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Research Article

On Hardy and Hermite-Hadamard inequalities for N-tuple diamond-alpha integral

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Abstract

In this paper, we aim to construct n dimensional Jensen, Hardy and Hermite-Hadamard type inequalities for multiple diamond-alpha integral on time scales. The cases of Hardy type inequality with a weighted function and Hermite-Hadamard type inequality with three variables are also considered minutely.

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Keywords. Hardy type inequality, Hermite-Hadamard type inequality, diamond-alpha integral, diamond-alpha derivative, time scales

1. Introduction

In order to unify the representation of discrete and continuous inequalities, Hilger established the theory of time scales, which received growing attention [14]. Considerable studies that involve inequalities and dynamic equations have been reported [4, 5, 7, 13, 15, 17–21, 24, 29, 30, 32, 38, 39]. We refer readers to the papers [8–10] for more details.

In the past two decades, a number of researchers have concerned inequalities through the diamond-alpha integral, which is defined as a linear combination of Δ and ∇ integral. The following theorems consider Hardy's diamond-alpha integral inequalities.

Theorem 1.1 (Two dimensional weighted Hardy-Knopp type inequality,[23]). *Assume that $A : [a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \to [0, \infty)$ is a continuous function with diamond-$\alpha_i (i = 1, 2)$ partial derivative exists and*

$$\int_{t_2}^{b_2}\int_{t_1}^{b_1} \frac{A(v_1, v_2)}{(v_1 - a_1)(\sigma(v_1) - a_1)(v_2 - a_2)(\sigma(v_2) - a_2)} \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 < \infty,$$

for all $t_1, t_2 \in [a_1, b_1) \times [a_2, b_2)$. We denote the function B as

$$B(t_1, t_2) = (t_1 - a_1)(t_2 - a_2)\int_{t_2}^{b_2}\int_{t_1}^{b_1} \frac{A(v_1, v_2) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2}{(v_1 - a_1)(\sigma_1(v_1) - a_1)(v_2 - a_2)(\sigma_2(v_2) - a_2)}.$$

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If ϕ is a convex and positive function, and f is a diamond-$\alpha_i(i=1,2)$ integrable, nonnegative, bounded function with diamond-$\alpha_i(i=1,2)$ partial derivative exists, then the following inequality holds.

$$\begin{aligned}&\int_{a_2}^{b_2}\int_{a_1}^{b_1} A(v_1,v_2)\\ &\phi\Big(\frac{1}{(\sigma_1(v_1)-a_1)(\sigma_2(v_2)-a_2)}\int_{a_2}^{\sigma_2(v_2)}\int_{a_1}^{\sigma_1(v_1)} f(t_1,t_2)\diamond_{\alpha_1} t_1\diamond_{\alpha_2} t_2\Big)\frac{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2}{(v_1-a_1)(v_2-a_2)}\\ \le\ &\int_{a_2}^{b_2}\int_{a_1}^{b_1} B(v_1,v_2)\phi(f(v_1,v_2))\frac{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2}{(v_1-a_1)(v_2-a_2)}.\end{aligned}$$

Theorem 1.2 (Hardy type inequalities, [31]). *Assume that $L(x_1,x_2)$, $f(x_1)$, $g(x_2)$, $\phi_1(x_1)$, $\phi_2(x_2)$ are nonnegative functions and*

$$A(x_1):=\int_a^b L(x_1,x_2)\phi_2^{-p}(x_2)\diamond_\alpha x_2,$$

$$B(x_2):=\int_a^b L(x_1,x_2)\phi_1^{-q}(x_1)\diamond_\alpha x_1,$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. Then the following two inequalities hold and they are equivalent.

$$\begin{aligned}&\int_a^b\int_a^b L(x_1,x_2)f(x_1)g(x_2)\diamond_\alpha x_1\diamond_\alpha x_2\\ \le\ &\Big(\int_a^b \phi_1^p(x_1)A(x_1)f^p(x_1)\diamond_\alpha x_1\Big)^{\frac{1}{p}}\Big(\int_a^b \phi_2^q(x_2)B(x_2)g^q(x_2)\diamond_\alpha x_2\Big)^{\frac{1}{q}}.\end{aligned}$$

$$\int_a^b B^{1-p}(x_2)\phi_2^{-p}(x_2)\Big(\int_a^b L(x_1,x_2)f(x_1)\diamond_\alpha x_1\Big)^p\diamond_\alpha x_2\le\int_a^b \phi_1^p(x_1)A(x_1)f^p(x_1)\diamond_\alpha x_1.$$

Meantime, a lot of works involving Hermite-Hadamard inequalities have been provided [1–3, 6, 12, 16, 22, 25–28]. Some results for Diamond-α integral are listed as follows.

Theorem 1.3 (Hermite-Hadamard inequality, [12]). *Assume that ϕ is a continuous convex function defined on $[a,b]_{\mathbb{T}}$, then*

$$\phi(\bar{x})\le\frac{1}{b-a}\int_a^b\phi(x)\diamond_\alpha x\le\frac{b-\bar{x}}{b-a}\phi(a)+\frac{\bar{x}-a}{b-a}\phi(b),\tag{1.1}$$

where $\bar{x}=\int_a^b x\diamond_\alpha x/(b-a)$.

Theorem 1.4 (Weighted Hermite-Hadamard inequality, [12]). *Assume that ϕ is a continuous convex function defined on $[a,b]_{\mathbb{T}}$, w is a continuous and nonnegative function with $\int_a^b w(x)\diamond_\alpha x<\infty$, then*

$$\phi(x^*)\le\frac{1}{\int_a^b w(x)\diamond_\alpha x}\int_a^b\phi(x)w(x)\diamond_\alpha x\le\frac{b-x^*}{b-a}\phi(a)+\frac{x^*-a}{b-a}\phi(b),$$

where $x^*=\int_a^b xw(x)\diamond_\alpha x/\int_a^b w(x)\diamond_\alpha x$.

Theorem 1.5 (Two dimensional Weighted Hermite-Hadamard inequality, [22]). *Assume that $\phi:[a_1,b_1]\times[a_2,b_2]\to\mathbb{T}$,*

$$\phi_2:[a_1,b_1]\to\mathbb{R},\qquad \phi_2(u)=\phi(u,x_2),$$

and

$$\phi_1:[a_2,b_2]\to\mathbb{R},\qquad \phi_1(v)=\phi(x_1,v),$$

are convex and continuous for all $x_1 \in [a_1, b_1]$ *and* $x_2 \in [a_2, b_2]$*. Then the following double inequalities hold.*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\phi(x_1,s_1)\diamond_\alpha x_1+\frac{1}{b_2-a_2}\int_{a_2}^{b_2}\phi(s_2,x_2)\diamond_\alpha x_2\right)\\
\leq\;& \frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\phi(x_1,x_2)\diamond_\alpha x_1\diamond_\alpha x_2\\
\leq\;& \frac{1}{2(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\Big((b_2-s_1)\phi(x_1,a_2)+(s_1-a_2)\phi(x_1,b_2)\Big)\diamond_\alpha x_1\\
+\;& \frac{1}{2(b_1-a_1)(b_2-a_2)}\int_{a_2}^{b_2}\Big((b_1-s_1)\phi(a_1,x_2)+(s_2-a_1)\phi(b_1,x_2)\Big)\diamond_\alpha x_2,
\end{aligned}$$

where

$$s_1=\frac{\int_{a_1}^{b_1}x\diamond_\alpha x}{b_1-a_1},\qquad s_2=\frac{\int_{a_2}^{b_2}x\diamond_\alpha x}{b_2-a_2}.$$

Tian et al. [33] provided the definition of the following multiple diamond-alpha integral by employing antiderivatives of single-variable functions:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\phi(v_1,v_2,\cdots,v_n)\diamond_\alpha v_1\diamond_\alpha v_2\cdots\diamond_\alpha v_n. \tag{1.2}$$

Moreover, some noted inequalities including Jensen, Hölder, Minkowski type inequalities have been given for n-tuple diamond-alpha integral on time scales.

Later on, Mao et al. [20] conceptualized the definition of this integral with different alpha, "multiple Diamond-α_i integral" for short, which was polished up (1.2):

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\phi(v_1,v_2,\cdots,v_n)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n.$$

The paper is organized as follows, in section 2, we will establish Hardy type inequalities for multiple Diamond-α_i integral and the case with a weighted function, which generalizes Theorem 1.1. Before that, Jensen's inequality will be provided first. In section 3, Hermite-Hadamard type inequalities for multiple Diamond-α_i integral of three and n variables will be minutely given, namely, we improve Theorems 1.3 and 1.5 to three and n dimensional.

2. Hardy type inequalities for multiple Diamond-α_i integral

In what follows, unless otherwise noted, we denote $[a, b]\cap\mathbb{T}$ as $[a, b]_{\mathbb{T}}$ and assume always that $a, b \in \mathbb{T}$. For more basic knowledge about time scales, readers can consult [8].

To establish the desired Theorem 2.4, we need the following Jensen's inequality for multiple Diamond-α_i integral.

Theorem 2.1 (Jensen's inequality for multiple Diamond-α_i integral)**.** *Assume that* $\omega, \psi : [\beta_1, \gamma_1]_{\mathbb{T}_1} \times [\beta_2, \gamma_2]_{\mathbb{T}_2} \times \cdots \times [\beta_n, \gamma_n]_{\mathbb{T}_n} \to \mathbb{R}$ *are diamond-*$\alpha_i (i = 1, 2, \cdots, n)$ *integrable functions with*

$$\int_{\beta_1}^{\gamma_1}\int_{\beta_2}^{\gamma_2}\cdots\int_{\beta_n}^{\gamma_n}|\omega(v_1,v_2,\cdots,v_n)|\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n<\infty.$$

If the function Φ *is bounded convex, then*

$$\begin{aligned}
&\Phi\left(\frac{\int_{\beta_1}^{\gamma_1}\int_{\beta_2}^{\gamma_2}\cdots\int_{\beta_n}^{\gamma_n}|\omega(v_1,v_2,\cdots,v_n)|\psi(v_1,v_2,\cdots,v_n)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n}{\int_{\beta_1}^{\gamma_1}\int_{\beta_2}^{\gamma_2}\cdots\int_{\beta_n}^{\gamma_n}|\omega(v_1,v_2,\cdots,v_n)|\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n}\right)\\
\leq\;&\frac{\int_{\beta_1}^{\gamma_1}\int_{\beta_2}^{\gamma_2}\cdots\int_{\beta_n}^{\gamma_n}|\omega(v_1,v_2,\cdots,v_n)|\Phi(\psi(v_1,v_2,\cdots,v_n))\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n}{\int_{\beta_1}^{\gamma_1}\int_{\beta_2}^{\gamma_2}\cdots\int_{\beta_n}^{\gamma_n}|\omega(v_1,v_2,\cdots,v_n)|\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n}.
\end{aligned}$$

Remark 2.2. The proof of Theorem 2.1 is similar to the proof of Theorem 2.3 in [33], hence we omit it here. It is also worth to point out that Theorem 2.1 is a generalization of [23, Theorem 4.1].

