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Binomial Transforms for Hybrid Numbers Defined Through Fibonacci and Lucas Number Components

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Abstract

The main purpose of covering this article has been to examine the hybrid numbers defined through Fibonacci and Lucas number components $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ and their binomial transforms $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$, respectively. Firstly, general sum formulas, binomial identities, which are not in the literature yet, about hybrid numbers $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ are discussed. Then, the recurrence relation is obtained for $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ and some results have been found for the new sequence.

Keywords: *Binomial transforms, Hybrid numbers, Lucas sequences of first and second kind.*
2010 Mathematics Subject Classification: *11B37, 11B39, 11B65, 11R52.*

1. Introduction

The sequences

$$\begin{aligned} U_n &= U_n(p,q) = pU_{n-1} - qU_{n-2} \\ V_n &= V_n(p,q) = pV_{n-1} - qV_{n-2} \end{aligned} \tag{1.1}$$

for $n \geq 2$ with initial values $U_0 = 0, U_1 = 1, V_0 = 2$ and $V_1 = p$ are defined with p and q that are non–zero integers such that $D = p^2 - 4q \neq 0$. The sequences U_n and V_n are called the (first and second) Lucas sequences with parameters p and q. The characteristic equation of U_n and V_n is $x^2 - px + q = 0$. Also the roots are $\alpha = \frac{p+\sqrt{D}}{2}$ and $\beta = \frac{p-\sqrt{D}}{2}$. So their Binet formulas are

$$\begin{aligned} U_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \\ V_n &= \alpha^n + \beta^n \end{aligned}$$

for $n \geq 0$ respectively. Moreover, $\alpha^2 = p\alpha - q$ and $\beta^2 = p\beta - q$.
The generating functions of U_n and V_n are

$$\begin{aligned} U(x) &= \frac{x}{1 - px + qx^2} \\ V(x) &= \frac{2 - px}{1 - px + qx^2} \end{aligned} \tag{1.2}$$

respectively.

Fibonacci, Lucas, Pell and Jacobsthal numbers can be derived from (1.1). In addition to, for $p = 1$ and $q = -1$, the numbers $U_n = U_n(1,-1)$ are called the Fibonacci numbers, while the numbers $V_n = V_n(1,-1)$ are called the Lucas numbers. Similarly, for $p = 2$ and $q = -1$, the numbers $U_n = U_n(2,-1)$ are called the Pell numbers, while the numbers $V_n = V_n(2,-1)$ are called the Pell–Lucas numbers (for further details see [1]). Also it follows that $\alpha^2 = p\alpha - q$ and $\beta^2 = p\beta - q$. If we generalize this situation even more $\alpha^n = \alpha U_n - qU_{n-1}$ and $\beta^n = \beta U_n - qU_{n-1}$ for all n is integer. In addition to the terms of these sequences can be extended to negative subscripts as $U_{-n} = -q^{-n}U_n$ and $V_{-n} = q^{-n}V_n$ (see [2]). Various articles on these sequences are available, one of which is [3].

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The Horadam numbers, which are the generalizations of Fibonacci, Lucas, Pell and Jacobsthal numbers are defined as follows:

$$W_n = pW_{n-1} - qW_{n-2} \tag{1.3}$$

for $n \geq 2$ with initial values $W_0 = a, W_1 = b$. Here, a, b, p, and q are integers. In addition to, Binet formula for the Horadam numbers is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

where also $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2-4q}}{2}$ are the roots of the characteristic equation for the sequence. We also know that A and B are

$$\begin{aligned} A &= b - a\beta \\ B &= b - a\alpha \end{aligned} \tag{1.4}$$

(see [4, 5]).
Özdemir define a new generalization of complex, hyperbolic and dual numbers different from another generalizations. This is a generalization that presents a system of such numbers that consists of all three number systems together that has to be four-dimensional. The set H of hybrid numbers z, has the form

$$H = \{z = a + bi + c\varepsilon + dh;\ a,b,c,d \in \mathbb{R}\} \tag{1.5}$$

where i, ε, h are operators such that

$$i^2 = -1,\ \varepsilon^2 = 0,\ \ ih = -hi = \varepsilon + i.$$

The conjugate of hybrid number z is defined by

$$\bar{z} = \overline{a + bi + c\varepsilon + dh} = a - bi - c\varepsilon - dh.$$

The multiplication of hybrid numbers is not commutative, but it has the property of associativity. The set H of hybrid numbers forms a non-commutative ring with respect to the addition and multiplication operations. The real number $C(z) = z\bar{z} = \bar{z}z = a^2 + b^2 - 2bc - d^2$ is called the character of the hybrid number z and the real number $\sqrt{|C(z)|}$ will be called the norm of the hybrid number z and it will be denoted by $\|z\|$. For more information about these operators see [6].
Now we summarize briefly, the relevant known facts about hybrid numbers. In [7], Fibonacci hybrid numbers were mentioned for the first time in the literature. Jacobsthal and Jacobsthal-Lucas hybrid numbers are studied by Liana and Wloch [8]. In [9, 10], the authors have adverted the special identities of the Horadam hybrid numbers. In [11], modified $k-$Pell hybrid sequence is defined and also some identities are obtained. In addition to Polatlı in [12], he defined hybrid numbers with Fibonacci and Lucas hybrid number coefficients. In [13], hybrid numbers created by using tribonacci and tribonacci-Lucas numbers, which are an integer sequences with a third order recurrence relation, have been investigated.
In [14], given a sequence $A = \{a_1, a_2, \cdots\}$, its binomial transform B is the sequence $B(A) = \{b_n\}$ defined as follows:

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i. \tag{1.6}$$

The binomial transforms of sequences can be defined, and many authors have studied this topic [15]. Actually few of them was studied by Chen [16] and later was studied by Falcon in [17], and also binomial transform of quadrapell quaternions in [18]. In [19], the authors obtained new quaternion sequences by using binomial transform and iterated binomial transform for quaternion sequences and obtained some results about these sequences. In recent years, another work done for binomial transform is [20]. Again, another author of binomial transforms for balancing polynomials is Yılmaz [21].
There are also some nice relationships between the transforms for example, the binomial transform and the inverse binomial transform. For the sequence A, the inverse binomial transform of A is defined to be the sequence $B^{-1}(A) = c_n$, where c_n is given by $c_n = \sum\limits_{n=0}^{\infty} \binom{n}{i}(-1)^{n-i}a_i$. They have been applied to the k–Fibonacci sequence in Falcon and Plaza [17]. The information in this article is as follows: Binomial transform $b_{k,n} = \sum_{i=0}^{n} \binom{n}{i} F_{k,i}$, falling k–binomial transform $f_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} F_{k,i}$, rising k–binomial transform $z_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i}$, k–binomial transform $w_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^n F_{k,i}$. For $k = 1$, these four transforms coincide and therefore when applied to the classical Fibonacci sequence, produce the bisection sequence of the classical Fibonacci sequence. There are articles that reveal the relations of binomial transforms with different disciplines. For example [22] is one of them.

