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# Numerical Solution of Conformable Time Fractional Generalized Burgers Equation with Proportional Delay by New Methods

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**Abstract**

By using two new methods, called the conformable fractional q-homotopy analysis transform method and the conformable Shehu homotopy perturbation method, the conformable time-fractional partial differential equations with proportional delay is analysed. The graphs of this equation's numerical solutions are drawn. According to numerical simulations, the proposed methods are effective and reliable.

**Keywords:** Conformable time fractional generalized Burgers equation, conformable q-homotopy analysis transform method, conformable Shehu homotopy perturbation method, proportional delay.

# Oransal Gecikmeli Uyumlu Zaman Kesirli Genelleştirilmiş Burgers Denkleminin Yeni Yöntemlerle Sayısal Çözümü

**Öz**

Uyumlu kesirli q-homotopi analiz dönüşümü yöntemi ve uyumlu Shehu homotopi pertürbasyon yöntemi olarak adlandırılan iki yeni yöntem kullanılarak, orantılı gecikmeli uyumlu zaman-fraksiyonel kısmi diferansiyel denklemler analiz edilmiştir. Bu denklemin sayısal çözümlerinin grafikleri çizilir. Sayısal simülasyonlara göre önerilen yöntemler etkili ve güvenilirdir.

**Anahtar Kelimeler:** Uyumlu zaman kesirli genelleştirilmiş Burgers denklemi, uyumlu q-homotopi analiz dönüşüm metodu, uyumlu Shehu homotopi pertürbasyon metodu, oransal gecikme.

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## 1. Introduction

Fractional calculus (FC) is a generalization of integer order calculus to arbitrary order. It was reviewed in an early letter in 1695 between the great mathematicians Leibniz and L' Hospital. Many authors have begun to examine fractional calculus in recent years because of its capacity to provide an exact description for various types of non-linear phenomena. Fractional order differential equations are a generalization of classical differential equations with non-local and genetic material property consequences. Many senior academics examined and defined the concept of fractional calculus, and they defined revolutionary definitions of fractional calculus that laid the foundations for fractional calculus (Liouville, 1832; Riemann, 1896; Caputo, 1969; Miller and Ross, 1993; Podlubny, 1999; Baleanu et al., 2012; Povstenko, 2015, Ala, 2022; Ala and Shaikhova, 2022). Nowadays, fractional partial differential equations are widely used in the development of nonlinear models and the investigation of dynamical systems. The theory of fractional-order calculus has been associated to practical projects and has been used to evaluate and study many phenomena such as chaos theory (Baleanu et al., 2017), financial models (Sweilam et al., 2017), a noisy environment (Liu et al., 2015), optics (Esen et al., 2018), and others (Veeresha et al., 2019a; Caponetto et al., 2010; Prakash et al., 2019; Veeresha et al., 2019b). The solutions of fractional differential equations are critical in characterizing the characteristics of nonlinear problems that occur in nature. Because exact solutions to fractional differential equations representing non-linear phenomena are difficult to acquire, we employ a variety of analytical and numerical techniques.

The conformable fractional derivative is a simple and valuable tool. It also allows us to understand how to characterize the behavior of physical phenomena. When it comes to solving complicated problems, the conformable fractional derivative is an incredibly useful tool. This derivative was discovered to be more strong and attractive than those that came before it. Because differential equations with conformable fractional derivatives are easier to solve numerically than those with Riemann-Liouville or Caputo fractional derivatives, the conformable fractional derivative is very useful when modeling many physical problems. Different fractional-order models are used in engineering and applied sciences because they provide a more precise explanation of real-world scenarios. Numerous academics have already applied conformable fractional derivatives in a range of fields (Gao and Chi, 2020). The conformable fractional operator overcomes some of the limitations of the existing fractional operators and provides traditional calculus with properties such as the mean value theorem, the chain rule, the product of two functions, the derivative of the quotient of two functions, and Rolle's theorem (Kilbas et al., 2006; Debnath, 2003; Khalil et al., 2014; Abdeljawad, 2015).

This investigation deals with the numerical solution of conformable time-fractional partial differential equations with proportional delay defined by

$$\begin{cases} D_t^{\alpha} w(x,t) = \psi\left(x, w(\rho_0 x, \sigma_0 t), \frac{\partial w(\rho_1 x, \sigma_1 t)}{\partial x}, \dots, \frac{\partial^m w(\rho_m x, \sigma_m t)}{\partial x^m}\right), \\ w^{(k)}(x,0) = \varphi_k(x). \end{cases} \quad (1)$$

where $\rho_i, \sigma_i \in (0,1)$ for all $i \in N$, $\varphi_k$ is initial value, $\psi$ differential operator and $D_t^{\alpha}$ is conformable time-fractional operator.

The partial functional differential equations with proportional delays are a type of delay partial differential equation that come up in biology, medicine, population ecology, control systems, climate models, and complex economic macrodynamics (Wu, 1996; Keller, 2010).

