olan yönlendirilmiş graf gruplarını $G_{\Gamma}(R)$ incelemiştir. Yapılan inceleme neticesinde, Cuno & Williams anlamlı 6 durumun var olduğunu ve bunlara karşılık gelen grupların çoğu durumda sonlu devirli ya da sonsuz olduğunu göstermiştir. Mevzubahis çalışmanın bir sonraki adımı olan $|V(\Gamma)| = |A(\Gamma)| - 1$ durumu bu makalede incelenmekte ve anlamlı 35 durumun var olduğu ispat edilmektedir.

Anahtar Kelimeler: Yönlendirilmiş graf grupları, en küçük daire, sonlu, sonsuz, devirli

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1. Introduction

Presentations with more generators than relators must define infinite groups, as seen when the groups are abelianized [3, page 84]. Cuno & Williams [1] examined digraph groups $G_\Gamma(R)$ in which the undirected graph is triangle free (i.e., the girth $g(\Gamma) \geq 4$) and $|V(\Gamma)| = |A(\Gamma)|$ (the number of generators and relators are equal). Cuno & Williams showed there are 6 cases to consider, and Cuno & Williams demonstrated that the corresponding group $G_\Gamma(R)$ is either infinite or finite cyclic in most cases. The purpose of this present work is to prove that there are 35 classes of digraphs in the case $|V(\Gamma)| = |A(\Gamma)| - 1$. We also state how the corresponding group for a digraph present as a digraph group.

We first explain certain terms for digraphs before making the theorem. A digraph whose underlying undirected graph is connected is referred to as a weakly connected digraph. Sources are vertices with positive outdegree and zero indegree, sinks are vertices with positive indegree and zero outdegree, and leaves are vertices whose indegree and outdegree total to one. Every leaf is either a source or a sink.

We will use a remark when constructing the graphs. Therefore, we state it here.

Remark [1. page 7]. “This statement introduces a **reflection principle**: Let Γ be any digraph, and $R(a, b)$ be any word. Then, the digraph Γ' may be defined as being formed by interchanging every arc with the opposite direction, and the word $R'(a, b)$ as the word that results from interchanging the letters a and b and replacing every letter with its inverse, thus α and β are also interchanged. Then, $G_\Gamma(R) \cong G_{\Gamma'}(R')$ ”.

2. Proving the main theorem

We need to refer to certain classes of digraphs, Therefore, the graphs are created under the following conditions.

(i). Γ is connected ($G_\Gamma(R) \cong G_{\Gamma_1}(R) * \dots * G_{\Gamma_N}(R)$ if Γ is disconnected with components $\Gamma_1, \Gamma_2, \dots, \Gamma_N$. We can therefore assume that Γ is connected) [2, page 13].

(ii). $|V(\Gamma)| = |A(\Gamma)| - 1$ (the group is infinite if $|V(\Gamma)| > |A(\Gamma)|$ [3], and Cuno and Williams demonstrated all possible digraphs for $|V(\Gamma)| = |A(\Gamma)|$ in [1]).

(iii). $0 \leq \sigma_1 \leq \sigma \leq 1$ and $0 \leq \tau_1 \leq \tau \leq 1$ where $\sigma, \sigma_1, \tau, \tau_1$ are the number of sources, source leaves, sinks, and sink leaves, respectively (it is explained why we should search between 0 and 1 in [1]).

(iv). $\sigma \geq \tau$ and if $\sigma = \tau$, then $\sigma_1 \geq \tau_1$ (by reflection principle).

Main Theorem:

There are 35 possible digraph families as indicated in Figure 3 and Figure 4 under above circumstances (i)-(iv).

Proof.

There are two options because the entire digraph is connected. Think about the scenario when the graph has no leaves at first. There are two options for the form of the underlying graph. The first one is that two cycles are fused together with a path connecting them to create the underlying

graph (as in Γ_1); the other one is to connect two cycles using a path that is shared by both cycles (as in Γ_2). The graphs are shown in Figure 1, where two specific vertices k and l are identified. Now, Γ_1 and Γ_2 in Figure 1 need to be directed to examine the possible digraph families.

