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# Existence of Positive Periodic Solutions of First Order Neutral Differential Equations

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**Abstract**

In this paper, we consider two classes of first order neutral nonlinear differential equations and we give some new sufficient conditions for the existence of positive periodic solutions of these equations by using the Krasnoselskii's fixed point theorem. An illustrative example is provided to support the theory developed in this study.

***Keywords:*** *Fixed point, Neutral equations, Positive periodic solution, First-order.*
***2010 Mathematics Subject Classification:*** *34C25.*

## 1. Introduction

In the present work, we provide new sufficient conditions for the existence of positive $\omega$-periodic solutions of the following first-order neutral differential equations

$$\left[r(t)\left[x(t)-\sum_{i=1}^{m}c_i(t)x(t-\tau_i(t))\right]\right]'=-q(t)x(t)+\sum_{j=1}^{n}f_j(t,x(t-\sigma_j(t))) \tag{1.1}$$

and

$$\left[r(t)\left[x(t)-\sum_{i=1}^{m}c_i(t)\int_{-\infty}^{0}P(\xi)x(t+h_i(\xi))d\xi\right]\right]'=-q(t)x(t)+\sum_{j=1}^{n}b_j(t)\int_{-\infty}^{0}P(\xi)f_j(t,x(t+g_j(\xi)))d\xi, \tag{1.2}$$

where $c_i,\tau_i,b_j,\sigma_j\in C(\mathbb{R},\mathbb{R})$ are $\omega$-periodic functions $(i=1,2,\dots,m; j=1,2,\dots,n.)$, $r\in C^1(\mathbb{R},\mathbb{R})$ with $r(t)>0$, $q\in C(\mathbb{R},\mathbb{R})$ with $q(t)>0$, $r,q$ are $\omega$-periodic functions, $f_j\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $f_j$ is $\omega$-periodic in $t$. In addition, $h_i,g_j\in C(\mathbb{R},\mathbb{R})$ $(i=1,2,\dots,m; j=1,2,\dots,n.)$, $P\in C(\mathbb{R},[0,\infty))$ with $\int_{-\infty}^{0}P(\xi)d\xi=1$.

In recent years, there has been a rapid growth of interest in the existence of positive periodic solutions of first-order neutral differential equations which appear in the control models, blood cell production models and population models. We also refer the reader to the papers [5, 7, 8] for applications of nonlinear neutral differential equations in mathematical, theoretical, and chemical physics. Existence of positive periodic solutions of the following neutral differential equations

$$\frac{d}{dt}\left[x(t)-cx(t-\tau(t))\right]=-a(t)x(t)+f(t,x(t-\tau(t))) \tag{1.3}$$

and

$$\frac{d}{dt}\left[x(t)-c\int_{-\infty}^{0}Q(r)x(t+r)dr\right]=-a(t)x(t)+b(t)\int_{-\infty}^{0}Q(r)f(t,x(t+r))dr, \tag{1.4}$$

where $0\leqslant c<1$ and $-1<c<0$, and

$$[g(t)(x(t)+c(t)x(t-\tau(t)))]'=-a(t)x(t)+f(t,x(t-\tau(t))), \tag{1.5}$$

and

$$\left[g(t)(x(t)+c(t)\int_{-\infty}^{0}Q(r)x(t+h(r))dr)\right]'=-a(t)x(t)+b(t)\int_{-\infty}^{0}Q(r)f(t,x(t+h(r)))dr, \tag{1.6}$$

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where $-c_1 \leqslant c(t) \leqslant c_2$ for $c_1, c_2 \geqslant 0$ with $c_1 + c_2 < 1$, were investigated in [11] and [10], respectively. As we see from the above, equations (1.3) and (1.5) are special forms of equation (1.1), and equations (1.4) and (1.6) are special forms of equation (1.2). Consequently, the results presented in this paper generalize the main results in [11] and [10]. We refer to [2, 3, 4, 6, 9, 12, 13] and references therein for recent studies of positive periodic solutions of neutral differential equations.
The following fixed point theorem will be used in proofs.