Taking $w(x_1, x_2, \cdots, x_n) = 1$, we obtain the following corollary.

Corollary 2.3. *Assume that* $\psi : [\beta_1, \gamma_1]_{\mathbb{T}_1} \times [\beta_2, \gamma_2]_{\mathbb{T}_2} \times \cdots \times [\beta_n, \gamma_n]_{\mathbb{T}_n} \to \mathbb{R}$ *is diamond-*$\alpha_i (i = 1, 2, \cdots, n)$ *integrable function. If the function* Φ *is bounded convex, then*

$$\begin{aligned} &\Phi\Big(\frac{\int_{\beta_1}^{\gamma_1} \int_{\beta_2}^{\gamma_2} \cdots \int_{\beta_n}^{\gamma_n} \psi(v_1, v_2, \cdots, v_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n}{\prod_{i=1}^n (\gamma_i - \beta_i)}\Big) \\ \le\ & \frac{\int_{\beta_1}^{\gamma_1} \int_{\beta_2}^{\gamma_2} \cdots \int_{\beta_n}^{\gamma_n} \Phi(\psi(v_1, v_2, \cdots, v_n)) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n}{\prod_{i=1}^n (\gamma_i - \beta_i)}. \end{aligned} \tag{2.1}$$

Now we consider Hardy type inequality for multiple Diamond-α_i integral. For more details about classical Hardy type inequality, readers can consult [11,34–37]. Before stating the following theorem, for convenience, we introduce following notations

$$\delta_i := v_i - a_i, \qquad \delta_i^\sigma := \sigma_i(v_i) - a_i, \qquad i = 1, 2, \cdots, n,$$

Theorem 2.4 (Hardy-knopp type inequality for multiple Diamond-α_i integral)**.** *Assume* $A, f : [a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n]_{\mathbb{T}_n} \to [0, \infty)$ *are continuous functions with diamond-*$\alpha_i (i = 1, 2, \cdots, n)$ *partial derivatives exist and*

$$\int_{t_1}^{b_1} \int_{t_2}^{b_2} \cdots \int_{t_n}^{b_n} \frac{A(v_1, v_2, \cdots, v_n)}{\prod_{i=1}^n \delta_i \delta_i^\sigma} \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n < \infty,$$

for all $t_1, t_2, \cdots, t_n \in [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$. *If* ϕ *is a convex and positive function, and* f *is nonnegative bounded function, then the following inequality holds.*

$$\begin{aligned} &\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \frac{A(v_1, v_2, \cdots, v_n)}{\prod_{i=1}^n \delta_i} \phi\Big(\frac{C(v_1, v_2, \cdots, v_n)}{\prod_{i=1}^n \delta_i^\sigma}\Big) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\ \le\ & \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} B(v_1, v_2, \cdots, v_n) \phi(f(v_1, v_2, \cdots, v_n)) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n, \end{aligned} \tag{2.2}$$

where the functions B, C *are respectively defined as*

$$B(t_1, t_2, \cdots, t_n) := \int_{t_1}^{b_1} \int_{t_2}^{b_2} \cdots \int_{t_n}^{b_n} \frac{A(v_1, v_2, \cdots, v_n)}{\prod_{i=1}^n \delta_i \delta_i^\sigma} \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n,$$

and

$$C(v_1, v_2, \cdots, v_n) := \int_{a_1}^{\sigma_1(v_1)} \int_{a_2}^{\sigma_2(v_2)} \cdots \int_{a_n}^{\sigma_n(v_n)} f(t_1, t_2, \cdots, t_n) \diamond_{\alpha_1} t_1 \diamond_{\alpha_2} t_2 \cdots \diamond_{\alpha_n} t_n.$$

Proof. By employing Corollary 2.3, we obtain

$$\begin{aligned} &\phi\Big(\frac{C(v_1, v_2, \cdots, v_n)}{\prod_{i=1}^n \delta_i^\sigma}\Big) \\ =\ & \phi\Big(\frac{\int_{a_1}^{\sigma_1(v_1)} \int_{a_2}^{\sigma_2(v_2)} \cdots \int_{a_n}^{\sigma_n(v_n)} f(t_1, t_2, \cdots, t_n) \diamond_{\alpha_1} t_1 \diamond_{\alpha_2} t_2 \cdots \diamond_{\alpha_n} t_n}{\prod_{i=1}^n (\sigma_i(v_i) - a_i)}\Big) \\ \le\ & \frac{\int_{a_1}^{\sigma_1(v_1)} \int_{a_2}^{\sigma_2(v_2)} \cdots \int_{a_n}^{\sigma_n(v_n)} \phi(f(t_1, t_2, \cdots, t_n)) \diamond_{\alpha_1} t_1 \diamond_{\alpha_2} t_2 \cdots \diamond_{\alpha_n} t_n}{\prod_{i=1}^n (\sigma_i(v_i) - a_i)}. \end{aligned} \tag{2.3}$$

Substituting inequality (2.3) into the left hand side of (2.2) and using Fubini's Theorem, we yield

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^n\delta_i}\phi\Big(\frac{C(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^n\delta_i^\sigma}\Big)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\
\le\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^n\delta_i}\\
&\frac{\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\phi(f(t_1,\cdots,t_n))\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n}{\prod_{i=1}^n(\sigma_i(v_i)-a_i)}\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\
=\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^n\delta_i\delta_i^\sigma}\\
&\Big(\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\phi(f(t_1,\cdots,t_n))\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n\Big)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\
=\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\phi(f(t_1,\cdots,t_n))\\
&\Big(\int_{v_1}^{b_1}\int_{v_2}^{b_2}\cdots\int_{v_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^n\delta_i\delta_i^\sigma}\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\Big)\diamond_{\alpha_1}t_1\diamond_{\alpha_2}t_2\cdots\diamond_{\alpha_n}t_n\\
=\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}B(v_1,v_2,\cdots,v_n)\phi(f(v_1,v_2,\cdots,v_n))\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n.
\end{aligned}$$

Thereby we complete the proof. □

If the function $A(v_1,v_2,\cdots)\equiv 1$, then

$$\begin{aligned}
&B(t_1,t_2,\cdots,t_n)=\int_{t_1}^{b_1}\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\frac{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n}{\prod_{i=1}^n\delta_i\delta_i^\sigma}\\
=\ &\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\frac{1}{\prod_{i=2}^n\delta_i\delta_i^\sigma}\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\int_{t_1}^{b_1}\frac{1}{(v_1-a_1)(\sigma_1(v_1-a_1))}\diamond_{\alpha_1}v_1\\
=\ &\frac{1}{v_1-a_1}\Big|_{b_1}^{t_1}\int_{t_2}^{b_2}\int_{t_3}^{b_3}\cdots\int_{t_n}^{b_n}\frac{1}{\prod_{i=2}^n\delta_i\delta_i^\sigma}\diamond_{\alpha_2}v_2\diamond_{\alpha_3}v_3\cdots\diamond_{\alpha_n}v_n\\
=\ &\cdots\\
=\ &\prod_{i=1}^n\Big(\frac{1}{v_i-a_i}\Big|_{b_i}^{t_i}\Big)=\prod_{i=1}^n\Big(\frac{1}{t_i-a_i}-\frac{1}{b_i-a_i}\Big).
\end{aligned}$$

Moreover, if every b_i is finite number, namely, $b_i<\infty (i=1,2,\cdots,n)$, then we have

$$B(t_1,t_2,\cdots,t_n)=\prod_{i=1}^n\Big(\frac{1}{t_i-a_i}-\frac{1}{b_i-a_i}\Big);$$

if $b_i=\infty (i=1,2,\cdots,n)$, then we get

$$B(t_1,t_2,\cdots,t_n)=\prod_{i=1}^n\frac{1}{t_i-a_i}.$$

Theorem 2.4 reduces to the following corollary by taking $A(v_1,v_2,\cdots,v_n)\equiv 1$.

Corollary 2.5. *Assume that $A(v_1,v_2,\cdots,v_n)\equiv 1$ and conditions in Theorem 2.4 hold, then inequality (2.2) changes into*

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{1}{\prod_{i=1}^n\delta_i}\phi\Big(\frac{C(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^n\delta_i^\sigma}\Big)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\
\le\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\prod_{i=1}^n\Big(\frac{1}{\delta_i}-\frac{1}{b_i-a_i}\Big)\phi(f(v_1,v_2,\cdots,v_n))\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n,
\end{aligned}$$

if $b_i < \infty (i = 1, 2, \cdots, n)$ *and it transforms into*

$$\int_{a_1}^{\infty}\int_{a_2}^{\infty}\cdots\int_{a_n}^{\infty}\frac{1}{\prod_{i=1}^{n}\delta_i}\phi\Big(\frac{C(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i^{\sigma}}\Big)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n$$
$$\leq \int_{a_1}^{\infty}\int_{a_2}^{\infty}\cdots\int_{a_n}^{\infty}\prod_{i=1}^{n}\frac{1}{\delta_i}\phi(f(v_1,v_2,\cdots,v_n))\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n,$$

if $b_i = \infty (i = 1, 2, \cdots, n)$.

Taking $\alpha_i (i = 1, 2, \cdots, n) = 1$ or 0, we obtain the following corollaries, respectively.

Corollary 2.6 (Hardy-knopp type inequality for multiple Δ-integral)**.** *Let* $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$. *Assume that* $A, f : [a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n]_{\mathbb{T}_n} \to [0, \infty)$ *are continuous functions with* $\Delta_i (i = 1, 2, \cdots, n)$ *partial derivatives exist and*

$$\int_{t_1}^{b_1}\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i\delta_i^{\sigma}}\Delta_1v_1\Delta_2v_2\cdots\Delta_nv_n<\infty,$$

for all $t_1, t_2, \cdots, t_n \in [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$. *If* ϕ *is a convex and positive function, and* f *is* $\Delta_i (i = 1, 2, \cdots, n)$ *integrable, nonnegative, bounded, then the following inequality holds.*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i}\phi\Big(\frac{C(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i^{\sigma}}\Big)\Delta_1v_1\Delta_2v_2\cdots\Delta_nv_n$$
$$\leq \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}B(v_1,v_2,\cdots,v_n)\phi(f(v_1,v_2,\cdots,v_n))\Delta_1v_1\Delta_2v_2\cdots\Delta_nv_n, \quad (2.4)$$

where functions B, C *are respectively defined by*

$$B(t_1,t_2,\cdots,t_n):=\int_{t_1}^{b_1}\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i\delta_i^{\sigma}}\Delta_1v_1\Delta_2v_2\cdots\Delta_nv_n,$$

and

$$C(v_1,v_2,\cdots,v_n):=\int_{a_1}^{\sigma_1(v_1)}\int_{a_2}^{\sigma_2(v_2)}\cdots\int_{a_n}^{\sigma_n(v_n)}f(t_1,t_2,\cdots,t_n)\Delta_1t_1\Delta_2t_2\cdots\Delta_nt_n.$$