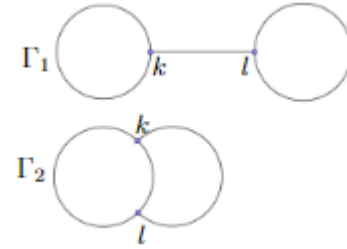


Figure 1: Two possible undirected graphs with $|V| = |A| - 1$

By conditions (iii) and (iv), we have $(\sigma, \tau) = (0, 0), (1, 0)$ or $(1, 1)$.

Case 1: $(\sigma, \tau) = (0, 0)$. Then $(\sigma_1, \tau_1) = (0, 0)$.

This case is with no source and no sink.

For Γ_1 , this is (i) in Figure 2: these 2 cycles do not have a source or a sink and thus they are the directed cycles and the path between them could be in any direction (basically from l to k or k to l , which do not matter since both cases are isomorphic to each other.

For Γ_2 , there are $2 \cdot 2 \cdot 2 = 8$ possibilities to direct this undirected graph because there are 2 possibilities in the first cycle, second cycle and intersection part of the cycles for each one (basically from k to l or l to k for each one).

(1) It is not feasible because k is a source, which is not possible, if k to l for the first cycle, the intersection part of the cycles and the second cycle.

(2) The one that is isomorphic to (vii) in Figure 2 is produced by converting the first cycle and the intersection part of the cycles from k to l , and the second cycle from l to k .

(3) The one that is isomorphic to (vii) in Figure 2 is produced by converting the first cycle and the second cycle from k to l , the intersection part of the cycles from l to k .

(4) The one that is isomorphic to (vii) in Figure 2 is produced by converting the first cycle from k to l , the intersection part of the cycles and the second cycle from l to k .

(5) The one that is isomorphic to (vii) in Figure 2 is produced by converting the first cycle from l to k , the intersection part of the cycles and the second cycle from k to l .

(6) The one that is isomorphic to (vii) in Figure 2 is obtained by converting the first cycle and the second cycle from l to k , and the intersection of the cycles from k to l .

(7) The one that is isomorphic to (vii) in Figure 2 is obtained by converting the first cycle and the intersection part of the cycles from l to k , and the second cycle from k to l .

(8) It is not feasible because k is a sink, which is not possible, if l to k for the first cycle, the intersection part of the cycles and the second cycle.

Case 2: $(\sigma, \tau) = (1, 0)$. Then $(\sigma_1, \tau_1) = (0, 0)$ or $(1, 0)$.

This case is with 1 source and the first possibility is that there are 1 source and no leaves.

Case 2(a): $(\sigma, \tau) = (1, 0)$ and $(\sigma_1, \tau_1) = (0, 0)$

For Γ_1 , this source could be on any of the cycles (it makes no difference because they are isomorphic) or on the path that connects the cycles. If the source is on the cycle (say on first cycle), then k cannot be a sink because the path that connects the cycles has to be from k to l and the second cycle cannot have one more sink or source since the entire digraph has 1 source and no sink in total and this second cycle is a directed cycle basically. Thus, this is result in (ii) in Figure 3.

If this source is on the path that connects the cycles (let t be a source $k \leq t \leq l$), then there is a path from t to k and t to l . These cycles cannot have one more sink or source since the entire digraph has 1 source and no sink in total and these cycles are directed cycles basically. Thus, this is result in (iii) in Figure 3.

For Γ_2 , this source can be on the first cycle, second cycle or intersection of these two cycles. These three possibilities are isomorphic to each other and so we will just consider the case when this source is on the first cycle. Since there is a source on the first cycle there is a path from the source to k and l . Now, to direct the undirected graph, there are $2 \cdot 2 = 4$ possibilities (2 possibilities for each one both intersection part of the cycles and second cycle, which are basically from k to l or l to k).

- (1) It is not feasible because l is a sink if k to l for both second cycle and intersection part of the cycles.
- (2) The one that is isomorphic to (viii) in Figure 3 is obtained by converting the second cycle from l to k and the intersection part of the cycles from k to l .
- (3) The one that is isomorphic to (viii) in Figure 3 is obtained by converting the second cycle from k to l and the intersection part of the cycles from l to k .
- (4) It is not feasible because k is a sink if l to k for both second cycle and intersection part of the cycles.