**Lemma 1.1.** *(Krasnoselskii's Fixed Point Theorem [1] p.8). Let $X$ be a Banach space, let $\Omega$ be a bounded closed and convex subset of $X$ and, let $T,S$ be maps of $\Omega$ into $X$ such that $Tx+Sy \in \Omega$ for every pair $x,y \in \Omega$. If $T$ is a contractive and $S$ is completely continuous, then the equation*

$$Tx + Sx = x$$

*has a solution in $\Omega$.*

## 2. Main Results

Let $\Phi = \{x(t) : x(t) \in C(\mathbb{R},\mathbb{R}), \quad x(t+\omega) = x(t), \quad t \in \mathbb{R}\}$ with the sup norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. It is clear that $\Phi$ is a Banach space. Let $r_0 = \min_{t\in[0,\omega]} r(t)$ and $r_1 = \max_{t\in[0,\omega]} r(t)$.

**Theorem 2.1.** *Assume that $c_i(t) \geqslant 0$, $i = 1,2,\dots,m$, $\sum_{i=1}^m c_i(t) \leqslant c < 1$ and there exist positive constants $M_1$ and $M_2$ with $0 < M_1 < M_2$ such that*

$$r_1 M_1 \leqslant r(t) \left[ \frac{\sum_{j=1}^n f_j(t,x_j)}{q(t)} - \sum_{i=1}^m c_i(t) y_i \right] \leqslant r_0(1-c) M_2, \tag{2.1}$$

*$\forall t \in [0,\omega]$ and $\forall x_j, y_i \in [M_1, M_2]$, $(j = 1,2,\dots n; i = 1,2,\dots m)$. Then (1.1) has at least one positive $\omega$-periodic solution $x(t) \in [M_1, M_2]$.*

*Proof.* We note that finding an $\omega$-periodic solution of (1.1) is equivalent to finding an $\omega$-periodic solution of the integral equation

$$x(t) = \sum_{i=1}^m c_i(t) x(t-\tau_i(t)) + \frac{1}{r(t)} \int_t^{t+\omega} G(t,s) \left[ \sum_{j=1}^n f_j(s, x(s-\sigma_j(s))) - q(s) \sum_{i=1}^m c_i(s) x(s-\tau_i(s)) \right] ds,$$

where

$$G(t,s) = \frac{\exp(\int_t^s \frac{q(u)}{r(u)} du)}{\exp(\int_0^\omega \frac{q(u)}{r(u)} du) - 1}.$$

Let $\Omega = \{x \in \Phi : M_1 \leqslant x(t) \leqslant M_2,\ t \in [0,\omega]\}$. It can be seen that $\Omega$ is a bounded, closed and convex subset of $\Phi$. Define two operators $T,S : \Omega \to \Phi$ as follows

$$(Tx)(t) = \sum_{i=1}^m c_i(t) x(t-\tau_i(t)) \tag{2.2}$$

and

$$(Sx)(t) = \frac{1}{r(t)} \int_t^{t+\omega} G(t,s) \left[ \sum_{j=1}^n f_j(s, x(s-\sigma_j(s))) - q(s) \sum_{i=1}^m c_i(s) x(s-\tau_i(s)) \right] ds. \tag{2.3}$$

For any $x \in \Omega$ and $t \in \mathbb{R}$, it follows from (2.2) and (2.3) that

$$(Tx)(t+\omega) = \sum_{i=1}^m c_i(t+\omega) x(t+\omega-\tau_i(t+\omega)) = \sum_{i=1}^m c_i(t) x(t-\tau_i(t)) = (Tx)(t)$$

and

$$\begin{aligned}
(Sx)(t+\omega) &= \frac{1}{r(t+\omega)} \int_{t+\omega}^{t+2\omega} G(t+\omega,s) \left[ \sum_{j=1}^n f_j(s, x(s-\sigma_j(s))) - q(s) \sum_{i=1}^m c_i(s) x(s-\tau_i(s)) \right] ds \\
&= \frac{1}{r(t+\omega)} \int_t^{t+\omega} G(t+\omega, v+\omega) \left[ \sum_{j=1}^n f_j(v+\omega, x(v+\omega-\sigma_j(v+\omega))) - q(v+\omega) \sum_{i=1}^m c_i(v+\omega) x(v+\omega-\tau_i(v+\omega)) \right] dv \\
&= \frac{1}{r(t)} \int_t^{t+\omega} G(t,v) \left[ \sum_{j=1}^n f_j(v, x(v-\sigma_j(v))) - q(v) \sum_{i=1}^m c_i(v) x(v-\tau_i(v)) \right] dv \\
&= (Sx)(t)
\end{aligned}$$