Corollary 2.7 (Hardy-knopp type inequality for multiple ∇-integral)**.** *Let* $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. *Assume that* $A, f : [a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n]_{\mathbb{T}_n} \to [0, \infty)$ *are continuous functions with* $\nabla_i (i = 1, 2, \cdots, n)$ *partial derivatives exist and*

$$\int_{t_1}^{b_1}\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i\delta_i^{\sigma}}\nabla_1v_1\nabla_2v_2\cdots\nabla_nv_n<\infty,$$

for all $t_1, t_2, \cdots, t_n \in [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$. *If* ϕ *is a convex and positive function, and* f *is* $\nabla_i (i = 1, 2, \cdots, n)$ *integrable, nonnegative and bounded, then the following inequality holds.*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i}\phi\Big(\frac{C(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i^{\sigma}}\Big)\nabla_1v_1\nabla_2v_2\cdots\nabla_nv_n$$
$$\leq \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}B(v_1,v_2,\cdots,v_n)\phi(f(v_1,v_2,\cdots,v_n))\nabla_1v_1\nabla_2v_2\cdots\nabla_nv_n, \quad (2.5)$$

where functions B, C *are respectively defined by*

$$B(t_1,t_2,\cdots,t_n):=\int_{t_1}^{b_1}\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{\prod_{i=1}^{n}\delta_i\delta_i^{\sigma}}\nabla_1v_1\nabla_2v_2\cdots\nabla_nv_n,$$

and

$$C(v_1,v_2,\cdots,v_n):=\int_{a_1}^{\sigma_1(v_1)}\int_{a_2}^{\sigma_2(v_2)}\cdots\int_{a_n}^{\sigma_n(v_n)}f(t_1,t_2,\cdots,t_n)\nabla_1t_1\nabla_2t_2\cdots\nabla_nt_n.$$

Noting that

$$\prod_{i=1}^{n}\delta_i^{\sigma}=\prod_{i=1}^{n}(\sigma_i(v_i)-a_i)=\int_{a_1}^{\sigma_1(v_1)}\int_{a_2}^{\sigma_2(v_2)}\cdots\int_{a_n}^{\sigma_n(v_n)}1\diamond_{\alpha_1}t_1\diamond_{\alpha_2}t_2\cdots\diamond_{\alpha_n}t_n, \tag{2.6}$$

replacing 1 by $\omega(t_1,t_2,\cdots,v_n)$ in the formula (2.6), it improves Theorem 2.4. We assume that $\omega(t_1,t_2,\cdots,v_n)>0$. For convenience, we denote

$$H(v_1,v_2,\cdots,v_n):=\int_{a_1}^{v_1}\int_{a_2}^{v_2}\cdots\int_{a_n}^{v_n}\omega(t_1,t_2,\cdots,v_n)\diamond_{\alpha_1}t_1\diamond_{\alpha_2}t_2\cdots\diamond_{\alpha_n}t_n,$$

and

$$\begin{aligned}&H^{\sigma}(v_1,v_2,\cdots,v_n):=H(\sigma_1(v_1),\sigma_2(v_2),\cdots,\sigma_n(v_n))\\ =&\int_{a_1}^{\sigma_1(v_1)}\int_{a_2}^{\sigma_2(v_2)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,t_2,\cdots,v_n)\diamond_{\alpha_1}t_1\diamond_{\alpha_2}t_2\cdots\diamond_{\alpha_n}t_n.\end{aligned}$$

Thus, we obtain the following theorem, which involves and improves Theorem 2.4.

Theorem 2.8 (Weighted Hardy-knopp type inequality for multiple Diamond-α_i integral)**.** *Assume that $A,f:[a_1,b_1]_{\mathbb{T}_1}\times[a_2,b_2]_{\mathbb{T}_2}\times\cdots\times[a_n,b_n]_{\mathbb{T}_n}\to[0,\infty)$ are continuous functions with diamond-$\alpha_i(i=1,2,\cdots,n)$ partial derivatives exist and*

$$\int_{t_1}^{b_1}\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{H(v_1,v_2,\cdots,v_n)H^{\sigma}(v_1,v_2,\cdots,v_n)}\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n<\infty,$$

for all $t_1,t_2,\cdots,t_n\in[a_1,b_1)\times[a_2,b_2)\times\cdots\times[a_n,b_n)$. If ϕ is a convex and positive function, f is nonnegative and bounded, then the following inequality holds.

$$\begin{aligned}&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{H(v_1,v_2,\cdots,v_n)}\phi\Big(\frac{C(v_1,v_2,\cdots,v_n)}{H^{\sigma}(v_1,v_2,\cdots,v_n)}\Big)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\ \le&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\omega(v_1,v_2,\cdots,v_n)B(v_1,v_2,\cdots,v_n)\phi(f(v_1,v_2,\cdots,v_n))\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n,\end{aligned} \tag{2.7}$$

where the functions B,C are respectively defined by

$$B(t_1,t_2,\cdots,t_n):=\int_{t_1}^{b_1}\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n}{H(v_1,v_2,\cdots,v_n)H^{\sigma}(v_1,v_2,\cdots,v_n)},$$

and

$$C(v_1,v_2,\cdots,v_n):=\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,\cdots,t_n)f(t_1,\cdots,t_n)\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n.$$

Proof. By employing Theorem 2.1, we obtain

$$\begin{aligned}&\phi\Big(\frac{C(v_1,v_2,\cdots,v_n)}{H^{\sigma}(v_1,v_2,\cdots,v_n)}\Big)\\ =&\ \phi\Big(\frac{\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,\cdots,v_n)f(t_1,\cdots,t_n)\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n}{\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,\cdots,v_n)\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n}\Big)\\ \le&\ \frac{\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,\cdots,v_n)\phi(f(t_1,\cdots,t_n))\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n}{\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,\cdots,v_n)\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n}\\ =&\ \frac{\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,\cdots,v_n)\phi(f(t_1,\cdots,t_n))\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n}{H^{\sigma}(v_1,v_2,\cdots,v_n)}.\end{aligned} \tag{2.8}$$

Substituting inequality (2.8) into the left hand side of (2.7) and using Fubini's Theorem, we yield

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{H(v_1,v_2,\cdots,v_n)}\phi\Big(\frac{C(v_1,v_2,\cdots,v_n)}{H^\sigma(v_1,v_2,\cdots,v_n)}\Big)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\
\le\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{H(v_1,v_2,\cdots,v_n)}\\
&\frac{\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,\cdots,t_n)\phi(f(t_1,\cdots,t_n))\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n}{H^\sigma(v_1,v_2,\cdots,v_n)}\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\
=\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\frac{A(v_1,v_2,\cdots,v_n)}{H(v_1,v_2,\cdots,v_n)H^\sigma(v_1,v_2,\cdots,v_n)}\\
&\Big(\int_{a_1}^{\sigma_1(v_1)}\cdots\int_{a_n}^{\sigma_n(v_n)}\omega(t_1,\cdots,t_n)\phi(f(t_1,\cdots,t_n))\diamond_{\alpha_1}t_1\cdots\diamond_{\alpha_n}t_n\Big)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\
=\ &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\omega(t_1,t_2,\cdots,t_n)\phi(f(t_1,t_2,\cdots,t_n))\Big(\int_{t_1}^{b_1}\int_{t_2}^{b_2}\cdots\int_{t_n}^{b_n}\\
&\frac{A(v_1,v_2,\cdots,v_n)}{H(v_1,v_2,\cdots,v_n)H^\sigma(v_1,v_2,\cdots,v_n)}\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\Big)\diamond_{\alpha_1}t_1\diamond_{\alpha_2}t_2\cdots\diamond_{\alpha_n}t_n\\
=\ &\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}B(v_1,\cdots,v_n)\omega(v_1,\cdots,v_n)\phi(f(v_1,\cdots,v_n))\diamond_{\alpha_1}v_1\cdots\diamond_{\alpha_n}v_n. \qquad (2.9)
\end{aligned}$$

Thereby we complete the proof. □

In the same way, we could establish weighted Hardy-knopp type inequality for multiple Δ and ∇ integrals by taking $\alpha_1=\alpha_2=\cdots=\alpha_n=1$ and 0, respectively.

3. Hermite-Hadamard type inequalities for multiple Diamond-α_i integral

In what follows, $f(v_1,v_2,\cdots,v_n)$ is convex on $[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$ means

$$f(c_1,c_2,\cdots,c_{i-1},v_i,c_{i+1},\cdots,c_n),\qquad i=1,2,\cdots,n,$$

are convex on $[a_i,b_i]$ for all $(c_1,c_2,\cdots,c_{i-1},c_{i+1},\cdots,c_n)\in[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_{i-1},b_{i-1}]\times[a_{i+1},b_{i+1}]\times\cdots\times[a_n,b_n]$.

In this section, we first present the Hermite-Hadamard type inequality for Diamond-α_i integral with three variables.

Theorem 3.1. *Assume that $f(v_1,v_2,v_3)$ is continuous and convex on $[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$, then the following inequalities hold.*