2. Hybrid Numbers Defined Through Fibonacci and Lucas Number Components

Definition 2.1. *Hybrid numbers defined through Fibonacci and Lucas number components $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ are defined by*

$$\begin{aligned} \hat{H}_{U,n+2} &= U_{n+2}1 + U_{n+3}i + U_{n+4}\varepsilon + U_{n+5}h, \\ \hat{H}_{V,n+2} &= V_{n+2}1 + V_{n+3}i + V_{n+4}\varepsilon + V_{n+5}h \end{aligned}$$

where U_n are called the first Lucas sequences and V_n are called the second Lucas sequences with parameters p and q, respectively. Initial conditions are

$$\begin{aligned} \hat{H}_{U,0} &= (0, 1, p, p^2 - q) \\ \hat{H}_{U,1} &= (1, p, p^2 - q, p^3 - 2pq) \end{aligned}$$

and

$$\begin{aligned} \hat{H}_{V,0} &= (2, p, p^2 - 2q, p^3 - 3pq) \\ \hat{H}_{V,1} &= (p, p^2 - 2q, p^3 - 3pq, p^4 - 4p^2q + 2q^2). \end{aligned}$$

Furthermore, recurrence relations for these numbers are

$$\hat{H}_{U,n+2} = p\hat{H}_{U,n+1} - q\hat{H}_{U,n} \tag{2.1}$$
$$\hat{H}_{V,n+2} = p\hat{H}_{V,n+1} - q\hat{H}_{V,n}. \tag{2.2}$$

Also

$$\alpha_1 = \frac{p+\sqrt{p^2-4q}}{2} \quad \text{and} \quad \alpha_2 = \frac{p-\sqrt{p^2-4q}}{2}$$

which are the roots of the characteristic equation. Note that,

$$\alpha_1 + \alpha_2 = p,$$
$$\alpha_1\alpha_2 = q$$

also

$$\Delta = \alpha_1 - \alpha_2 = \sqrt{p^2-4q}.$$

In addition to $\underline{\alpha_1}$ and $\underline{\alpha_2}$ defined as $\underline{\alpha_1} = 1 + i\alpha_1 + \varepsilon\alpha_1^2 + h\alpha_1^3$, $\underline{\alpha_2} = 1 + i\alpha_2 + \varepsilon\alpha_2^2 + h\alpha_2^3$.
The Horadam hybrid numbers will now be mentioned in a moment. Because the following results were found in [9, 10] and the hybrid numbers defined through generalized Fibonacci and Lucas numbers are actually included in Horadam hybrid numbers. Note that, the definition of Horadam hybrid numbers is reduced to $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ depending on the choice of the parameters. If we choose $a = 0, b = 1, p = p$ and $q = q$ in equation (1.3), it means that, these hybrid numbers are called hybrid numbers defined through Fibonacci number components $\{\hat{H}_{U,n}\}$. Similarly for hybrid numbers defined through Lucas number components $\{\hat{H}_{V,n}\}$ if we choose $a = 2, b = p, p = p$ and $q = q$, the same is true.

In [9], $n-$th Horadam hybrid number H_n is defined as

$$H_n = W_n + iW_{n+1} + \varepsilon W_{n+2} + hW_{n+3}.$$

W_n passing here, defined by the recursive equation (1.3).
Let's take a look at some of the properties found for the sequence.

Theorem 2.2. *[9] The generating function for Horadam hybrid numbers is*

$$\sum_{n=0}^{\infty} H_n t^n = \frac{H_0 + t(H_1 - pH_0)}{1 - pt + qt^2}.$$

Theorem 2.3. *[9] Binet's formula for Horadam Hybrid numbers is*

$$H_n = \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha - \beta}$$

where $\underline{\alpha} = 1 + i\alpha + \varepsilon\alpha^2 + h\alpha^3$, $\underline{\beta} = 1 + i\beta + \varepsilon\beta^2 + h\beta^3$ *and also A and B are denoted from equation (1.4).*

Theorem 2.4. *[10] For every positive integer n, the sum of Horadam Hybrid numbers is*

$$\sum_{t=1}^{n} H_t = \frac{1}{p-q-1}(H_{n+1} - qH_n - H_1 + qH_0).$$

Theorem 2.5. *[10] The exponential generating function for Horadam hybrid numbers is*

$$G_e(t) = \frac{A\underline{\alpha}e^{\alpha t} - B\underline{\beta}e^{\beta t}}{\alpha - \beta}$$

where $\underline{\alpha}$ *and* $\underline{\beta}$ *are denoted from Theorem 2.3.*

Theorem 2.6. *[10] (Vajda's identity) For any integers n, r and s, the Vajda's identity for the sequence is*

$$H_{n+r}H_{n+s} - H_nH_{n+r+s} = ABq^n u_r(2U_s - u_s\vartheta - v_s(U_0 + q\eta))$$

where $\vartheta = 1 + q - pq - q^3$ *and* $\eta = -iu_2 + \varepsilon(qu_1 - u_2) + hu_1$*. Here for* $a = 1, b = p$ *Horadam sequence is* $\{U_n\}$ *and* $a = 2, b = p$ *Horadam sequence is* $\{V_n\}$.