The second possibility is that there are 1 source and 1 leaf.

Case 2(b): $(\sigma, \tau) = (1, 0)$ and $(\sigma_1, \tau_1) = (1, 0)$

In addition to Case 2(a), there is now a source leaf. To acquire the possible digraphs, we must add a source leaf to Figure 1. Figure 2 depicts the possibilities by labeling three vertices t, k, l .

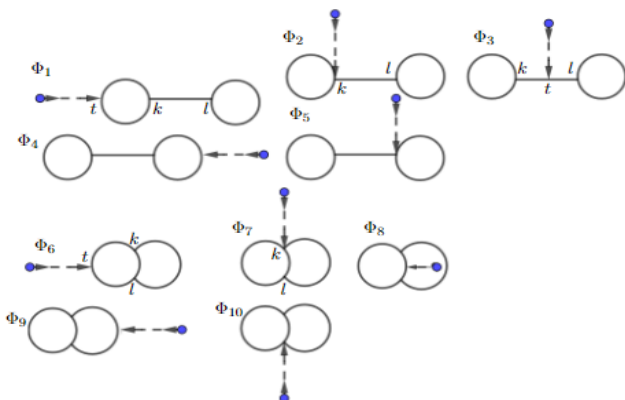


Figure 2: Possible undirected graphs with a source leaf for $|V| = |A| - 1$.

It can be seen that $[\Phi_1$ and $\Phi_4]$, $[\Phi_2$ and $\Phi_5]$, $[\Phi_6, \Phi_8$ and $\Phi_9]$, $[\Phi_7$ and $\Phi_{10}]$, are isomorphic. As a result, while assigning directions to arcs, it is sufficient to evaluate the graphs $\Phi_1, \Phi_2, \Phi_3, \Phi_6, \Phi_7$. It should be highlighted that there cannot be one more source or sink by (iii) since we already have a source leaf in Figure 2.

There are $4 \cdot 2 \cdot 1 = 8$ possibilities (4 possibilities for first cycle-basically t to k or k to t for upper part of the cycle and also for lower part of the cycle-, 2 possibilities for intersection part of the cycles (basically k to l or l to k) and only 1 possibility for second cycle (it has to be a directed cycle) to direct the undirected graph Φ_1 in Figure 2.

- (1) The one that is isomorphic to (xii) in Figure 4 is obtained by converting upper part and lower part of the first cycle from t to k and intersection part of the cycles from k to l .
- (2) It is not feasible because k is a sink, which is not possible, if t to k for upper part and lower part of the first cycle l to k for intersection part of the cycles.
- (3) The one that is isomorphic to (xiii) in Figure 4 is obtained by converting upper part of the first cycle from t to k , lower part of the first cycle from k to t and intersection part of the cycles from k to l .
- (4) The one that is isomorphic to (xiv) in Figure 4 is obtained by converting upper part of the first cycle from t to k , lower part of the first cycle from k to t and intersection part of the cycles from l to k .
- (5) The one that is isomorphic to (xiii) in Figure 4 is obtained by converting upper part of the first cycle from k to t , lower part of the first cycle from t to k and intersection part of the cycles from k to l .
- (6) The one that is isomorphic to (xiv) in Figure 4 is obtained by converting upper part of the first cycle from k to t , lower part of the first cycle from t to k and intersection part of the cycles from l to k .
- (7) It is not feasible because k is a source, which is not possible, if k to t for upper part and lower part of the first cycle k to l for intersection part of the cycles.
- (8) It is not feasible because t is a sink, which is not possible, if k to t for upper part and lower part of the first cycle l to k for intersection part of the cycles.

When the graphs in Figure 2 are directed by using same method, we have (xvii) and (xviii) for the graph Φ_2 , (xv), (xvi) for the graph Φ_3 , (xxviii), (xxix), (xxx) for the graph Φ_6 and (xxvi), (xxvii) for the graph Φ_7 .