$$\begin{aligned}
&f(t_1^*,t_2^*,t_3^*) \qquad (3.1)\\
\le\ &\frac{1}{3}\Big(\frac{1}{b_1-a_1}\int_{a_1}^{b_1}f(v_1,t_2^*,t_3^*)\diamond_{\alpha_1}v_1+\frac{1}{b_2-a_2}\int_{a_2}^{b_2}f(t_1^*,v_2,t_3^*)\diamond_{\alpha_2}v_2\\
&+\frac{1}{b_3-a_3}\int_{a_3}^{b_3}f(t_1^*,t_2^*,v_3)\diamond_{\alpha_3}v_3\Big) \qquad (3.2)\\
\le\ &\frac{1}{3}\Big(\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(v_1,v_2,t_3^*)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\\
&+\frac{1}{(b_1-a_1)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_3}^{b_3}f(v_1,t_2^*,v_3)\diamond_{\alpha_1}v_1\diamond_{\alpha_3}v_3\\
&+\frac{1}{(b_2-a_2)(b_3-a_3)}\int_{a_2}^{b_2}\int_{a_3}^{b_3}f(t_1^*,v_2,v_3)\diamond_{\alpha_2}v_2\diamond_{\alpha_3}v_3\Big) \qquad (3.3)\\
\le\ &\frac{1}{\prod_{i=1}^3(b_i-a_i)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}f(v_1,v_2,v_3)\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\diamond_{\alpha_3}v_3 \qquad (3.4)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{3\prod_{i=1}^{3}(b_i-a_i)} \\
&\quad \Big(\int_{a_2}^{b_2}\int_{a_3}^{b_3} \eta_{1,1} f(a_2,v_2,v_3) + \eta_{1,2} f(b_2,v_2,v_3) \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\
&\quad + \int_{a_1}^{b_1}\int_{a_3}^{b_3} \eta_{2,1} f(v_2,a_2,v_3) + \eta_{2,2} f(v_2,b_2,v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_3} v_3 \\
&\quad + \int_{a_1}^{b_1}\int_{a_2}^{b_2} \eta_{3,1} f(v_2,v_2,a_3) + \eta_{3,2} f(v_2,v_2,b_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2\Big) \qquad (3.5) \\
&\le \frac{1}{3\prod_{i=1}^{3}(b_i-a_i)} \\
&\quad \Big(\int_{a_1}^{b_1} \eta_{2,1}\eta_{3,1} f(v_1,a_2,a_3) + \eta_{2,1}\eta_{3,2} f(v_1,a_2,b_3) \\
&\qquad + \eta_{2,2}\eta_{3,1} f(v_1,b_2,a_3) + \eta_{2,2}\eta_{3,2} f(v_1,b_2,b_3) \diamond_{\alpha_1} v_1 \\
&\quad + \int_{a_2}^{b_2} \eta_{1,1}\eta_{3,1} f(a_1,v_2,a_3) + \eta_{1,1}\eta_{3,2} f(a_1,v_2,b_3) \\
&\qquad + \eta_{1,2}\eta_{3,1} f(b_1,v_2,a_3) + \eta_{1,2}\eta_{3,2} f(b_1,v_2,b_3) \diamond_{\alpha_2} v_2 \\
&\quad + \int_{a_3}^{b_3} \eta_{1,1}\eta_{2,1} f(a_1,a_2,v_3) + \eta_{1,1}\eta_{2,2} f(a_1,b_2,v_3) \\
&\qquad + \eta_{1,2}\eta_{2,1} f(b_1,a_2,v_3) + \eta_{1,2}\eta_{2,2} f(b_1,b_2,v_3) \diamond_{\alpha_3} v_3\Big) \qquad (3.6) \\
&\le \frac{1}{\prod_{i=1}^{3}(b_i-a_i)} \Big(\eta_{1,1}\eta_{2,1}\eta_{3,1} f(a_1,a_2,a_3) + \eta_{1,1}\eta_{2,1}\eta_{3,2} f(a_1,a_2,b_3) \\
&\quad \eta_{1,1}\eta_{2,2}\eta_{3,1} f(a_1,b_2,a_3) + \eta_{1,1}\eta_{2,2}\eta_{3,2} f(a_1,b_2,b_3) + \eta_{1,2}\eta_{2,1}\eta_{3,1} f(b_1,a_2,a_3) \\
&\quad \eta_{1,2}\eta_{2,1}\eta_{3,2} f(b_1,a_2,b_3) + \eta_{1,2}\eta_{2,2}\eta_{3,1} f(b_1,b_2,a_3) + \eta_{1,2}\eta_{2,2}\eta_{3,2} f(b_1,b_2,b_3)\Big), \qquad (3.7)
\end{aligned}
$$

where $\eta_{i,j} (i = 1, 2, 3; j = 1, 2)$ *are defined as follows*

$$
\eta_{i,1} = b_i - t_i^*, \qquad i = 1,2,3,
$$
$$
\eta_{i,2} = t_i^* - a_i, \qquad i = 1,2,3,
$$

with

$$
t_i^* = \frac{\int_{a_i}^{b_i} s \diamond_{\alpha_i} s}{b_i - a_i}, \qquad i = 1,2,3.
$$

Proof. Since $f(v_1, v_2, v_3)$ is convex on $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, we have
(a)$f_1(v_1) := f(v_1, c_2, c_3)$ is convex on $[a_1, b_1]$ for all $(c_2, c_3) \in [a_2, b_2] \times [a_3, b_3]$;
(b)$f_2(v_2) := f(c_1, v_2, c_3)$ is convex on $[a_2, b_2]$ for all $(c_1, c_3) \in [a_1, b_1] \times [a_3, b_3]$;
(c)$f_3(v_3) := f(c_1, c_2, v_3)$ is convex on $[a_3, b_3]$ for all $(c_1, c_2) \in [a_1, b_1] \times [a_2, b_2]$.

And due to that $f(v_1, v_2, v_3)$ is continuous on $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, thus
(a)$f_1(v_1) := f(v_1, c_2, c_3)$ is continuous on $[a_1, b_1]$ for all $(c_2, c_3) \in [a_2, b_2] \times [a_3, b_3]$;
(b)$f_2(v_2) := f(c_1, v_2, c_3)$ is continuous on $[a_2, b_2]$ for all $(c_1, c_3) \in [a_1, b_1] \times [a_3, b_3]$;
(c)$f_3(v_3) := f(c_1, c_2, v_3)$ is continuous on $[a_3, b_3]$ for all $(c_1, c_2) \in [a_1, b_1] \times [a_2, b_2]$.

Then the conditions in Theorem 1.3 hold, we get

$$
f(t_1^*, c_2, c_3) \le \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} f(v_1, c_2, c_3) \diamond_{\alpha_1} v_1, \qquad (3.8)
$$

$$
f(c_1, t_2^*, c_3) \le \frac{1}{b_2 - a_2} \int_{a_2}^{b_2} f(c_1, v_2, c_3) \diamond_{\alpha_2} v_2, \qquad (3.9)
$$

and

$$
f(c_1, c_2, t_3^*) \le \frac{1}{b_3 - a_3} \int_{a_3}^{b_3} f(c_1, c_2, v_3) \diamond_{\alpha_3} v_3. \qquad (3.10)
$$

In fact, inequalities (3.8), (3.9) and (3.10) hold for all $c_j (j = 1, 2, 3)$, which could replaced by a_j, b_j, t_j^* or v_j.

(1) The proof of inequality "(3.1) $\leq$ (3.2)". Taking c_2, c_3 equal to t_2^*, t_3^* in Inequality (3.8), we have

$$f(t_1^*, t_2^*, t_3^*) \leq \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} f(v_1, t_2^*, t_3^*) \diamond_{\alpha_1} v_1.$$

Similarly, the following inequalities hold based on inequalities (3.9) and (3.10).

$$f(t_1^*, t_2^*, t_3^*) \leq \frac{1}{b_2 - a_2} \int_{a_2}^{b_2} f(t_1^*, v_2, t_3^*) \diamond_{\alpha_2} v_2,$$

$$f(t_1^*, t_2^*, t_3^*) \leq \frac{1}{b_3 - a_3} \int_{a_3}^{b_3} f(t_1^*, t_2^*, v_3) \diamond_{\alpha_3} v_3.$$

Adding them up, we complete the proof of inequality "(3.1) $\leq$ (3.2)".

(2) The proof of inequality "(3.2) $\leq$ (3.3)". Taking $c_2 = v_2$ and $c_3 = t_3^*$ in inequality (3.8), we have

$$f(t_1^*, v_2, t_3^*) \leq \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} f(v_1, v_2, t_3^*) \diamond_{\alpha_1} v_1. \tag{3.11}$$

Taking diamond-α_2 integral of both sides with respect of v_2, inequality (3.11) leads to

$$\int_{a_2}^{b_2} f(t_1^*, v_2, t_3^*) \diamond_{\alpha_2} v_2 \leq \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(v_1, v_2, t_3^*) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2. \tag{3.12}$$

Similarly, we immediately obtain

$$\int_{a_3}^{b_3} f(t_1^*, t_2^*, v_3) \diamond_{\alpha_3} v_3 \leq \frac{1}{b_2 - a_2} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(t_1^*, v_2, v_3) \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3, \tag{3.13}$$

and

$$\int_{a_1}^{b_1} f(v_1, t_2^*, t_3^*) \diamond_{\alpha_1} v_1 \leq \frac{1}{b_3 - a_3} \int_{a_1}^{b_1} \int_{a_3}^{b_3} f(v_1, t_2^*, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_3} v_3. \tag{3.14}$$

The sum of inequalities (3.12), (3.13) and (3.14) multiplying by $\frac{1}{b_2 - a_2}, \frac{1}{b_3 - a_3}$ and $\frac{1}{b_1 - a_1}$ respectively arrives at inequality "(3.2) $\leq$ (3.3)".

(3) The proof of inequality "(3.3) $\leq$ (3.4)". Taking $c_2 = v_2, c_3 = v_3$ in inequality (3.8), we obtain

$$f(t_1^*, v_2, v_3) \leq \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} f(v_1, v_2, v_3) \diamond_{\alpha_1} v_1. \tag{3.15}$$

Taking diamond-α_2 and diamond-α_3 integrals of both sides, inequality (3.15) gives

$$\int_{a_2}^{b_2} \int_{a_3}^{b_3} f(t_1^*, v_2, v_3) \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \leq \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(v_1, v_2, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3,$$

namely,

$$\begin{aligned} &\frac{1}{(b_2 - a_2)(b_3 - a_3)} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(t_1^*, v_2, v_3) \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\ \leq\ &\frac{1}{\prod_{i=1}^{3}(b_i - a_i)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(v_1, v_2, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3. \end{aligned}$$

Clearly, we also have

$$\begin{aligned} &\frac{1}{(b_1 - a_1)(b_3 - a_3)} \int_{a_1}^{b_1} \int_{a_3}^{b_3} f(v_2, t_2^*, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_3} v_3 \\ \leq\ &\frac{1}{\prod_{i=1}^{3}(b_i - a_i)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(v_1, v_2, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3, \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(v_2,v_2,t_3^*)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \\ \leq\ & \frac{1}{\prod_{i=1}^3(b_i-a_i)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1,v_2,v_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3. \end{aligned}$$

Thereby we complete the proof of inequality "(3.3) $\leq$ (3.4)" immediately.

By employing the right hand side of inequality (1.1) in Theorem 1.3 and the definition of η, we obtain

$$\begin{aligned} \int_{a_1}^{b_1} f(v_1,c_2,c_3)\diamond_{\alpha_1} v_1 &\leq (b_1-t_1^*)f(a_1,c_2,c_3)+(t_1^*-a_1)f(b_1,c_2,c_3) \\ &= \eta_{1,1}f(a_1,c_2,c_3)+\eta_{1,2}f(b_1,c_2,c_3), \end{aligned} \tag{3.16}$$

$$\begin{aligned} \int_{a_2}^{b_2} f(c_1,v_2,c_3)\diamond_{\alpha_2} v_2 &\leq (b_2-t_2^*)f(c_1,a_2,c_3)+(t_2^*-a_2)f(c_1,b_2,c_3) \\ &= \eta_{2,1}f(c_1,a_2,c_3)+\eta_{2,2}f(c_1,b_2,c_3), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \int_{a_3}^{b_3} f(c_1,c_2,v_3)\diamond_{\alpha_3} v_3 &\leq (b_3-t_3^*)f(c_1,c_2,a_3)+(t_2^*-a_2)f(c_1,c_2,b_3) \\ &= \eta_{3,1}f(c_1,c_2,a_3)+\eta_{3,1}f(c_1,c_2,b_3), \end{aligned} \tag{3.18}$$

(4) The proof of inequality "(3.4) $\leq$ (3.5)". Setting $c_2=v_2$ and $c_3=v_3$ in inequality (3.16), we have

$$\int_{a_1}^{b_1} f(v_1,v_2,v_3)\diamond_{\alpha_1} v_1 \leq \eta_{1,1}f(a_1,v_2,v_3)+\eta_{1,2}f(b_1,v_2,v_3), \tag{3.19}$$

Taking diamond-α_2 and diamond-α_3 integrals of both sides, inequality (3.19) gives

$$\begin{aligned} & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1,v_2,v_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\ =\ & \int_{a_2}^{b_2}\int_{a_3}^{b_3}\Big(\eta_{1,1}f(a_1,v_2,v_3)+\eta_{1,2}f(b_1,v_2,v_3)\Big)\diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3. \end{aligned} \tag{3.20}$$

In the same way, we have

$$\begin{aligned} & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1,v_2,v_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\ \leq\ & \int_{a_1}^{b_1}\int_{a_3}^{b_3}\Big(\eta_{2,1}f(v_1,a_2,v_3)+\eta_{2,2}f(v_1,b_2,v_3)\Big)\diamond_{\alpha_1} v_1 \diamond_{\alpha_3} v_3, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1,v_2,v_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\ \leq\ & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\Big(\eta_{3,1}f(v_1,v_2,a_3)+\eta_{3,2}f(v_1,v_2,b_3)\Big)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2, \end{aligned} \tag{3.22}$$

Then the proof is completed based on inequalities (3.20), (3.21) and (3.22).