With the Vajda identity in above theorem, Catalan's identity, Cassini's identity and d'Ocagne identity have also been shown.
As we tried to explain above, all results in the above theorems given by Horadam hybrid numbers are valid for the sequence $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ which we are studying in this study, because Horadam sequence provides results that include all of them. We will try to find different results for the hybrid numbers defined through Fibonacci and Lucas number components, that support the above theorems.
Now, in the two following theorems, different binomial identities will be given.

Theorem 2.7. *For $k \in \mathbb{N}$ and $s \in \mathbb{Z}$ and also hybrid numbers $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$, we obtain*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \hat{H}_{U,2k+s} &= \hat{H}_{U,n+s} \\ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \hat{H}_{V,2k+s} &= \hat{H}_{V,n+s} \end{aligned}$$

respectively.

Proof. If the Binet formula and binomial formula is used again with the same think

$$\begin{aligned} & \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} q^{n-k} \hat{H}_{U,2k+s} \right] \frac{x^n}{n!} \\ = & \left(\sum_{n=0}^{\infty} q^n \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \hat{H}_{U,2n+s} \frac{x^n}{n!} \right) \\ = & \left(\sum_{n=0}^{\infty} q^n \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{A_U \underline{\alpha_1} \alpha_1^{2n+s} - B_U \underline{\alpha_2} \alpha_2^{2n+s}}{\Delta} \frac{x^n}{n!} \right). \end{aligned}$$

Considering the binomial part,

$$\begin{aligned} & e^{qx} \left(\frac{\underline{\alpha_1} \alpha_1^s e^{\alpha_1^2 x} - \underline{\alpha_2} \alpha_2^s e^{\alpha_2^2 x}}{\Delta} \right) \\ = & \left(\frac{\underline{\alpha_1} \alpha_1^s}{\Delta} \right) e^{\alpha_1 x} - \left(\frac{\underline{\alpha_2} \alpha_2^s}{\Delta} \right) e^{\alpha_2 x} \\ = & \frac{\underline{\alpha_1} \alpha_1^s}{\Delta} \sum_{n=0}^{\infty} \frac{(\alpha_1 x)^n}{n!} - \frac{\underline{\alpha_2} \alpha_2^s}{\Delta} \sum_{n=0}^{\infty} \frac{(\alpha_2 x)^n}{n!} \\ = & \sum_{n=0}^{\infty} \frac{\underline{\alpha_1} \alpha_1^{n+s} - \underline{\alpha_2} \alpha_2^{n+s}}{\Delta} \frac{x^n}{n!}, \end{aligned}$$

we get the result. Similar proof is made within $\{\hat{H}_{V,n}\}$. □

Theorem 2.8. *For $k \in \mathbb{N}$ and $s \in \mathbb{Z}$ and also hybrid numbers $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$, we obtain*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-q)^{n-k} \hat{H}_{U,k+s} &= \hat{H}_{U,2n+s} \\ \sum_{k=0}^{n} \binom{n}{k} (-q)^{n-k} \hat{H}_{V,k+s} &= \hat{H}_{V,2n+s} \end{aligned}$$

respectively.

Proof. The proof will be made with the help of Binet's formula, which is the most useful method we use to find such identities.

$$\begin{aligned} & \sum_{n=0}^{\infty} \hat{H}_{U,2n+s} \frac{x^n}{n!} \\ = & \sum_{n=0}^{\infty} \frac{\underline{\alpha_1} \alpha_1^{2n+s} - \underline{\alpha_2} \alpha_2^{2n+s}}{\Delta} \frac{x^n}{n!} \\ = & \frac{\underline{\alpha_2} \alpha_1^s e^{\alpha_1^2 x} - \underline{\alpha_2} \alpha_2^s e^{\alpha_2^2 x}}{\Delta} \\ = & e^{-qx} \frac{\underline{\alpha_1} \alpha_1^s e^{\alpha_1 x} - \underline{\alpha_2} \alpha_2^s e^{\alpha_2 x}}{\Delta}. \end{aligned}$$

Let's continue the proof using the expansion of e^x,

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} (-q)^n \frac{x^n}{n!} \right) \left(\frac{\underline{\alpha_1} \alpha_1^s}{\Delta} \sum_{n=0}^{\infty} \alpha_1^n \frac{x^n}{n!} - \frac{\underline{\alpha_2} \alpha_2^s}{\Delta} \sum_{n=0}^{\infty} \alpha_2^n \frac{x^n}{n!} \right) \\ = & \left(\sum_{n=0}^{\infty} (-q)^n \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \hat{H}_{U,n+s} \frac{x^n}{n!} \right) \\ = & \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} (-q)^{n-k} \hat{H}_{U,k+s} \right) \frac{x^n}{n!}. \end{aligned}$$

So

$\sum_{k=0}^{n}\binom{n}{k}(-q)^{n-k}\hat{H}_{U,k+s}=\hat{H}_{U,2n+s}.$

With the help of the operations performed, the result is clear. Similar proof is made within $\{\hat{H}_{V,n}\}$. □

Theorem 2.9. *For $k,n\in\mathbb{N}$ and also hybrid numbers $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$, we obtain*

$$\begin{aligned}\hat{H}_{U,k}\hat{H}_{U,n+1}-q\hat{H}_{U,k-1}\hat{H}_{U,n} &= \frac{\underline{\alpha_1}^2\alpha_1^{k+n}-\underline{\alpha_2}^2\alpha_2^{k+n}}{\Delta}\\ \hat{H}_{V,k}\hat{H}_{V,n+1}-q\hat{H}_{v,k-1}\hat{H}_{V,n} &= \frac{\underline{\alpha_1}^2\alpha_1^{k+n}-\underline{\alpha_2}^2\alpha_2^{k+n}}{\Delta}\end{aligned}$$

respectively.