The solution to (1) could be the voltage, temperature, densities of various particles, such as chemicals, cells, and so on. One notable application of the model, the Korteweg-de Vries (KdV) equation, in the study of shallow water waves is as follows:

$$D_t^{\alpha} w(x,t) + a w w_x(\rho_0 x, \sigma_0 t) + w_{xxx}(\rho_1 x, \sigma_1 t) = 0, 0 < \alpha \leq 1, \quad (2)$$

where $a$ is a constant. Another well-known model, the time fractional nonlinear Klein-Gordon equation with proportional delay, is used to describe nonlinear wave interaction in quantum field theory:

$$D_t^{\alpha} w(x,t) = w_{xx}(\rho_0 x, \sigma_0 t) - b w(\rho_1 x, \sigma_1 t) - F\big(w(\rho_2 x, \sigma_2 t)\big) + k(x,t), 1 < \alpha \leq 2, \quad (3)$$

where $b$ is a constant, $k(x,t)$ is a source term and $F(w(\rho_2 x, \sigma_2 t))$ is a nonlinear function of $w(x,t)$.

To the best of my knowledge, there is minimal literature on methods for solving TFPDE with delay, such as the Chebyshev pseudospectral method (Zubik-Kawal, 2000), homotopy analysis method (Alkan, 2022), spectral collocation and waveform relaxation methods (Zubik-Kawal and Jackiewicz, 2006), and iterated pseudospectral method (Mead and Zubik-Kawal, 2005) for nonlinear delay partial differential equations. Using RDTM, Abazari and Ganji (2011) found approximate solutions to PDEs with proportional delay. Abazari and Kilicman (2014) used DTM to obtain analytical solutions to nonlinear integro-differential equations with proportional delay. Tanthanuch (2012) used a group analysis method to solve the non-homogeneous inviscid Burgers equation with proportional delay. Sakar et al. (2016) and Biazar ad Ghanbari (2012) used the homotopy perturbation method to find analytical solutions to TFPDE with proportional delay. Chen and Wang (2010) solved a neutral functional-differential equation with proportional delays using the variational iteration method (VIM). Singh and Kumar (2017) used an additional variational iteration method (AVIM) to achieve an alternative approximate solution of the initial valued autonomous system of TFPDE with proportional delay. The primary reason for writing this paper is to propose two new methods: conformable q-homotopy analysis transform method (Cq-HATM) and conformable Shehu homotopy perturbation method (CSHPM). The second motivation is to use these novel methods to get numerical solutions for the conformable time-fractional partial differential equations with proportional delay.

The rest of the study is listed below. Section 2 presents basic concepts of conformable fractional calculus as well as Shehu transform (Maitama and Zhao, 2019). Section 3 introduces the conformable q-homotopy analysis transform method and conformable Shehu homotopy perturbation method. Section 4 shows an application for conformable time fractional partial differential equations with proportional delay. Section 5 presents the conclusion.

## 2. Preliminaries

We remember what conformable fractional calculus and the Shehu transform represent and how they should be used in this article.

**Definition 2.1 (Khalil et al., 2014; Abdeljawad, 2015; Gözütok and Gözütok, 2017)** Given a function $f:[0,\infty)\to\mathbb{R}$. Then, the conformable fractional derivative of $f$ order $\alpha$ is described by

$$T_\alpha(f)(x)=\lim_{\varepsilon\to 0}\frac{f\left(x+\varepsilon x^{1-\alpha}\right)-f(x)}{\varepsilon}, \tag{4}$$

for all $x>0, \alpha\in(0,1]$.

**Theorem 2.1 (Khalil et al., 2014; Abdeljawad, 2015; Gözütok and Gözütok, 2017).** Let $\alpha\in(0,1]$ and $f,g$ be $\alpha-$differentiable at a point $x>0$. Then

**(i)** $T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$, for all $a,b\in\mathbb{R}$, (5)

**(ii)** $T_\alpha(x^p)=px^{p-1}$, for all $p\in\mathbb{R}$, (6)

**(iii)** $T_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$, (7)

**(iv)** $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$, (8)

**(v)** $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$. (9)

If $f$ is differentiable, then the derivative of the polynomial $t$ is obtained as

$$T_\alpha(f)(t) = t^{1-\alpha}\frac{d}{dt}f(t). \tag{10}$$

**Definition 2.2 (Khalil et al., 2014).** Let $f$ be an $n-$times differentiable at $x$. Then, the conformable fractional derivative of $f$ order $\alpha$ is defined by:

$$T_\alpha(f)(x) = \lim_{\varepsilon\to 0}\frac{f^{([\alpha]-1)}\left(x+\varepsilon x^{([\alpha]-\alpha)}\right) - f^{([\alpha]-1)}(x)}{\varepsilon}, \tag{11}$$

for all $x > 0, \alpha \in (n, n+1]$, $[\alpha]$ is the smallest integer greater than or equal to $\alpha$.