(5) The proof of inequality "(3.5) $\leq$ (3.6)". Taking $c_2=v_2$ and $c_3=a_3$ in inequality (3.16), we have

$$\int_{a_1}^{b_1} f(v_1,v_2,a_3)\diamond_{\alpha_1} v_1 \leq \eta_{1,1}f(a_1,v_2,a_3)+\eta_{1,2}f(b_1,v_2,a_3). \tag{3.23}$$

Taking Diamond-α_2 integral of both sides gives with respect of v_2, inequality (3.23) gives

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(v_1,v_2,a_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \leq \int_{a_2}^{b_2} \eta_{1,1}f(a_1,v_2,a_3)+\eta_{1,2}f(b_1,v_2,a_3)\diamond_{\alpha_2} v_2. \quad (3.24)$$

Similarly, we obtain the following inequality if taking $c_2 = v_2, c_3 = b_3$ in inequality (3.16).

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(v_1,v_2,b_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \leq \int_{a_2}^{b_2} \eta_{1,1}f(a_1,v_2,b_3)+\eta_{1,2}f(b_1,v_2,b_3)\diamond_{\alpha_2} v_2. \quad (3.25)$$

In the same way, we have

$$\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(a_1,v_2,v_3)\diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \leq \int_{a_3}^{b_3} \eta_{2,1}f(a_1,a_2,v_3)+\eta_{2,2}f(a_1,b_2,v_3)\diamond_{\alpha_3} v_3, \quad (3.26)$$

$$\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(b_1,v_2,v_3)\diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \leq \int_{a_3}^{b_3} \eta_{2,1}f(b_1,a_2,v_3)+\eta_{2,2}f(b_1,b_2,v_3)\diamond_{\alpha_3} v_3, \quad (3.27)$$

$$\int_{a_1}^{b_1}\int_{a_3}^{b_3} f(v_1,a_2,v_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_3} v_3 \leq \int_{a_1}^{b_1} \eta_{3,1}f(v_1,a_2,a_3)+\eta_{3,2}f(v_1,a_2,b_3)\diamond_{\alpha_1} v_1, \quad (3.28)$$

and

$$\int_{a_1}^{b_1}\int_{a_3}^{b_3} f(v_1,b_2,v_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_3} v_3 \leq \int_{a_1}^{b_1} \eta_{3,1}f(v_1,b_2,a_3)+\eta_{3,2}f(v_1,b_2,b_3)\diamond_{\alpha_1} v_1. \quad (3.29)$$

Clearly, (3.24) $\times\eta_{3,1}+$ (3.25) $\times\eta_{3,2}+$ (3.26) $\times\eta_{1,1}+$ (3.27) $\times\eta_{1,2}+$ (3.28) $\times\eta_{2,1}+$ (3.29) $\times\eta_{3,2}$ leads to

$$\begin{aligned}
&\int_{a_2}^{b_2}\int_{a_3}^{b_3} \eta_{1,1}f(a_2,v_2,v_3)+\eta_{1,2}f(b_2,v_2,v_3)\diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3\\
&+\int_{a_1}^{b_1}\int_{a_3}^{b_3} \eta_{2,1}f(v_2,a_2,v_3)+\eta_{2,2}f(v_2,b_2,v_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_3} v_3\\
&+\int_{a_1}^{b_1}\int_{a_2}^{b_2} \eta_{3,1}f(v_2,v_2,a_3)+\eta_{3,2}f(v_2,v_2,b_3)\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2\\
\leq\ &\int_{a_1}^{b_1} \eta_{2,1}\eta_{3,1}f(v_1,a_2,a_3)+\eta_{2,1}\eta_{3,2}f(v_1,a_2,b_3)\\
&\qquad +\eta_{2,2}\eta_{3,1}f(v_1,b_2,a_3)+\eta_{2,2}\eta_{3,2}f(v_1,b_2,b_3)\diamond_{\alpha_1} v_1\\
&+\int_{a_2}^{b_2} \eta_{1,1}\eta_{3,1}f(a_1,v_2,a_3)+\eta_{1,1}\eta_{3,2}f(a_1,v_2,b_3)\\
&\qquad +\eta_{1,2}\eta_{3,1}f(b_1,v_2,a_3)+\eta_{1,2}\eta_{3,2}f(b_1,v_2,b_3)\diamond_{\alpha_2} v_2\\
&+\int_{a_3}^{b_3} \eta_{1,1}\eta_{2,1}f(a_1,a_2,v_3)+\eta_{1,1}\eta_{2,2}f(a_1,b_2,v_3)\\
&\qquad +\eta_{1,2}\eta_{2,1}f(b_1,a_2,v_3)+\eta_{1,2}\eta_{2,2}f(b_1,b_2,v_3)\diamond_{\alpha_3} v_3.
\end{aligned}$$

Thereby we complete the proof of inequality "(3.5) $\leq$ (3.6)".

(6) The proof of inequality "(3.6) $\leq$ (3.7)". By employing inequalities (3.16), (3.17) and (3.18), it gives

$$\begin{aligned}
&\int_{a_1}^{b_1} \eta_{2,1}\eta_{3,1}f(v_1,a_2,a_3)+\eta_{2,1}\eta_{3,2}f(v_1,a_2,b_3)\diamond_{\alpha_1} v_1\\
\leq\ &\eta_{1,1}\eta_{2,1}\eta_{3,1}f(a_1,a_2,a_3)+\eta_{1,2}\eta_{2,1}\eta_{3,1}f(b_1,a_2,a_3)\\
+\ &\eta_{1,1}\eta_{2,1}\eta_{3,2}f(a_1,a_2,b_3)+\eta_{1,2}\eta_{2,1}\eta_{3,2}f(b_1,a_2,b_3), \qquad (3.30)
\end{aligned}$$

$$\begin{aligned}
&\int_{a_1}^{b_1} \eta_{2,2}\eta_{3,1}f(v_1,b_2,a_3)+\eta_{2,2}\eta_{3,2}f(v_1,b_2,b_3)\diamond_{\alpha_1} v_1\\
\leq\ & \eta_{1,1}\eta_{2,2}\eta_{3,1}f(a_1,b_2,a_3)+\eta_{1,2}\eta_{2,2}\eta_{3,1}f(b_1,b_2,a_3)\\
+\ & \eta_{1,1}\eta_{2,2}\eta_{3,2}f(a_1,b_2,b_3)+\eta_{1,2}\eta_{2,2}\eta_{3,2}f(b_1,b_2,b_3),
\end{aligned}\tag{3.31}$$

$$\begin{aligned}
&\int_{a_2}^{b_2} \eta_{1,1}\eta_{3,1}f(a_1,v_2,a_3)+\eta_{1,1}\eta_{3,2}f(a_1,v_2,b_3)\diamond_{\alpha_2} v_2\\
\leq\ & \eta_{1,1}\eta_{2,1}\eta_{3,1}f(a_1,a_2,a_3)+\eta_{1,1}\eta_{2,2}\eta_{3,1}f(a_1,b_2,a_3)\\
+\ & \eta_{1,1}\eta_{2,1}\eta_{3,2}f(a_1,a_2,b_3)+\eta_{1,1}\eta_{2,2}\eta_{3,2}f(a_1,b_2,b_3),
\end{aligned}\tag{3.32}$$

$$\begin{aligned}
&\int_{a_2}^{b_2} \eta_{1,2}\eta_{3,1}f(b_1,v_2,a_3)+\eta_{1,2}\eta_{3,2}f(b_1,v_2,b_3)\diamond_{\alpha_2} v_2\\
\leq\ & \eta_{1,2}\eta_{2,1}\eta_{3,1}f(b_1,a_2,a_3)+\eta_{1,2}\eta_{2,2}\eta_{3,1}f(b_1,b_2,a_3)\\
+\ & \eta_{1,2}\eta_{2,1}\eta_{3,2}f(b_1,a_2,b_3)+\eta_{1,2}\eta_{2,2}\eta_{3,2}f(b_1,b_2,b_3),
\end{aligned}\tag{3.33}$$

$$\begin{aligned}
&\int_{a_3}^{b_3} \eta_{1,1}\eta_{2,1}f(a_1,a_2,v_3)+\eta_{1,1}\eta_{2,2}f(a_1,b_2,v_3)\diamond_{\alpha_3} v_3\\
\leq\ & \eta_{1,1}\eta_{2,1}\eta_{3,1}f(a_1,a_2,a_3)+\eta_{1,1}\eta_{2,1}\eta_{3,2}f(a_1,a_2,b_3)\\
+\ & \eta_{1,1}\eta_{2,2}\eta_{3,1}f(a_1,b_2,a_3)+\eta_{1,1}\eta_{2,2}\eta_{3,2}f(a_1,b_2,b_3),
\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}
&\int_{a_3}^{b_3} \eta_{1,2}\eta_{2,1}f(b_1,a_2,v_3)+\eta_{1,2}\eta_{2,2}f(b_1,b_2,v_3)\diamond_{\alpha_3} v_3\\
\leq\ & \eta_{1,2}\eta_{2,1}\eta_{3,1}f(b_1,a_2,a_3)+\eta_{1,2}\eta_{2,1}\eta_{3,2}f(b_1,a_2,b_3)\\
+\ & \eta_{1,2}\eta_{2,2}\eta_{3,1}f(b_1,b_2,a_3)+\eta_{1,2}\eta_{2,2}\eta_{3,2}f(b_1,b_2,b_3).
\end{aligned}\tag{3.35}$$

Adding inequalities (3.30), (3.31), (3.32), (3.33), (3.34) and (3.35) up, we can complete the proof of inequality "(3.5) $\leq$ (3.6)". Hence we have proved the theorem. □

Undoubtedly, the corresponding Hermite-Hadamard type inequalities for Δ and ∇ integrals of three variables can be obtained easily if we take all $\alpha_i = 1$ or 0 in Theorem 3.1.