Proof. With the help of Binet formula, we get

$$\begin{aligned}&\hat{H}_{U,k}\hat{H}_{U,n+1}-q\hat{H}_{U,k-1}\hat{H}_{U,n}\\ =\ &\left(\frac{\underline{\alpha_1}\alpha_1^{k}-\underline{\alpha_2}\alpha_2^{k}}{\Delta}\right)\left(\frac{\underline{\alpha_1}\alpha_1^{n+1}-\underline{\alpha_2}\alpha_2^{n+1}}{\Delta}\right)\\ &-q\left(\frac{\underline{\alpha_1}\alpha_1^{k-1}-\underline{\alpha_2}\alpha_2^{k-1}}{\Delta}\right)\left(\frac{\underline{\alpha_1}\alpha_1^{n}-\underline{\alpha_2}\alpha_2^{n}}{\Delta}\right).\end{aligned}$$

Then we calculate the result

$$\begin{aligned}&\hat{H}_{U,k}\hat{H}_{U,n+1}-q\hat{H}_{U,k-1}\hat{H}_{U,n}\\ =\ &\frac{1}{\Delta^2}\left\{\begin{array}{c}\underline{\alpha_1}\alpha_1^{n+k+1}-\underline{\alpha_1}\underline{\alpha_2}\alpha_1^{k}\alpha_2^{n+1}+\underline{\alpha_2}^2\alpha_2^{n+k+1}\\ -\underline{\alpha_2}\underline{\alpha_1}\alpha_2^{k}\alpha_1^{n+1}-q\underline{\alpha_1}^2\alpha_1^{n+k-1}+q\underline{\alpha_1}\underline{\alpha_2}\alpha_1^{k-1}\alpha_2^{n}\\ -q\underline{\alpha_2}^2\alpha_2^{n+k-1}+q\underline{\alpha_2}\underline{\alpha_1}\alpha_2^{k-1}\alpha_1^{n}\end{array}\right\}\\ =\ &\frac{1}{\Delta^2}\left\{\begin{array}{c}\underline{\alpha_1}^2\alpha_1^{n+k}(\alpha_1-\frac{q}{\alpha_1})-\underline{\alpha_1}\underline{\alpha_2}\alpha_1^{k-1}\alpha_2^{n}(\alpha_1\alpha_2-q)\\ -\underline{\alpha_2}\underline{\alpha_1}\alpha_2^{k-1}\alpha_1^{n}(\alpha_1\alpha_2-q)-\underline{\alpha_2}^2\alpha_2^{n+k}(-\alpha_2+\frac{q}{\alpha_2})\end{array}\right\}\\ =\ &\frac{\underline{\alpha_1}^2\alpha_1^{k+n}-\underline{\alpha_2}^2\alpha_2^{k+n}}{\Delta}.\end{aligned}$$

Similar proof is made within $\{\hat{H}_{V,n}\}$. □

In this theorem a general formula for all sums will be found.

Theorem 2.10. *For all $n\in\mathbb{N}$ and $m,s\in\mathbb{Z}$ and also hybrid numbers $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$, we obtain*

$$\begin{aligned}\sum_{k=0}^{n}\hat{H}_{U,mk+s} &= \frac{q^m\hat{H}_{U,mn+s}-\hat{H}_{U,mk+m+s}-q^m\hat{H}_{U,s-m}+\hat{H}_{U,s}}{q^m-\left(\alpha_1^m+\alpha_2^m\right)+1}\\ \sum_{k=0}^{n}\hat{H}_{V,mk+s} &= \frac{q^m\hat{H}_{V,mn+s}-\hat{H}_{V,mk+m+s}-q^m\hat{H}_{V,s-m}+\hat{H}_{V,s}}{q^m-\left(\alpha_1^m+\alpha_2^m\right)+1}\end{aligned}$$

respectively.

Proof. Using the Binet formula, we have

$$\begin{aligned}&\sum_{k=0}^{n}\hat{H}_{U,mk+s}\\ =\ &\sum_{k=0}^{n}\frac{\underline{\alpha_1}\alpha_1^{mk+s}-\underline{\alpha_2}\alpha_2^{mk+s}}{\Delta}\\ =\ &\frac{\underline{\alpha_1}\alpha_1^{s}}{\Delta}\sum_{k=0}^{n}\alpha_1^{mk}-\frac{\underline{\alpha_2}\alpha_2^{s}}{\Delta}\sum_{k=0}^{n}\alpha_2^{mk}\\ =\ &\frac{1}{q^m-(\alpha_1^m+\alpha_2^m)+1}\left\{\begin{array}{c}q^m\frac{\underline{\alpha_1}\alpha_1^{mn+s}-\underline{\alpha_2}\alpha_2^{mn+s}}{\Delta}-\frac{\underline{\alpha_1}\alpha_1^{s}-\underline{\alpha_2}\alpha_2^{mn+m+s}}{\Delta}\\ -q^m\frac{\underline{\alpha_1}\alpha_1^{s-m}-\underline{\alpha_2}\alpha_2^{s-m}}{\Delta}+\frac{\underline{\alpha_1}\alpha_1^{s}-\underline{\alpha_2}\alpha_2^{s}}{\Delta}\end{array}\right\}\\ =\ &\frac{q^m\hat{H}_{U,mn+s}-\hat{H}_{U,mk+m+s}-q^m\hat{H}_{U,s-m}+\hat{H}_{U,s}}{q^m-\left(\alpha_1^m+\alpha_2^m\right)+1}\end{aligned}$$

Similar proof is made within $\{\hat{H}_{V,n}\}$. □

3. Binomial Transforms for Hybrid Numbers $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$

In this section, a new hybrid number sequence will be obtained by using the relations (2.1) and (2.2). First, the recurrence relation of this new sequence will be found, and then after the Binet formula is obtained, general results will be passed. Some of these general results will be created in parallel with the results given for $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ so that the relationships between them can be seen more clearly.
The binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ are

$$\begin{aligned} b\hat{H}_{U,n} &= \sum_{l=0}^{n}\binom{n}{l}\hat{H}_{U,n} \\ b\hat{H}_{V,n} &= \sum_{l=0}^{n}\binom{n}{l}\hat{H}_{V,n} \end{aligned} \tag{3.1}$$

respectively, for $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ hybrid numbers like (1.6).