Case 3: $(\sigma, \tau) = (1, 1)$. Then $(\sigma_1, \tau_1) = (0, 0), (1, 0)$ or $(1, 1)$.

This case is with 1 source and 1 sink and the first possibility is that there are no source leaves and no sink leaves.

Case 3(a): $(\sigma, \tau) = (1, 1)$ and $(\sigma_1, \tau_1) = (0, 0)$

The resulting digraphs are (iv), (v), (vi) in Figure 3 after directing the graphs Γ_1 in Figure 1 and (ix), (x), (xi) in Figure 4 after directing the graphs Γ_2 in Figure 2.

The second possibility is that there is 1 source leaf and no sink leaves.

Case 3(b): $(\sigma, \tau) = (1, 1)$ and $(\sigma_1, \tau_1) = (1, 0)$

The resulting digraphs are (xix), (xx) and (xxi) in Figure 4 after directing the graphs Φ_1 in Figure 2, (xxiv) and (xxv) in Figure 4 after directing the graphs Φ_2 in Figure 2, (xxii), (xxiii) in Figure 4 after directing the graphs Φ_3 in Figure 2, (xxxiii), (xxxiv), (xxxv) in Figure 4 after directing the graphs Φ_6 in Figure 2 and (xxxi), (xxxii) in Figure 4 after directing the graphs Φ_7 in Figure 2.

Case 3(c): $(\sigma, \tau) = (1, 1)$ and $(\sigma_1, \tau_1) = (1, 1)$

The last possibility is that there are 1 source leaf and 1 sink leaf. Because there must be an arc between every source and sink, it is impossible for this situation to exist. Otherwise, the digraph with two vertices and one arc between them would have more vertices than arcs.

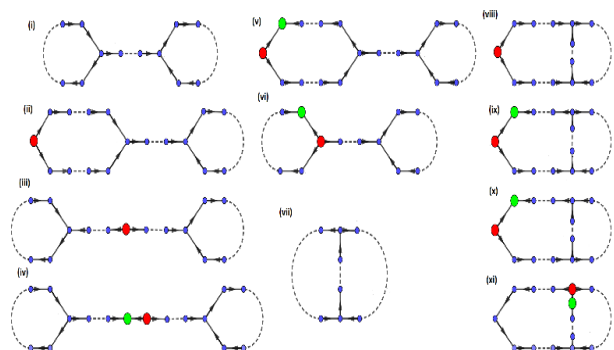


Figure 3: Classes of digraphs without leaf

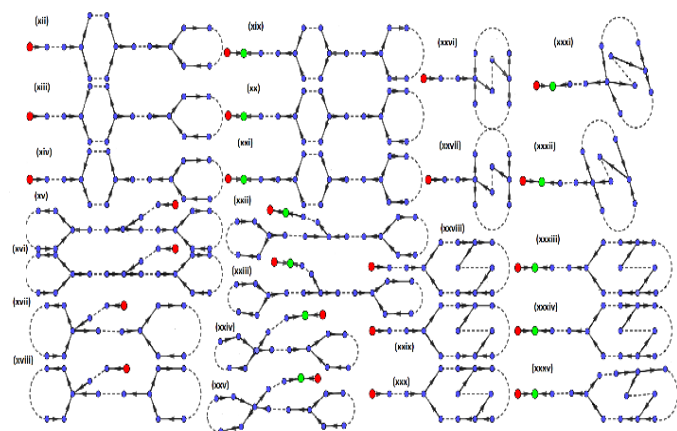


Figure 4: Classes of digraphs with leaf

3. Digraph groups

We will now specify how to construct a group presentation from a digraph. Let Γ be a finite digraph with vertex set $V(\Gamma)$ and (directed) arc set $A(\Gamma)$. The vertices $v \in V(\Gamma)$ correspond to the generators x_v and the arcs $(u, v) \in A(\Gamma)$ correspond to the relators $R(x_u, x_v)$ so that the group $G_\Gamma(R)$ is defined by the presentation

$$P_\Gamma(R) = \langle x_v \ (v \in V(\Gamma)) \mid R(x_u, x_v) \ ((u, v) \in A(\Gamma)) \rangle$$

A group is called a **digraph group** if it is isomorphic to $G_\Gamma(R)$ for some Γ and R [1].