In fact, the inequalities given in Theorem 3.1 have the following equivalent form:

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(t_1^*,t_2^*,t_3^*)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\diamond_{\alpha_3} v_3\\
\leq\ & \frac{1}{3}\Big(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1,t_2^*,t_3^*)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\diamond_{\alpha_3} v_3\\
&+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(t_1^*,v_2,t_3^*)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\diamond_{\alpha_3} v_3\\
&+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(t_1^*,t_2^*,v_3)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\diamond_{\alpha_3} v_3\Big)\\
\leq\ & \frac{1}{3}\Big(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1,v_2,t_3^*)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\diamond_{\alpha_3} v_3\\
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1,t_2^*,v_3)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\diamond_{\alpha_3} v_3
\end{aligned}$$

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(t_1^*, v_2, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3\Big) \\
\leq\;& \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1, v_2, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\
\leq\;& \frac{1}{3}\Big(\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \eta_{1,1} f(a_2, v_2, v_3) + \eta_{1,2} f(b_2, v_2, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\
&+\frac{1}{b_2-a_2}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \eta_{2,1} f(v_2, a_2, v_3) + \eta_{2,2} f(v_2, b_2, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\
&+\frac{1}{b_3-a_3}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \eta_{3,1} f(v_2, v_2, a_3) + \eta_{3,2} f(v_2, v_2, b_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3\Big) \\
\leq\;& \frac{1}{3}\Big(\frac{1}{(b_2-a_2)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \eta_{2,1}\eta_{3,1} f(v_1, a_2, a_3) + \eta_{2,1}\eta_{3,2} f(v_1, a_2, b_3) \\
&\quad +\eta_{2,2}\eta_{3,1} f(v_1, b_2, a_3) + \eta_{2,2}\eta_{3,2} f(v_1, b_2, b_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\
&+\frac{1}{(b_1-a_1)(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \eta_{1,1}\eta_{3,1} f(a_1, v_2, a_3) + \eta_{1,1}\eta_{3,2} f(a_1, v_2, b_3) \\
&\quad +\eta_{1,2}\eta_{3,1} f(b_1, v_2, a_3) + \eta_{1,2}\eta_{3,2} f(b_1, v_2, b_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3 \\
&+\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \eta_{1,1}\eta_{2,1} f(a_1, a_2, v_3) + \eta_{1,1}\eta_{2,2} f(a_1, b_2, v_3) \\
&\quad +\eta_{1,2}\eta_{2,1} f(b_1, a_2, v_3) + \eta_{1,2}\eta_{2,2} f(b_1, b_2, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3\Big) \\
\leq\;& \eta_{1,1}\eta_{2,1}\eta_{3,1} f(a_1, a_2, a_3) + \eta_{1,1}\eta_{2,1}\eta_{3,2} f(a_1, a_2, b_3) \\
& \eta_{1,1}\eta_{2,2}\eta_{3,1} f(a_1, b_2, a_3) + \eta_{1,1}\eta_{2,2}\eta_{3,2} f(a_1, b_2, b_3) + \eta_{1,2}\eta_{2,1}\eta_{3,1} f(b_1, a_2, a_3) \\
& \eta_{1,2}\eta_{2,1}\eta_{3,2} f(b_1, a_2, b_3) + \eta_{1,2}\eta_{2,2}\eta_{3,1} f(b_1, b_2, a_3) + \eta_{1,2}\eta_{2,2}\eta_{3,2} f(b_1, b_2, b_3).
\end{aligned}$$

Now we consider the case of n variables. Before starting our theorem, some necessary notations that simplify our proof are given. Similar to Theorem 3.1, $t_i^*(i = 1, 2, \cdots, n)$ and $\eta'_{i,j}(i = 1, 2, \cdots, n, j = -1, 0, 1, 2)$ are defined as

$$t_i^* = \frac{\int_{a_i}^{b_i} s \diamond_{\alpha_i} s}{b_i - a_i}, \quad \eta'_{i,-1} = 1, \quad \eta'_{i,0} = 1, \quad \eta'_{i,1} = \frac{b_i - t_i^*}{b_i - a_i}, \quad \eta'_{i,2} = \frac{t_i^* - a_i}{b_i - a_i}.$$

Let $D := (d_1, d_2, \cdots, d_n) \in \{0, 1, 2, 3\} \times \{0, 1, 2, 3\} \times \cdots \times \{0, 1, 2, 3\}$ and

$$I_n\{f\}(d_1, d_2, \cdots, d_n) = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f(\psi_1(d_1), \psi_2(d_2), \cdots, \psi_n(d_n)) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n,$$

where $\psi_i(x)(i = 1, 2, \cdots, n)$ are defined on $\{0, 1, 2, 3\}$ as follows

$$\psi_i(x) = \begin{cases} v_i, & \text{if} \quad x = 0, \\ t_i^*, & \text{if} \quad x = 1, \\ a_i, & \text{if} \quad x = 2, \\ b_i, & \text{if} \quad x = 3. \end{cases}$$

For example, $I_3\{f\}(0, 2, 3)$ means

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1, a_2, b_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3,$$

and $I_3\{f\}(1, 1, 0)$ means

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(t_1^*, t_2^*, v_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3.$$

We also denote

$$J_n\{f\}(d_1, d_2, \cdots, d_n) = \Big(\prod_{i=1}^{n} \eta'_{i,d_i-1}\Big) I_n\{f\}(d_1, d_2, \cdots, d_n),$$

for instance, $J_3\{f\}(0, 2, 3)$ means

$$\begin{aligned} &\eta'_{1,-1}\eta'_{2,1}\eta'_{3,2} I_n\{f\}(0, 2, 3) \\ = \;& \eta'_{2,1}\eta'_{3,2} \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(v_1, a_2, b_3) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \diamond_{\alpha_3} v_3. \end{aligned}$$

Clearly, if every $d_i (i = 1, 2, \cdots, n)$ equal to 0 or 1, then

$$J_n\{f\}(d_1, d_2, \cdots, d_n) = I_n\{f\}(d_1, d_2, \cdots, d_n).$$

Theorem 3.2 (Hermite-Hadamard type inequality for multiple Diamond-α_i integral). *Assume that f is continuous and convex on $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, Λ_i means that i parameters in J_n equal to 0 and others equal to 1, Υ_i means that i parameters in J_n equal to 2 or 3 and others equal to 0. And we define $L_i^n (i = 0, 1, 2, \cdots, n)$ as follows*

$$L_i^n\{f\} = \frac{1}{C_n^i} \sum_{\Lambda_i} J_n\{f\}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

where $\lambda_j (j = 1, 2, \cdots, n) = 0$ or 1 and $C_n^i = \frac{n!}{i!(n-i)!}$, define $K_i^n (i = 0, 1, 2, \cdots, n)$ as follows

$$K_i^n\{f\} = \frac{1}{C_n^i} \sum_{\Upsilon_i} J_n\{f\}(\tau_1, \tau_2, \cdots, \tau_n),$$

where $\tau_j (j = 1, 2, \cdots, n) = 0$, 2 or 3.

Then

$$L_0^n\{f\} \le L_1^n\{f\} \le \cdots \le L_n^n\{f\} = K_0^n\{f\} \le K_1^n\{f\} \le \cdots \le K_n^n\{f\}.$$

Proof. It is obvious that $L_n^n\{f\} = K_0^n\{f\}$. It follows from that $f(v_1, v_2, \cdots, v_n)$ is continuous and convex on $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ that

$$f(c_1, c_2, \cdots, c_{i-1}, v_i, c_{i+1}, \cdots, c_n),$$

is continuous and convex on $[a_i, b_i]$ for all $(c_1, c_2, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n) \in [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n]$.

Then the conditions in Theorem 1.3 hold, for all $i = 1, 2, \cdots, n$, we have

$$\begin{aligned} &f(c_1, c_2, \cdots, c_{i-1}, t_i^*, c_{i+1}, \cdots, c_n) \\ \le &\frac{1}{b_i - a_i} \int_{a_i}^{b_i} f(c_1, c_2, \cdots, c_{i-1}, v_i, c_{i+1}, \cdots, c_n) \diamond_{\alpha_i} v_i. \end{aligned} \tag{3.36}$$

Taking diamond-$\alpha_j (j \ne i)$ integral with the respect of $v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n$ and regarding c_j as functions of v_j, we yield

$$\begin{aligned} &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f(c_1, c_2, \cdots, c_{i-1}, t_i^*, c_{i+1}, \cdots, c_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n \\ \le \;& \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f(c_1, c_2, \cdots, c_{i-1}, v_i, c_{i+1}, \cdots, c_n) \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n. \end{aligned} \tag{3.37}$$

Since that inequality (3.37) holds for all $c_j (j \ne i)$, as special cases, taking $c_j = d_j (j \ne i)$, we obtain

$$I_n\{f\}(d_1, d_2, \cdots, d_{i-1}, 1, d_{i+1}, \cdots, d_n) \le I_n\{f\}(d_1, d_2, \cdots, d_{i-1}, 0, d_{i+1}, \cdots, d_n),$$

holds for all $d_j (j \ne i) = 0, 1, 2, 3$ and $i = 1, 2, \cdots, n$.