Lemma 3.1. *Let $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ be the binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$. Then we have*

$$\begin{aligned} b\hat{H}_{U,n+1} &= \sum_{l=0}^{n}\binom{n}{l}(\hat{H}_{U,l}+\hat{H}_{U,l+1}) \\ b\hat{H}_{V,n+1} &= \sum_{l=0}^{n}\binom{n}{l}(\hat{H}_{V,l}+\hat{H}_{V,l+1}) \end{aligned}$$

respectively.

Proof. Since we know the (3.1) equation , we get

$$b\hat{H}_{U,n+1} = \sum_{l=0}^{n+1}\binom{n+1}{l}\hat{H}_{U,l} = \sum_{l=1}^{n+1}\binom{n+1}{l}\hat{H}_{U,l}+\hat{H}_{U,0}.$$

Also the binomial rules of $\binom{n}{l-1}+\binom{n}{l}=\binom{n+1}{l}$ and $\binom{n}{n+1}=n$ are known, we get

$$b\hat{H}_{U,n+1} = \sum_{l=1}^{n+1}\left[\binom{n}{l-1}+\binom{n}{l}\right]\hat{H}_{U,l}+\hat{H}_{U,0}.$$

Then the following result will be obtained

$$b\hat{H}_{U,n+1} = \sum_{l=0}^{n}\binom{n}{l}(\hat{H}_{U,l}+\hat{H}_{U,l+1}).$$

Similar proof is made within $\{b\hat{H}_{V,n}\}$. □

From Lemma 3.1, we can give the following main result.

Theorem 3.2. *The binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ states following recurrence relation*

$$b\hat{H}_{U,n} = (p+2)b\hat{H}_{U,n-1}+(-q-p-1)b\hat{H}_{U,n-2} \tag{3.2}$$
$$b\hat{H}_{V,n} = (p+2)b\hat{H}_{V,n-1}+(-q-p-1)b\hat{H}_{V,n-2} \tag{3.3}$$

for $n\geq 2$, where

$$b\hat{H}_{U,0} = (0,1,p,p^2-q) \tag{3.4}$$
$$b\hat{H}_{V,0} = (2,p,p^2-2q,p^3-3pq) \tag{3.5}$$

and

$$b\hat{H}_{U,1} = (1,p+1,p^2+p-q,p^3+p^2-2pq-q) \tag{3.6}$$
$$b\hat{H}_{V,1} = (2+p,\ p^2+p-2q,\ p^3+p^2-2q-3pq,\ p^4+p^3-4p^2q+2q^2-3pq) \tag{3.7}$$

respectively.

Proof. With the help of Lemma 3.1 we get,

$$\begin{aligned} b\hat{H}_{U,n+1} &= \sum_{l=0}^{n}\binom{n}{l}(\hat{H}_{U,l}+\hat{H}_{U,l+1}) \\ &= \sum_{l=1}^{n}\binom{n}{l}(\hat{H}_{U,l}+\hat{H}_{U,l+1})+\hat{H}_{U,0}+\hat{H}_{U,1} \\ &= (p+1)\sum_{l=1}^{n}\binom{n}{l}\hat{H}_{U,l}-q\sum_{l=1}^{n}\binom{n}{l}\hat{H}_{U,l}+\hat{H}_{U,0}+\hat{H}_{U,1}\pm(p+1)\hat{H}_{U,0} \\ &= (p+1)b\hat{H}_{U,n}-q\sum_{l=1}^{n}\binom{n}{l}\hat{H}_{U,l-1}+\hat{H}_{U,1}-p\hat{H}_{U,0}. \end{aligned}$$

If we take $n \to n-1$, then we get

$$\begin{aligned} b\hat{H}_{U,n} &= (p+1)b\hat{H}_{U,n-1} - q\sum_{l=1}^{n-1}\binom{n-1}{l}\hat{H}_{U,l-1} + \hat{H}_{U,1} - p\hat{H}_{U,0} \\ &= p(b\hat{H}_{U,n-1}) + \sum_{l=1}^{n}\left[\binom{n-1}{l-1} - q\binom{n-1}{l}\right]\hat{H}_{U,l-1} + \hat{H}_{U,1} - p\hat{H}_{U,0} \\ &= p(b\hat{H}_{U,n-1}) + \sum_{l=1}^{n}\left[\binom{n-1}{l-1} - q\binom{n-1}{l} \pm q\binom{n-1}{l-1}\right]\hat{H}_{U,l-1} + \hat{H}_{U,1} - p\hat{H}_{U,0} \\ &= p(b\hat{H}_{U,n-1}) + (1+q)(b\hat{H}_{U,n-1}) - q\sum_{l=1}^{n}\binom{n}{l}\hat{H}_{U,l-1} + \hat{H}_{U,1} - p\hat{H}_{U,0}. \end{aligned}$$

Moreover, if the process continues, we get

$$-q\sum_{l=1}^{n}\binom{n}{l}\hat{H}_{U,l-1} + \hat{H}_{U,1} - p\hat{H}_{U,0} = b\hat{H}_{U,n} + (-q-p-1)b\hat{H}_{U,n-1}.$$

So we obtain $b\hat{H}_{U,n+1} = (p+2)b\hat{H}_{U,n} - (q+p+1)b\hat{H}_{U,n-1}$ as we wanted. Similar proof is made within $\{b\hat{H}_{V,n}\}$. □

Characteristic equation of (3.2) and also (3.3) is

$$x^2 - (p+2)x + (q+p+1) = 0. \tag{3.8}$$

The roots of equation (3.8) are

$$\gamma = \alpha_1 + 1 = \frac{p+2+\sqrt{p^2-4q}}{2} \quad \text{and} \quad \delta = \alpha_2 + 1 = \frac{p+2-\sqrt{p^2-4q}}{2}.$$

Throughout the article, α_1+1 and α_2+1 will be used instead of γ and δ to provide similarity.

Binet formulas for the $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ are in the following theorem.