Notational Convention 3.1 [1]. “We use α and β to represent the exponent sums of a and b in a cyclically reduced word $R(a, b)$, respectively, and K is used to indicate a group defined *e-ISSN: 2148-2683*

by the presentation $\langle a, b \mid R(a, b) \rangle$. As far as cyclic permutation is considered, the word R has the form $a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_t} b^{\beta_t}$ with $t \geq 1$ and $\alpha_i, \beta_i \in \mathbb{Z} \setminus \{0\}$ ($1 \leq i \leq t$).”

Pride defined the following property in [4, page 246]. “If no non-empty word of the form $a^k b^{-l}$ ($k, l \in \mathbb{Z}$) is equal to the identity in that group, then a two-generator group with generators a and b is said to have Property W_1 (with respect to a and b). Under the hypothesis that the girth of the underlying undirected graph of Γ is at least 4”. The conclusion that $G_\Gamma(R)$ is neither finite of rank 3 nor trivial cannot necessarily be drawn if the underlying undirected graph of Γ , or the undirected graph created from by replacing each directed edge with an undirected edge, has a cycle of length 2 or 3. We therefore put forth the hypothesis that the undirected graph of Γ has a girth of at least 4 [1].

Corollary 3.2 [4, Theorem 4]. “Let Γ be a non-empty finite digraph whose underlying undirected graph has the girth $g(\Gamma) \geq 4$ and let $R(a, b)$ be as in notational convention. If K has Property W_1 , then $G_\Gamma(R)$ is infinite”.

Therefore, it is crucial to research groups that do not have Property W_1 .

Proposition 3.3 [4, page 248]. If $k, l \in \mathbb{Z} \setminus \{0\}$ with $a^k = b^l$ in K are present, then, $a^\alpha = b^\beta$ in K , where $\alpha \neq 0, \beta \neq 0$.

Thus, $\alpha \neq 0, \beta \neq 0$ and $a^\alpha = b^\beta$ in K if and only if K does not have Property W_1 .

Now we know that our relator is $a^\alpha = b^\beta$. We will explain how it works in an example to understand how a digraph group can present.

Example 3.4. Let Λ be a digraph as in Figure 5 (this is just one of the possibilities for Γ_1 in Figure 1), below.

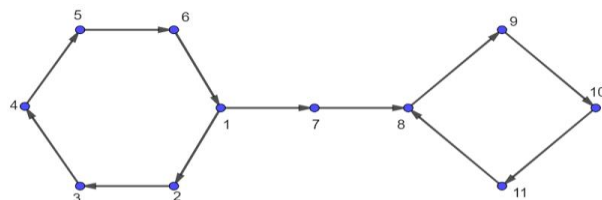


Figure 5: An example of a diigraph

The small circles are the generators and the lines between them are the relators. The corresponding digraph groups $G_\Lambda(R)$ can present as

$$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, \mid R(x_1, x_2), R(x_2, x_3), R(x_3, x_4), R(x_4, x_5), R(x_5, x_6), R(x_6, x_1), R(x_1, x_7), R(x_7, x_8), R(x_8, x_9), R(x_9, x_{10}), R(x_{10}, x_{11}), R(x_{11}, x_8) \rangle,$$

where each relator $R(a, b) = a^\alpha b^{-\beta}$ ($\alpha \neq 0, \beta \neq 0$).

Now the question is whether the corresponding digraph group write in terms of one generator. If yes, then we could say this digraph group is finite cyclic (the author assures that this answer and more is true).

4. Conclusions and Recommendations

In this paper, we proved that there are 35 classes of digraphs to consider in the case $|V(\Gamma)| = |A(\Gamma)| - 1$. We also describe how a class of digraphs is represented by a corresponding group, known as a digraph group.

Open question. When we think in general, the example 3.4 above is just one small case for one digraph family in 35 possible digraph families as we already show. Can this find a way to show all of them are finite or infinite. If yes, then what the order is for finite ones.

5. Acknowledge

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