Whereupon, noting that there are totally $n + 1$ parameters in I_{n+1}, we obtain

$$I_{n+1}\{f\}(0, 0, \cdots, 0) \ge I_{n+1}\{f\}(1, 0, \cdots, 0),$$

$$\begin{aligned} I_{n+1}\{f\}(0,0,\cdots,0) &\geq I_{n+1}\{f\}(0,1,\cdots,0), \\ &\cdots \\ I_{n+1}\{f\}(0,0,\cdots,0) &\geq I_{n+1}\{f\}(0,0,\cdots,1), \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} & I_{n+1}\{f\}(0,0,\cdots,0) \\ \geq\ & \frac{1}{n+1}\Big(I_{n+1}\{f\}(1,0,\cdots,0)+I_{n+1}\{f\}(0,1,\cdots,0)+\cdots+I_{n+1}\{f\}(0,0,\cdots,1)\Big). \end{aligned}$$

On the other hand, it follows from Theorem 1.3 that

$$\begin{aligned} & \frac{1}{b_i-a_i}\int_{a_i}^{b_i} f(c_1,c_2,\cdots,c_{i-1},v_i,c_{i+1},\cdots,c_n)\diamond_{\alpha_i} v_i \\ \leq\ & \eta'_{i,1}f(c_1,c_2,\cdots,c_{i-1},a_i,c_{i+1},\cdots,c_n)+\eta'_{i,2}f(c_1,c_2,\cdots,c_{i-1},b_i,c_{i+1},\cdots,c_n). \end{aligned}$$

Taking diamond-$\alpha_j (j\neq i)$ integral with respect to $v_1,v_2,\cdots,v_{i-1},v_{i+1},\cdots,v_n$, we yield

$$\begin{aligned} & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f(c_1,c_2,\cdots,c_{i-1},v_i,c_{i+1},\cdots,c_n)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n \\ \leq\ & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} \eta'_{i,1}f(c_1,c_2,\cdots,c_{i-1},a_i,c_{i+1},\cdots,c_n)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n \\ +\ & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} \eta'_{i,2}f(c_1,c_2,\cdots,c_{i-1},b_i,c_{i+1},\cdots,c_n)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_n} v_n. \end{aligned}$$

Thus we obtain

$$\begin{aligned} & I_n\{f\}(d_1,d_2,\cdots,d_{i-1},0,d_{i+1},\cdots,d_n) \\ \leq\ & \eta'_{i,1}I_n\{f\}(d_1,d_2,\cdots,d_{i-1},2,d_{i+1},\cdots,d_n) \\ +\ & \eta'_{i,2}I_n\{f\}(d_1,d_2,\cdots,d_{i-1},3,d_{i+1},\cdots,d_n), \end{aligned}$$

holds for all $d_j(j\neq i)=0,1,2,3$, $i=1,2,\cdots,n$. Likewise,

$$\begin{aligned} I_{n+1}\{f\}(0,0,\cdots,0) &\leq \eta_{1,1}I_{n+1}\{f\}(2,0,\cdots,0)+\eta_{1,2}I_{n+1}\{f\}(3,0,\cdots,0), \\ I_{n+1}\{f\}(0,0,\cdots,0) &\leq \eta_{2,1}I_{n+1}\{f\}(0,2,\cdots,0)+\eta_{2,2}I_{n+1}\{f\}(0,3,\cdots,0), \\ &\cdots \\ I_{n+1}\{f\}(0,0,\cdots,0) &\leq \eta_{n+1,1}I_{n+1}\{f\}(0,0,\cdots,2)+\eta_{n+1,2}I_{n+1}\{f\}(0,0,\cdots,3). \end{aligned} \tag{3.39}$$

We complete our proof by induction. The cases of $n=1,2,3$ have been given in Theorems 1.3, 1.5 and 3.1, respectively. We suppose the case of $n=k$ holds, which means

$$L_0^k\{f\}\leq L_1^k\{f\}\leq\cdots\leq L_k^k\{f\},$$

and

$$K_0^k\{f\}\leq K_1^k\{f\}\leq\cdots\leq K_k^k\{f\}.$$

We firstly give the proof of

$$L_0^{k+1}\{f\}\leq L_1^{k+1}\{f\}\leq L_2^{k+1}\{f\}\leq\cdots\leq L_{k+1}^{k+1}\{f\}. \tag{3.40}$$

By using inequality (3.38), we have

$$\begin{aligned} & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{k+1}}^{b_{k+1}} f(v_1,v_2,\cdots,v_k,t_{k+1}^*)\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_{k+1}} v_{k+1} \\ \leq\ & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{k+1}}^{b_{k+1}} f(v_1,v_2,\cdots,v_k,v_{k+1})\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_{k+1}} v_{k+1}. \end{aligned}$$

It is equivalent to

$$(b_{k+1}-a_{k+1})\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_k}^{b_k} f(v_1,v_2,\cdots,v_k,t^*_{k+1})\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_k} v_k$$
$$\leq \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{k+1}}^{b_{k+1}} f(v_1,v_2,\cdots,v_k,v_{k+1})\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_{k+1}} v_{k+1}. \quad (3.41)$$

If we set $g(v_1,v_2,\cdots,v_k)=f(v_1,v_2,\cdots,v_k,t^*_{k+1})$, then inequality (3.41) can be rewritten as

$$(b_{k+1}-a_{k+1})I_k\{g\}(0,0,\cdots,0)\leq I_{k+1}\{f\}(0,0,\cdots,0,0). \quad (3.42)$$

Noting that $I_k\{g\}(0,0,\cdots,0)=L_k^k\{g\}$, together with inequality (3.42), we yield

$$(b_{k+1}-a_{k+1})L_0^k\{g\}\leq\cdots\leq(b_{k+1}-a_{k+1})L_k^k\{g\}\leq I_{k+1}\{f\}(0,0,\cdots,0)=L_{k+1}^{k+1}\{f\}. \quad (3.43)$$

According to the definitions of L, we see that

$$(b_{k+1}-a_{k+1})L_i^k\{g\}=\frac{b_{k+1}-a_{k+1}}{C_k^i}\sum_{\Lambda_i} J_k\{g\}(\lambda_1,\lambda_2,\cdots,\lambda_k)$$
$$=\frac{b_{k+1}-a_{k+1}}{C_k^i}\sum_{\Lambda_i} I_k\{g\}(\lambda_1,\lambda_2,\cdots,\lambda_k)$$
$$=\frac{b_{k+1}-a_{k+1}}{C_k^i}\sum_{\Lambda_i}$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_k}^{b_k} g(\psi_1(\lambda_1),\psi_2(\lambda_2),\cdots,\psi_k(\lambda_k))\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_k} v_k,$$
$$=\frac{b_{k+1}-a_{k+1}}{C_k^i}\sum_{\Lambda_i}$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_k}^{b_k} f(\psi_1(\lambda_1),\psi_2(\lambda_2),\cdots,\psi_k(\lambda_k),\psi_{k+1}(1))\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_k} v_k,$$
$$=\frac{1}{C_k^i}\sum_{\Lambda_i}$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{k+1}}^{b_{k+1}} f(\psi_1(\lambda_1),\psi_2(\lambda_2),\cdots,\psi_k(\lambda_k),\psi_{k+1}(1))\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_{k+1}} v_{k+1},$$
$$=\frac{1}{C_k^i}\sum_{\Lambda_i} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,1). \quad (3.44)$$

Substituting inequality (3.44) into (3.43), we obtain

$$\frac{1}{C_k^0}\sum_{\Lambda_0} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,1)\leq\frac{1}{C_k^1}\sum_{\Lambda_1} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,1)$$
$$\leq\frac{1}{C_k^2}\sum_{\Lambda_2} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,1)\leq\cdots\leq$$
$$\leq\frac{1}{C_k^k}\sum_{\Lambda_k} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,1)\leq L_{k+1}^{k+1}\{f\}.$$

In the same method, we also have

$$\begin{aligned}
&\frac{1}{C_k^0}\sum_{\Lambda_0} J_{k+1}\{f\}(1,\lambda_1,\lambda_2,\cdots,\lambda_k) \le \frac{1}{C_k^1}\sum_{\Lambda_1} J_{k+1}\{f\}(1,\lambda_1,\lambda_2,\cdots,\lambda_k)\\
\le\ &\frac{1}{C_k^2}\sum_{\Lambda_2} J_{k+1}\{f\}(1,\lambda_1,\lambda_2,\cdots,\lambda_k) \le \cdots \le\\
\le\ &\frac{1}{C_k^k}\sum_{\Lambda_k} J_{k+1}\{f\}(1,\lambda_1,\lambda_2,\cdots,\lambda_k) \le L_{k+1}^{k+1}\{f\},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{C_k^0}\sum_{\Lambda_0} J_{k+1}\{f\}(\lambda_1,1,\lambda_2,\cdots,\lambda_k) \le \frac{1}{C_k^1}\sum_{\Lambda_1} J_{k+1}\{f\}(\lambda_1,1,\lambda_2,\cdots,\lambda_k)\\
\le\ &\frac{1}{C_k^2}\sum_{\Lambda_2} J_{k+1}\{f\}(\lambda_1,1,\lambda_2,\cdots,\lambda_k) \le \cdots \le\\
\le\ &\frac{1}{C_k^k}\sum_{\Lambda_k} J_{k+1}\{f\}(\lambda_1,1,\lambda_2,\cdots,\lambda_k) \le L_{k+1}^{k+1}\{f\},
\end{aligned}$$

$$\cdots$$

$$\begin{aligned}
&\frac{1}{C_k^0}\sum_{\Lambda_0} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,1,\lambda_k) \le \frac{1}{C_k^1}\sum_{\Lambda_1} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,1,\lambda_k)\\
\le\ &\frac{1}{C_k^2}\sum_{\Lambda_2} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,1,\lambda_k) \le \cdots \le\\
\le\ &\frac{1}{C_k^k}\sum_{\Lambda_k} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,1,\lambda_k) \le L_{k+1}^{k+1}\{f\}.
\end{aligned}$$

Adding them all up, we can arrive at the desired inequality (3.40). In fact, that's according to the facts that

$$\begin{aligned}
&\frac{1}{C_k^i}\sum_{\Lambda_i} J_{k+1}\{f\}(1,\lambda_1,\lambda_2,\cdots,\lambda_k) + \frac{1}{C_k^i}\sum_{\Lambda_i} J_{k+1}\{f\}(\lambda_1,1,\lambda_2,\cdots,\lambda_k)\\
+\ &\frac{1}{C_k^i}\sum_{\Lambda_i} J_{k+1}\{f\}(\lambda_1,\lambda_2,1,\cdots,\lambda_k) + \cdots + \frac{1}{C_k^i}\sum_{\Lambda_i} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,1)\\
=\ &\frac{1}{C_k^i}\sum_{\Lambda_i}\Big(J_{k+1}\{f\}(1,\lambda_1,\lambda_2,\cdots,\lambda_k) + J_{k+1}\{f\}(\lambda_1,1,\lambda_2,\cdots,\lambda_k)\\
&\quad +J_{k+1}\{f\}(\lambda_1,\lambda_2,1,\cdots,\lambda_k) + \cdots + J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,1)\Big)\\
=\ &(k+1-i)\frac{1}{C_k^i}\sum_{\Lambda_i} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,\lambda_{k+1})\\
=\ &\frac{1}{k+1}\frac{1}{C_{k+1}^i}\sum_{\Lambda_i} J_{k+1}\{f\}(\lambda_1,\lambda_2,\cdots,\lambda_k,\lambda_{k+1})\\
=\ &\frac{1}{k+1}L_i^{k+1}.
\end{aligned}$$

And then we give the proof of

$$K_0^k\{f\} \le K_1^k\{f\} \le \cdots \le K_{k+1}^{k+1}\{f\}. \tag{3.45}$$

By using inequality (3.39), we have

$$\begin{aligned}&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{k+1}}^{b_{k+1}} f(v_1,v_2,\cdots,v_k,v_{k+1})\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_{k+1}} v_{k+1}\\ \leq\ & \eta'_{n+1,1}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{k+1}}^{b_{k+1}} f(v_1,v_2,\cdots,v_k,a_{k+1})\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_{k+1}} v_{k+1}\\ +\ & \eta'_{n+1,2}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{k+1}}^{b_{k+1}} f(v_1,v_2,\cdots,v_k,b_{k+1})\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_{k+1}} v_{k+1}.\end{aligned}$$