which implies that $T(\Omega) \subset \Phi$ and $S(\Omega) \subset \Phi$. Now, we show that $Tx+Sy \in \Omega$ for all $x,y \in \Omega$ and $t \in \mathbb{R}$. By using (2.1), (2.2) and (2.3), we have

$$\begin{aligned}(Tx)(t)+(Sy)(t) &= \sum_{i=1}^{m} c_i(t)x(t-\tau_i(t)) + \frac{1}{r(t)}\int_t^{t+\omega} G(t,s)\left[\sum_{j=1}^{n} f_j(s,y(s-\sigma_j(s))) - q(s)\sum_{i=1}^{m} c_i(s)y(s-\tau_i(s))\right]ds \\ &\leqslant \sum_{i=1}^{m} c_i(t)M_2 + \frac{1}{r_0}\int_t^{t+\omega} G(t,s)q(s)\left[\frac{\sum_{j=1}^{n} f_j(s,y(s-\sigma_j(s)))}{q(s)} - \sum_{i=1}^{m} c_i(s)y(s-\tau_i(s))\right]ds \\ &\leqslant cM_2 + (1-c)M_2\int_t^{t+\omega} G(t,s)\frac{q(s)}{r(s)}ds = M_2\end{aligned}$$

and

$$\begin{aligned}(Tx)(t)+(Sy)(t) &= \sum_{i=1}^{m} c_i(t)x(t-\tau_i(t)) + \frac{1}{r(t)}\int_t^{t+\omega} G(t,s)\left[\sum_{j=1}^{n} f_j(s,y(s-\sigma_j(s))) - q(s)\sum_{i=1}^{m} c_i(s)y(s-\tau_i(s))\right]ds \\ &\geqslant \frac{1}{r_1}\int_t^{t+\omega} G(t,s)q(s)\left[\frac{\sum_{j=1}^{n} f_j(s,y(s-\sigma_j(s)))}{q(s)} - \sum_{i=1}^{m} c_i(s)y(s-\tau_i(s))\right]ds \\ &\geqslant M_1\int_t^{t+\omega} G(t,s)\frac{q(s)}{r(s)}ds = M_1.\end{aligned}$$

Thus, $Tx+Sy \in \Omega$, for all $x,y \in \Omega$.
Next, we show that $T$ is a contraction mapping. For $x,y \in \Omega$, we have

$$\begin{aligned}|(Tx)(t)-(Ty)(t))| &= \left|\sum_{i=1}^{m} c_i(t)x(t-\tau_i(t)) - \sum_{i=1}^{m} c_i(t)y(t-\tau_i(t))\right| \\ &= \left|\sum_{i=1}^{m} c_i(t)\Big[x(t-\tau_i(t)) - y(t-\tau_i(t))\Big]\right| \\ &\leqslant \sum_{i=1}^{m} c_i(t)\Big|x(t-\tau_i(t)) - y(t-\tau_i(t))\Big|.\end{aligned}$$

By taking the sup norm of both sides, we see that

$$\|Tx-Ty\| \leqslant c\|x-y\|.$$

Since $c<1$, $T$ is a contraction mapping.
Finally, we show that $S$ is completely continuous. First, we shall show that $S$ is continuous. Let $\{x_k\} \in \Omega$ be a convergent sequence of functions such that $x_k(t) \to x(t)$ as $k \to \infty$. Since $\Omega$ is closed, $x \in \Omega$. For $t \in [0,\omega]$, we have

$$\begin{aligned}|(Sx_k)(t)-(Sx)(t)| &= \left|\frac{1}{r(t)}\int_t^{t+\omega} G(t,s)\left[\sum_{j=1}^{n} f_j(s,x_k(s-\sigma_j(s))) - q(s)\sum_{i=1}^{m} c_i(s)x_k(s-\tau_i(s))\right]ds\right. \\ &\quad \left. - \frac{1}{r(t)}\int_t^{t+\omega} G(t,s)\left[\sum_{j=1}^{n} f_j(s,x(s-\sigma_j(s))) - q(s)\sum_{i=1}^{m} c_i(s)x(s-\tau_i(s))\right]ds\right| \\ &\leqslant \frac{1}{r_0}\int_t^{t+\omega} G(t,s)\sum_{j=1}^{n}\left|f_j(s,x_k(s-\sigma_j(s))) - f_j(s,x(s-\sigma_j(s)))\right|ds \\ &\quad + \frac{1}{r_0}\int_t^{t+\omega} G(t,s)q(s)\sum_{i=1}^{m}\Big[c_i(s)\left|x_k(s-\tau_i(s)) - x(s-\tau_i(s))\right|\Big]ds.\end{aligned}$$