Theorem 3.3. *Let $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ be the binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$. Binet formulas of $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ are*

$$b\hat{H}_{U,n} = \frac{A_U\alpha_1^n + B_U\alpha_2^n}{\Delta} \tag{3.9}$$

$$b\hat{H}_{V,n} = \frac{A_V\alpha_1^n + B_V\alpha_2^n}{\Delta} \tag{3.10}$$

where

$$\begin{aligned} A_U &= b\hat{H}_{U,1} - b\hat{H}_{U,0}(\alpha_2+1) \\ A_V &= b\hat{H}_{V,1} - b\hat{H}_{V,0}(\alpha_2+1) \end{aligned}$$

and

$$\begin{aligned} B_U &= b\hat{H}_{U,0}(\alpha_1+1) - b\hat{H}_{U,1} \\ B_V &= b\hat{H}_{V,0}(\alpha_1+1) - b\hat{H}_{V,1} \end{aligned}$$

respectively.

Proof. There is a general solution in the form of $b\hat{H}_{U,n} = d_1\gamma^n + d_2\delta^n$ that is $b\hat{H}_{U,n} = d_1(\alpha_1+1)^n + d_2(\alpha_2+1)^n$. If $n=0$, then $b\hat{H}_{U,0} = d_1 + d_2$ and also if $n=1$, then $b\hat{H}_{U,1} = d_1(\alpha_1+1) + d_2(\alpha_2+1)$. Therefore

$$d_1 = \frac{b\hat{H}_{U,1} - b\hat{H}_{U,0}\alpha_2}{\Delta} \quad \text{and} \quad d_2 = \frac{b\hat{H}_{U,0}\alpha_1 - b\hat{H}_{U,1}}{\Delta}.$$

If $d_1 = \frac{b\hat{H}_{U,1} - b\hat{H}_{U,0}(\alpha_2+1)}{\Delta}$ and $d_2 = \frac{b\hat{H}_{U,0}(\alpha_1+1) - b\hat{H}_{U,1}}{\Delta}$ are substituted into $b\hat{H}_{U,n} = d_1\alpha_1^n + d_2\alpha_2^n$, then

$$b\hat{H}_{U,n} = \frac{b\hat{H}_{U,1} - b\hat{H}_{U,0}(\alpha_2+1)}{\Delta}\alpha_1^n + \frac{b\hat{H}_{U,0}(\alpha_1+1) - b\hat{H}_{U,1}}{\Delta}\alpha_2^n$$

such that $A_U = b\hat{H}_{U,1} - b\hat{H}_{U,0}(\alpha_2+1)$ and $B_U = b\hat{H}_{U,0}(\alpha_1+1) - b\hat{H}_{U,1}$. Then, we obtained Binet formulas of $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$

$$b\hat{H}_{U,n} = \frac{A_U\alpha_1^n + B_U\alpha_2^n}{\Delta} \tag{3.11}$$

where

$$\begin{aligned} A_U &= b\hat{H}_{U,1} - b\hat{H}_{U,0}(\alpha_2+1) \\ B_U &= b\hat{H}_{U,0}(\alpha_1+1) - b\hat{H}_{U,1}. \end{aligned}$$

Similar proof is made within $\{b\hat{H}_{V,n}\}$. □

We formulate the generating function and exponential generating function for the $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$.

Theorem 3.4. *Let $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ be the binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$. The generating functions of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$ are*

$$\begin{aligned} b\hat{H}_{U,n}(x) &= \frac{x(b\hat{H}_{U,1}-pb\hat{H}_{U,0})}{qx^2-px+1} \\ b\hat{H}_{V,n}(x) &= \frac{x(b\hat{H}_{V,1}-pb\hat{H}_{V,0})}{qx^2-px+1} \end{aligned}$$

where (3.4), (3.5) and also (3.6),(3.7) respectively.

Proof. Using equation (3.11), geometric series formula, we get that

$$\begin{aligned} &\sum_{n=0}^{\infty}\left(\frac{A_U\alpha_1^n+B_U\alpha_2^n}{\Delta}\right)x^n \\ &= \sum_{n=0}^{\infty}\frac{A_U(\alpha_1 x)^n}{\Delta}+\sum_{n=0}^{\infty}\frac{B_U(\alpha_2 x)^n}{\Delta} \\ &= \frac{1}{\Delta}\left(\frac{A_U}{1-\alpha_1 x}+\frac{B_U}{1-\alpha_2 x}\right) \\ &= \frac{\left\{\begin{array}{c} b\hat{H}_{U,1}-b\hat{H}_{U,0}(\alpha_2+1)-(b\hat{H}_{U,1}-b\hat{H}_{U,0}\alpha_2-b\hat{H}_{U,0})\alpha_2 x \\ b\hat{H}_{U,0}\alpha_1+\hat{H}_{U,0}-b\hat{H}_{U,1}-(b\hat{H}_{U,0}\alpha_1+b\hat{H}_{U,0}-b\hat{H}_{U,1})\alpha_1 x \end{array}\right\}}{\Delta(qx^2-px+1)} \\ &= \frac{1}{\Delta}\left(\frac{x\Delta(b\hat{H}_{U,1}-pb\hat{H}_{U,0})}{qx^2-px+1}\right) \\ &= \frac{x(b\hat{H}_{U,1}-pb\hat{H}_{U,0})}{qx^2-px+1} \end{aligned}$$

as we claimed. Similar proof is made within $\{b\hat{H}_{V,n}\}$. □

Theorem 3.5. *(Exponential generating function) Let $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ be the binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$. Then the exponential generating functions for $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ are*

$$\begin{aligned} \sum_{n=0}^{\infty}(b\hat{H}_{U,n})\frac{t^n}{n!} &= \frac{A_U e^{\alpha_1 t}+B_U e^{\alpha_2 t}}{\Delta} \\ \sum_{n=0}^{\infty}(b\hat{H}_{V,n})\frac{t^n}{n!} &= \frac{A_V e^{\alpha_1 t}+B_V e^{\alpha_2 t}}{\Delta} \end{aligned}$$

respectively.