It is equivalent to

$$\begin{aligned}&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{k+1}}^{b_{k+1}} f(v_1,v_2,\cdots,v_k,v_{k+1})\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_{k+1}} v_{k+1}\\ \leq\ & (b_{k+1}-a_{k+1})\\ & \Big(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_k}^{b_k}\eta'_{n+1,1}f(v_1,v_2,\cdots,v_k,a_{k+1})\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_k} v_k\\ & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_k}^{b_k}\eta'_{n+1,2}f(v_1,v_2,\cdots,v_k,b_{k+1})\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_k} v_k\Big). \qquad (3.46)\end{aligned}$$

If we set

$$g_1(v_1,v_2,\cdots,v_k)=f(v_1,v_2,\cdots,v_k,a_{k+1})$$

and

$$g_2(v_1,v_2,\cdots,v_k)=f(v_1,v_2,\cdots,v_k,b_{k+1}),$$

then inequality (3.46) can be rewritten as

$$\begin{aligned}& I_{k+1}\{f\}(0,0,\cdots,0)\\ \leq\ & (b_{k+1}-a_{k+1})\eta'_{n+1,1}I_k\{g_1\}(0,0,\cdots,0)\\ +\ & (b_{k+1}-a_{k+1})\eta'_{n+1,2}I_k\{g_2\}(0,0,\cdots,0). \qquad (3.47)\end{aligned}$$

Noting that $I_k\{g_1\}(0,0,\cdots,0)=K_0^k\{g_1\}$ and $I_k\{g_2\}(0,0,\cdots,0)=K_0^k\{g_2\}$, together with inequality (3.47), we yield

$$\begin{aligned}& K_0^{k+1}\{f\}=I_{k+1}\{f\}(0,0,\cdots,0)\\ \leq\ & (b_{k+1}-a_{k+1})\eta'_{n+1,1}K_0^k\{g_1\}+(b_{k+1}-a_{k+1})\eta'_{n+1,2}K_0^k\{g_2\}\\ \leq\ & (b_{k+1}-a_{k+1})\eta'_{n+1,1}K_1^k\{g_1\}+(b_{k+1}-a_{k+1})\eta'_{n+1,2}K_1^k\{g_2\}\\ \leq\ & \cdots\\ \leq\ & (b_{k+1}-a_{k+1})\eta'_{n+1,1}K_k^k\{g_1\}+(b_{k+1}-a_{k+1})\eta'_{n+1,2}K_k^k\{g_2\}, \qquad (3.48)\end{aligned}$$

According to the definitions of L, I and J, for $i=0,1,\cdots,k$, we see that

$$(b_{k+1}-a_{k+1})\eta'_{n+1,1}K_i^k\{g_1\}=\eta'_{n+1,1}\frac{b_{k+1}-a_{k+1}}{C_k^i}\sum_{\Upsilon_i}J_k\{g_1\}(\tau_1,\tau_2,\cdots,\tau_k)$$

$$\begin{aligned}=\ & \eta'_{n+1,1}\frac{b_{k+1}-a_{k+1}}{C_k^i}\sum_{\Upsilon_i}\prod_{i=1}^k\eta'_{i,\tau_i-1}I_k\{g_1\}(\tau_1,\tau_2,\cdots,\tau_k)\\ =\ & \eta'_{n+1,1}\prod_{i=1}^k\eta'_{i,\tau_i-1}\frac{b_{k+1}-a_{k+1}}{C_k^i}\sum_{\Upsilon_i}\\ & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_k}^{b_k} g_1(\psi_1(\tau_1),\psi_2(\tau_2),\cdots,\psi_k(\tau_k))\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_k} v_k,\\ =\ & \eta'_{n+1,1}\prod_{i=1}^k\eta'_{i,\tau_i-1}\frac{b_{k+1}-a_{k+1}}{C_k^i}\sum_{\Upsilon_i}\end{aligned}$$

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_k}^{b_k} f(\psi_1(\tau_1),\psi_2(\tau_2),\cdots,\psi_k(\tau_k),\psi_{k+1}(2))\diamond_{\alpha_1} v_1\diamond_{\alpha_2} v_2\cdots\diamond_{\alpha_k} v_k,\\
= \;& \eta'_{n+1,1}\prod_{i=1}^{k}\eta'_{i,\tau_i-1}\frac{1}{C_k^i}\sum_{\Upsilon_i} I_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,2)\\
= \;& \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,2), \qquad (3.49)
\end{aligned}$$

In the same way, we have

$$(b_{k+1}-a_{k+1})\eta'_{n+1,2}K_i^k\{g_2\} = \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,3), \qquad (3.50)$$

Substituting inequalities (3.49) and (3.50) into (3.48), we get

$$\begin{aligned}
& K_0^{k+1}\{f\}\\
\le\; & \frac{1}{C_k^1}\sum_{\Upsilon_1} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,2)+\frac{1}{C_k^1}\sum_{\Upsilon_1} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,3)\\
\le\; & \frac{1}{C_k^2}\sum_{\Upsilon_2} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,2)+\frac{1}{C_k^2}\sum_{\Upsilon_2} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,3)\\
\le\; & \cdots\\
\le\; & \frac{1}{C_k^k}\sum_{\Upsilon_k} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,2)+\frac{1}{C_k^k}\sum_{\Upsilon_k} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,3).
\end{aligned}$$

In the same method, we also have

$$\begin{aligned}
& K_0^{k+1}\{f\}\\
\le\; & \frac{1}{C_k^1}\sum_{\Upsilon_1} J_{k+1}\{f\}(2,\tau_1,\tau_2,\cdots,\tau_k)+\frac{1}{C_k^1}\sum_{\Upsilon_1} J_{k+1}\{f\}(3,\tau_1,\tau_2,\cdots,\tau_k)\\
\le\; & \frac{1}{C_k^2}\sum_{\Upsilon_2} J_{k+1}\{f\}(2,\tau_1,\tau_2,\cdots,\tau_k)+\frac{1}{C_k^2}\sum_{\Upsilon_2} J_{k+1}\{f\}(3,\tau_1,\tau_2,\cdots,\tau_k)\\
\le\; & \cdots\\
\le\; & \frac{1}{C_k^k}\sum_{\Upsilon_k} J_{k+1}\{f\}(2,\tau_1,\tau_2,\cdots,\tau_k)+\frac{1}{C_k^k}\sum_{\Upsilon_k} J_{k+1}\{f\}(3,\tau_1,\tau_2,\cdots,\tau_k),
\end{aligned}$$

$$\begin{aligned}
& K_0^{k+1}\{f\}\\
\le\; & \frac{1}{C_k^1}\sum_{\Upsilon_1} J_{k+1}\{f\}(\tau_1,2,\tau_2,\cdots,\tau_k)+\frac{1}{C_k^1}\sum_{\Upsilon_1} J_{k+1}\{f\}(\tau_1,3,\tau_2,\cdots,\tau_k)\\
\le\; & \frac{1}{C_k^2}\sum_{\Upsilon_2} J_{k+1}\{f\}(\tau_1,2,\tau_2,\cdots,\tau_k)+\frac{1}{C_k^2}\sum_{\Upsilon_2} J_{k+1}\{f\}(\tau_1,3,\tau_2,\cdots,\tau_k)\\
\le\; & \cdots\\
\le\; & \frac{1}{C_k^k}\sum_{\Upsilon_k} J_{k+1}\{f\}(\tau_1,2,\tau_2,\cdots,\tau_k)+\frac{1}{C_k^k}\sum_{\Upsilon_k} J_{k+1}\{f\}(\tau_1,3,\tau_2,\cdots,\tau_k),
\end{aligned}$$

$$\cdots$$

$$
\begin{aligned}
&K_0^{k+1}\{f\} \\
\leq\ & \frac{1}{C_k^1}\sum_{\Upsilon_1} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,2,\tau_k) + \frac{1}{C_k^1}\sum_{\Upsilon_1} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,3,\tau_k) \\
\leq\ & \frac{1}{C_k^2}\sum_{\Upsilon_2} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,2,\tau_k) + \frac{1}{C_k^2}\sum_{\Upsilon_2} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,3,\tau_k) \\
\leq\ & \cdots \\
\leq\ & \frac{1}{C_k^k}\sum_{\Upsilon_k} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,2,\tau_k) + \frac{1}{C_k^k}\sum_{\Upsilon_k} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,3,\tau_k).
\end{aligned}
$$

Add them all up, we can arrive at the desired inequality (3.45). In fact, that's accords to the facts that

$$
\begin{aligned}
& \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(2,\tau_1,\tau_2,\cdots,\tau_k) + \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,2,\tau_2,\cdots,\tau_k) \\
+\ & \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,\tau_2,2,\cdots,\tau_k) + \cdots + \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,2) \\
+\ & \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(3,\tau_1,\tau_2,\cdots,\tau_k) + \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,3,\tau_2,\cdots,\tau_k) \\
+\ & \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,\tau_2,3,\cdots,\tau_k) + \cdots + \frac{1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,3) \\
=\ & \frac{1}{C_k^i}\sum_{\Upsilon_i} \Big(J_{k+1}\{f\}(2,\tau_1,\tau_2,\cdots,\tau_k) + J_{k+1}\{f\}(\tau_1,2,\tau_2,\cdots,\tau_k) \\
& \qquad + J_{k+1}\{f\}(\tau_1,\tau_2,2,\cdots,\tau_k) + \cdots + J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,2) \Big) \\
+\ & \frac{1}{C_k^i}\sum_{\Upsilon_i} \Big(J_{k+1}\{f\}(3,\tau_1,\tau_2,\cdots,\tau_k) + J_{k+1}\{f\}(\tau_1,3,\tau_2,\cdots,\tau_k) \\
& \qquad + J_{k+1}\{f\}(\tau_1,\tau_2,3,\cdots,\tau_k) + \cdots + J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,3) \Big) \\
=\ & \frac{i+1}{C_k^i}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,\tau_{k+1}) \\
=\ & \frac{1}{k+1}\frac{1}{C_{k+1}^{i+1}}\sum_{\Upsilon_i} J_{k+1}\{f\}(\tau_1,\tau_2,\cdots,\tau_k,\tau_{k+1}) \\
=\ & \frac{1}{k+1} K_{i+1}^{k+1}.
\end{aligned}
$$

□

Clearly, some corollaries for Δ and ∇ integral can be established if we set all $\alpha_i (i = 1, 2, \cdots, n) = 1$ or 0, respectively.

Remark 3.3. Undoubtedly, we can use the method in Theorem 3.1 to prove Theorem 3.2. In fact, we provided two methods to prove Hermite-Hadamard type inequality for multiple Diamond-α_i integral.

Remark 3.4. Although Theorem 1.5 doesn't give a complete inequality of the case of $n = 2$, we can obtain the case of $n = 2$ together with Jensen's inequality and Theorem 3.9 in [22] easily.

4. Conclusions

In this paper, we generalize Jensen, Hardy, and Hermite-Hadamard type inequalities to multiple diamond-alpha integral on time scales. Moreover, we present Hardy type inequality with a weighted function and Hermite-Hadamard type inequality with three variables.

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