Since $|f_j(t,x_k(t-\sigma_j(t))) - f_j(t,x(t-\sigma_j(t)))| \to 0$, $j=1,2,\ldots n$, and $|x_k(t-\tau_i(t)) - x(t-\tau_i(t))| \to 0$, $i=1,2,\ldots m$, as $k \to \infty$, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{k\to\infty} \|(Sx_k)-(Sx)\| = 0$$

and therefore $S$ is continuous. Second, we prove that $S\Omega$ is relatively compact. It suffices to show that the family of functions $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[0,\omega]$. From (2.3), we see that

$$\begin{aligned}|(Sx)(t)| &= \left|\frac{1}{r(t)}\int_t^{t+\omega} G(t,s)\left[\sum_{j=1}^{n} f_j(s,x(s-\sigma_j(s))) - q(s)\sum_{i=1}^{m} c_i(s)x(s-\tau_i(s))\right]ds\right| \\ &\leqslant \frac{1}{r_0}\int_t^{t+\omega} G(t,s)q(s)\left[\frac{\sum_{j=1}^{n} f_j(s,x(s-\sigma_j(s)))}{q(s)} - \sum_{i=1}^{m} c_i(s)x(s-\tau_i(s))\right]ds \\ &\leqslant (1-c)M_2\int_t^{t+\omega} G(t,s)\frac{q(s)}{r(s)}ds = (1-c)M_2\end{aligned}$$

and it follows that

$$\|Sx\| \leqslant (1-c)M_2.$$

On the other hand, using (2.3), we obtain

$$\begin{aligned}
|(Sx)'(t)| &= \left| \frac{-q(t)x(t)+\sum_{j=1}^{n} f_j(t,x(t-\sigma_j(t)))-r'(t)(Sx)(t)}{r(t)} \right| \\
&\leqslant \frac{1}{r_0}\left[ |q(t)||x(t)|+|\sum_{j=1}^{n} f_j(t,x(t-\sigma_j(t)))|+|r'(t)||(Sx)(t)| \right] \\
&\leqslant \frac{M_2}{r_0}\left[ 2\|q\|+(1-c)\|r'\| \right],
\end{aligned}$$

which shows that $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[0,\omega]$. Therefore $S\Omega$ is relatively compact. By Lemma 1.1, there is an $x \in \Omega$ such that $Tx+Sx = x$. Hence, $x(t)$ is a positive $\omega$-periodic solution of (1.1). This completes the proof. □

**Theorem 2.2.** *Assume that $c_i(t) \leqslant 0$, $i = 1,2,\dots,m$, $-1 < c \leqslant \sum_{i=1}^{m} c_i(t)$, $-r_1c < r_0$ and there exist positive constants $M_1$ and $M_2$ with $0 < M_1 < M_2$ such that*

$$r_1(M_1-cM_2) \leqslant r(t)\left[ \frac{\sum_{j=1}^{n} f_j(t,x_j)}{q(t)} - \sum_{i=1}^{m} c_i(t)y_i \right] \leqslant r_0M_2,$$

*$\forall t \in [0,\omega]$ and $\forall x_j, y_i \in [M_1,M_2]$, $(j = 1,2,\dots n; i = 1,2,\dots m)$. Then (1.1) has at least one positive $\omega$-periodic solution $x(t) \in [M_1,M_2]$.*

**Theorem 2.3.** *Assume that $c_i(t) \geqslant 0$, $i = 1,2,\dots,m$, $\sum_{i=1}^{m} c_i(t) \leqslant c < 1$ and there exist positive constants $M_1$ and $M_2$ with $0 < M_1 < M_2$ such that*

$$r_1M_1 \leqslant r(t)\left[ \frac{\sum_{j=1}^{n} b_j(t)f_j(t,x_j)}{q(t)} - \sum_{i=1}^{m} c_i(t)y_i \right] \leqslant r_0(1-c)M_2,$$