Proof. Applying Binet formula, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} b_n\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{A_U\alpha_1^n+B_U\alpha_2^n}{\Delta}\right)\frac{t^n}{n!} \\ &= \frac{A_U}{\Delta}\sum_{n=0}^{\infty}\frac{(\alpha_1 t)^n}{n!}+\frac{B_U}{\Delta}\sum_{n=0}^{\infty}\frac{(\alpha_2 t)^n}{n!} \\ &= \frac{A_U}{\Delta}e^{\alpha_1 t}+\frac{B_U}{\Delta}e^{\alpha_2 t} \\ &= \frac{A_U e^{\alpha_1 t}+B_U e^{\alpha_2 t}}{\Delta}. \end{aligned}$$

Similar proof is made within $\{b\hat{H}_{V,n}\}$. □

Theorem 3.6. *Let $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ be the binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$. Then*

$$\begin{aligned} \sum_{k=0}^{n} b\hat{H}_{U,mk+s} &= \frac{q^m(b\hat{H}_{U,mn+s}-b\hat{H}_{U,s-m})-b\hat{H}_{U,mn+m+s}+b\hat{H}_{U,s}}{q^m-(\alpha_1^m+\alpha_2^m)+1} \\ \sum_{k=0}^{n} b\hat{H}_{V,mk+s} &= \frac{q^m(b\hat{H}_{V,mn+s}-b\hat{H}_{V,s-m})-b\hat{H}_{V,mn+m+s}+b\hat{H}_{V,s}}{q^m-(\alpha_1^m+\alpha_2^m)+1} \end{aligned}$$

for all $n\in\mathbb{N}$ and $m,s\in\mathbb{Z}, s>m,$ respectively.

Proof. From (3.11), we get

$$\begin{aligned}
\sum_{k=0}^{n} b\hat{H}_{U,mk+s} &= \sum_{k=0}^{n} \frac{A_U\alpha_1^{mk+s}+B_U\alpha_2^{mk+s}}{\Delta} \\
&= \frac{A_U\alpha_1^{s}}{\Delta}\sum_{k=0}^{n}\alpha_1^{mk}+\frac{B_U\alpha_2^{s}}{\Delta}\sum_{k=0}^{n}\alpha_2^{mk} \\
&= \frac{A_U^s\alpha_1^{s}}{\Delta}\left(\frac{(\alpha_1^m)^{n+1}-1}{\alpha_1^m-1}\right)+\frac{B_U\alpha_2^{s}}{\Delta}\left(\frac{(\alpha_2^m)^{n+1}-1}{\alpha_2^m-1}\right) \\
&= \frac{A_U(\alpha_1^{mn+m+s}-\alpha_1^s)(\alpha_2^m-1)}{\Delta(\alpha_1^m\alpha_2^m-\alpha_1^m-\alpha_2^m+1)}+\frac{B_U(\alpha_2^{mn+m+s}-\alpha_2^s)(\alpha_1^m-1)}{\Delta(\alpha_1^m\alpha_2^m-\alpha_1^m-\alpha_2^m+1)} \\
&= \frac{A_U(\alpha_1^{mn+s}(\alpha_1\alpha_2)^m-\alpha_1^s\alpha_2^m-\alpha_1^{mn+m+s}+\alpha_1^s)}{\Delta(\alpha_1^m\alpha_2^m-\alpha_1^m-\alpha_2^m+1)} \\
&\quad+\frac{B_U(\alpha_2^{mn+s}(\alpha_1\alpha_2)^m-\alpha_1^m\alpha_2^s-\alpha_2^{mn+m+s}+\alpha_2^s)}{\Delta(\alpha_1^m\alpha_2^m-\alpha_1^m-\alpha_2^m+1)} \\
&= \frac{\left\{\begin{array}{c} q^m\frac{A_U\alpha_1^{mn+s}+B_U\alpha_2^{mn+s}}{\Delta}-q^m\frac{A_U\alpha_1^{s-m}+B_U\alpha_2^{s-m}}{\Delta} \\ -\frac{A_U\alpha_1^{mn+m+s}+B_U\alpha_2^{mn+m+s}}{\Delta}+\frac{A_U\alpha_1^{s}+B_U\alpha_2^{s}}{\Delta}\end{array}\right\}}{q^m-(\alpha_1^m+\alpha_2^m)+1} \\
&= \frac{q^m(b\hat{H}_{U,mn+s}-b\hat{H}_{U,s-m})-b\hat{H}_{U,mn+m+s}+b\hat{H}_{U,s}}{q^m-(\alpha_1^m+\alpha_2^m)+1}.
\end{aligned}$$

Similar proof is made within $\{b\hat{H}_{V,n}\}$. □

Theorem 3.7. *Let $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ be the binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$. Then the Catalan identity for $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ is*

$$\begin{aligned}
b\hat{H}_{U,n-k}b\hat{H}_{U,n+k}-(b\hat{H}_{U,n})^2 &= \frac{q^{n-k}(\alpha_1^k-\alpha_2^k)(B_UA_U\alpha_1^k-A_UB_U\alpha_2^k)}{\Delta^2} \\
b\hat{H}_{V,n-k}b\hat{H}_{V,n+k}-(b\hat{H}_{V,n})^2 &= \frac{q^{n-k}(\alpha_1^k-\alpha_2^k)(B_VA_V\alpha_1^k-A_VB_V\alpha_2^k)}{\Delta^2}
\end{aligned}$$

for $n,k\in\mathbb{N}$ with $n\geq k$, respectively.