*$\forall t \in [0,\omega]$ and $\forall x_j, y_i \in [M_1,M_2]$, $(j = 1,2,\dots n; i = 1,2,\dots m)$. Then (1.2) has at least one positive $\omega$-periodic solution $x(t) \in [M_1,M_2]$.*

*Proof.* As in the proof of Theorem 2.1, it is clear that finding an $\omega$-periodic solution of (1.2) is equivalent to finding an $\omega$-periodic solution of the integral equation

$$\begin{aligned}
x(t) &= \sum_{i=1}^{m} c_i(t)\int_{-\infty}^{0} P(\xi)x(t+h_i(\xi))d\xi \\
&+ \frac{1}{r(t)}\int_{t}^{t+\omega} G(t,s)\left[ \sum_{j=1}^{n} b_j(s)\int_{-\infty}^{0} P(\xi)f_j(s,x(s+g_j(\xi)))d\xi - q(s)\sum_{i=1}^{m} c_i(s)\int_{-\infty}^{0} P(\xi)x(s+h_i(\xi))d\xi \right]ds.
\end{aligned}$$

We define $\Omega$ as in the proof of Theorem 2.1 and $T$, $S$ and $G(t,s)$ as in the following, respectively,

$$(Tx)(t) = \sum_{i=1}^{m} c_i(t)\int_{-\infty}^{0} P(\xi)x(t+h_i(\xi))d\xi$$

and

$$(Sx)(t) = \frac{1}{r(t)}\int_{t}^{t+\omega} G(t,s)\left[ \sum_{j=1}^{n} b_j(s)\int_{-\infty}^{0} P(\xi)f_j(s,x(s+g_j(\xi)))d\xi - q(s)\sum_{i=1}^{m} c_i(s)\int_{-\infty}^{0} P(\xi)x(s+h_i(\xi))d\xi \right]ds,$$

where

$$G(t,s) = \frac{\exp(\int_t^s \frac{q(u)}{r(u)}du)}{\exp(\int_0^\omega \frac{q(u)}{r(u)}du)-1}.$$

The remaining part of the proof follows the same lines as in the proof of Theorem 2.1. □

**Theorem 2.4.** *Assume that $c_i(t) \leqslant 0$, $i = 1,2,\dots,m$, $-1 < c \leqslant \sum_{i=1}^{m} c_i(t)$, $-r_1c < r_0$ and there exist positive constants $M_1$ and $M_2$ with $0 < M_1 < M_2$ such that*

$$r_1(M_1-cM_2) \leqslant r(t)\left[ \frac{\sum_{j=1}^{n} b_j(t)f_j(t,x_j)}{q(t)} - \sum_{i=1}^{m} c_i(t)y_i \right] \leqslant r_0M_2,$$

*$\forall t \in [0,\omega]$ and $\forall x_j, y_i \in [M_1,M_2]$, $(j = 1,2,\dots n; i = 1,2,\dots m)$. Then (1.2) has at least one positive $\omega$-periodic solution $x(t) \in [M_1,M_2]$.*

**Example 2.5.** *Consider the first-order neutral differential equation*

$$\begin{aligned}
&\left[ \left(1+\frac{\cos t}{10}\right)\left[ x(t) - \frac{e^{\cos t}}{100}x(t-e^{\sin t}) - \frac{e^{\sin t}}{100}x(t+e^{\cos t}) \right] \right]' = \\
&-\left(1+\frac{\sin t}{10}\right)x(t)+20+e^{\sin t}+\sin(x(t-e^{\sin t}))+10+e^{\cos t}-\cos(x(t+e^{\cos t})).
\end{aligned} \tag{2.4}$$

*Note that (2.4) is of the form (1.1) with $m = n = 2$, $\omega = 2\pi$, $r(t) = 1+\frac{\cos t}{10}$, $c_1(t) = \frac{e^{\cos t}}{100}$, $c_2(t) = \frac{e^{\sin t}}{100}$, $q(t) = 1+\frac{\sin t}{10}$, $f_1(t,x) = 20+e^{\sin t}+\sin(x)$, $f_2(t,x) = 10+e^{\cos t}-\cos(x)$, $\tau_1(t) = \sigma_1(t) = e^{\sin t}$, $\tau_2(t) = \sigma_2(t) = -e^{\cos t}$. It is easy to verify that the conditions of Theorem 2.1 are satisfied with $M_1 = 15$ and $M_2 = 60$. Thus (2.4) has at least one positive $2\pi$-periodic solution.*

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