Proof. Applying (3.9), we find that

$$\begin{aligned}
&b\hat{H}_{U,n-k}b\hat{H}_{U,n+k}-(b\hat{H}_{U,n})^2 \\
&= \left(\frac{A_U\alpha_1^{n-k}+B_U\alpha_2^{n-k}}{\Delta}\right)\left(\frac{A_U\alpha_1^{n+k}+B_U\alpha_2^{n+k}}{\Delta}\right)-\left(\frac{A_U\alpha_1^{n}+B_U\alpha_2^{n}}{\Delta}\right)\left(\frac{A_U\alpha_1^{n}+B_U\alpha_2^{n}}{\Delta}\right) \\
&= \frac{A_UA_U\alpha_1^{2n}+A_UB_U\alpha_1^{n-k}\alpha_2^{n+k}+B_UA_U\alpha_2^{n-k}\alpha_1^{n+k}+B_UB_U\alpha_2^{2n}}{\Delta^2} \\
&\quad-\frac{A_UA_U\alpha_1^{2n}+A_UB_U\alpha_1^{n}\alpha_2^{n}+B_UA_U\beta^{n}\alpha_1^{n}+B_UB_U\alpha_2^{2n}}{\Delta^2} \\
&= \frac{A_UB_Uq^n\frac{\alpha_2^k}{\alpha_1^k}+B_UA_Uq^n\frac{\alpha_1^k}{\alpha_2^k}-A_UB_Uq^n-B_UA_Uq^n}{\Delta^2} \\
&= \frac{q^n\left(A_UB_U\left(\frac{\alpha_2^k-\alpha_1^k}{\alpha_1^k}\right)+B_UA_U\left(\frac{\alpha_1^k-\alpha_2^k}{\alpha_2^k}\right)\right)}{\Delta^2} \\
&= \frac{q^n\left(A_UB_U(\alpha_2^k-\alpha_1^k)\alpha_2^k+B_UA_U(\alpha_1^k-\alpha_2^k)\alpha_1^k\right)}{q^k\Delta^2} \\
&= \frac{q^{n-k}(\alpha_1^k-\alpha_2^k)(B_UA_U\alpha_1^k-A_UB_U\alpha_2^k)}{\Delta^2} \\
&= \frac{q^{n-k}(\alpha_1^k-\alpha_2^k)(B_UA_U\alpha_1^k-A_UB_U\alpha_2^k)}{\Delta^2}.
\end{aligned}$$

Similar proof is made within $\{b\hat{H}_{V,n}\}$. □

Theorem 3.8. *Let $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ be the binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$. Then the Cassini identity for $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ is*

$$\begin{aligned}
b\hat{H}_{U,n-1}b\hat{H}_{U,n+1}-(b\hat{H}_{U,n})^2 &= \frac{q^{n-1}\Delta(B_UA_U\alpha_1-A_UB_U\alpha_2)}{\Delta^2} \\
b\hat{H}_{V,n-1}b\hat{H}_{V,n+1}-(b\hat{H}_{V,n})^2 &= \frac{q^{n-1}\Delta(B_VA_V\alpha_1-A_VB_V\alpha_2)}{\Delta^2}
\end{aligned}$$

for $n\in\mathbb{N}$, respectively.

Proof. If we take $k=1$ in Catalan identity, the we get the desired result. The other can be proved similarly. □

Theorem 3.9. *Let $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ be the binomial transforms of $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$. Then the d'Ocagne identity for $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$ is*

$$\begin{aligned}(b\hat{H}_{U,k})(b\hat{H}_{U,n+1})-(b\hat{H}_{U,k+1})(b\hat{H}_{U,n}) &= \frac{B_U A_U\,\alpha_2^k\alpha_1^n - A_U B_U\,\alpha_1^k\alpha_2^n}{\Delta}\\ (b\hat{H}_{V,k})(b\hat{H}_{V,n+1})-(b\hat{H}_{V,k+1})(b\hat{H}_{V,n}) &= \frac{B_V A_V\,\alpha_2^k\alpha_1^n - A_V B_V\,\alpha_1^k\alpha_2^n}{\Delta}\end{aligned}$$

for $n\geq k$, respectively.

Proof. Using Binet formula, we get

$$\begin{aligned}&(b\hat{H}_{U,k})(b\hat{H}_{U,n+1})-(b\hat{H}_{U,k+1})(b\hat{H}_{U,n})\\ &=\left(\frac{A_U\alpha_1^k+B_U\alpha_2^k}{\Delta}\right)\left(\frac{A_U\alpha_1^{n+1}+B_U\alpha_2^{n+1}}{\Delta}\right)-\left(\frac{A_U\alpha_1^{k+1}+B_U\alpha_2^{k+1}}{\Delta}\right)\left(\frac{A_U\alpha_1^{n}+B_U\alpha_2^{n}}{\Delta}\right)\\ &=\frac{A_UA_U\,\alpha_1^{k+n+1}+A_UB_U\,\alpha_1^k\alpha_2^{n+1}+B_UA_U\,\alpha_2^k\alpha_1^{n+1}+B_UB_U\,\alpha_2^{k+n+1}}{\Delta^2}\\ &\quad-\frac{A_UA_U\,\alpha_1^{k+1+n}+A_UB_U\,\alpha_1^{k+1}\alpha_2^{n}+B_UA_U\,\alpha_2^{k+1}\alpha_1^{n}+B_UB_U\,\alpha_2^{k+1+n}}{\Delta^2}\\ &=\frac{A_UB_U\,\alpha_1^k\alpha_2^{n+1}+B_UA_U\,\alpha_2^k\alpha_1^{n+1}-A_UB_U\,\alpha_1^{k+1}\alpha_2^{n}-B_UA_U\,\alpha_2^{k+1}\alpha_1^{n}}{\Delta^2}\\ &=\frac{-A_UB_U\,\alpha_1^k\alpha_2^n\Delta+B_UA_U\,\alpha_2^k\alpha_1^n\Delta}{\Delta^2}\\ &=\frac{\Delta(B_UA_U\,\alpha_2^k\alpha_1^n-A_UB_U\,\alpha_1^k\alpha_2^n)}{\Delta^2}\\ &=\frac{B_UA_U\,\alpha_2^k\alpha_1^n-A_UB_U\,\alpha_1^k\alpha_2^n}{\Delta}\end{aligned}$$

This completes the proof. Similar proof is made within $\{b\hat{H}_{V,n}\}$. □

4. Conclusion

In this article, two different types of sequences are discussed in two separate sections. First of all, results that were not in the literature were found about the $\{\hat{H}_{U,n}\}$ and $\{\hat{H}_{V,n}\}$, which is a subsequence of Horadam hybrid number sequences that already existed. Some of these include general formula for all sums and also binomial identities. In the second part, a subject on binomial transforms of such numbers, that is, hybrid numbers, has not been studied before. In this section, firstly, the recurrence relation for the $\{b\hat{H}_{U,n}\}$ and $\{b\hat{H}_{V,n}\}$, which are a new sequence obtained with the sequence of these given hybrid numbers, and then with the help of Binet formula, results about the generating function, sum formulas, explicit formulas and some special identities which is Catalan identity, Cassini identity and d'Ocagne identity, are found.

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