**Theorem 2.2 (Khalil et al., 2014).** Let $f$ be an $n-$times differentiable at $x$. Then, it is obtained as

$$T_\alpha(f(x)) = x^{[\alpha]-\alpha}f^{[\alpha]}(x), \tag{12}$$

for all $x > 0, \alpha \in (n, n+1]$.

**Definition 2.3 (Mittag-Leffler, 1903).**

The Mittag-Leffler function $E_a$ is given as follows:

$$E_a(z) = \sum_{n=0}^{\infty}\frac{z^a}{\Gamma(na+1)}. \tag{13}$$

**Definition 2.4 (Benattia and Belghaba, 2021).** Let $0 < \alpha \leq 1$, $f: [0,\infty) \to \mathbb{R}$ be real valued function. Then, the conformable fractional Shehu transform (CFST) of order $\alpha$ of $f$ is defined by

$$_cS_\alpha[f(t)] = V_\alpha(s;u) = \int_0^\infty exp\left(\frac{-st^\alpha}{u\alpha}\right)f(t)t^{\alpha-1}dt. \tag{14}$$

**Definition 2.5 (Benattia and Belghaba, 2021).** Let $0 < \alpha \leq 1$, $f: [0,\infty) \to \mathbb{R}$ be real valued function. The conformable Shehu transform for the conformable fractional-order derivative of the function $f(t)$ is described by

$$V_\alpha[T_\alpha f(t)](v) = \frac{s}{u} V_\alpha(s; u) - f(0). \quad (15)$$

## 3. The New Numerical Methods

In this section, we discuss concerning Cq-HATM and CSHPM in a few statements.

### Case (i) Conformable q-Homotopy Analysis Transform Method

Now, to explain the main idea behind CFEDM, we consider the conformable fractional order nonlinear partial differential equation with proportional delay:

$${}_tT_\alpha w(x,t) + Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) = f(x,t), t > 0, n-1 < \alpha \leq n, \quad (16)$$

where $M$ is a linear operator, $N$ is a nonlinear operator, $f(x,t)$ is a source term, $\rho_i, \sigma_i \in (0,1)$ and ${}_tT_\alpha$ is a conformable fractional derivative of order $\alpha$.

Applying the conformable Laplace transform to Eq. (16) and utilizing the initial condition, then Eq. (18) get

$$s\mathcal{L}_\alpha[w(x,t)] - w(x,0) + \mathcal{L}_\alpha[Mw(\rho_i x, \sigma_i t)] + \mathcal{L}_\alpha[Nw(\rho_i x, \sigma_i t)] = \mathcal{L}_\alpha[f(x,t)]. \quad (17)$$

By simplifying the Eq. (17), we get

$$\mathcal{L}_\alpha[w(x,t)] - \frac{1}{s} w(x,0) + \frac{1}{s}\mathcal{L}_\alpha[Mw(\rho_i x, \sigma_i t)] + \frac{1}{s}\mathcal{L}_\alpha[Nw(\rho_i x, \sigma_i t)] - \frac{1}{s}\mathcal{L}_\alpha[f(x,t)] = 0.$$
(18)

With the help of HAM, we can describe the nonlinear operator for real function $\varphi(x,t;q)$ as follows:

$$N[\varphi(x,t;q)] = \mathcal{L}_\alpha[\varphi(x,t;q)] - \frac{1}{s}\varphi(x,t;q)\,(0^+) + \frac{1}{s}(\mathcal{L}_\alpha[M\varphi(\rho_i x, \sigma_i t; q)]$$

$$+\mathcal{L}_\alpha[N\varphi(\rho_i x,\sigma_i t;q)]-\mathcal{L}_\alpha[f(x,t)]), \tag{19}$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

We construct a homotopy as follows:

$$(1-nq)\mathcal{L}_\alpha[\varphi(x,t;q)-w_0(x,t)]=hqH(x,t)N[\varphi(\rho_i x,\sigma_i t;q)], \tag{20}$$

where, $h\neq 0$ is an auxiliary parameter and $\mathcal{L}_\alpha$ represents conformable Laplace transform. For $q=0$ and $q=\frac{1}{n}$, the results of Eq. (20) are as follows:

$$\varphi(x,t;0)=w_0(x,t),\varphi\left(x,t;\frac{1}{n}\right)=w(x,t), \tag{21}$$

Thus, by amplifying $q$ from 0 to $\frac{1}{n}$, then the solution $\varphi(x,t;q)$ converges from $w_0(x,t)$ to the solution $w(x,t)$. Using the Taylor theorem around $q$ and then expanding $\varphi(x,t;q)$, we get

$$\varphi(x,t;q)=w_0(x,t)+\sum_{i=1}^{\infty}w_m(x,t)q^m, \tag{22}$$

where

$$w_m(x,t)=\frac{1}{m!}\frac{\partial^m\varphi(x,t;q)}{\partial q^m}|_{q=0}. \tag{23}$$

Eq. (22) converges at $q=\frac{1}{n}$ for the appropriate $w_0(x,t)$, $n$ and $h$. Then, we get one of the solutions of the original nonlinear equation of the form

$$w(x,t)=w_0(x,t)+\sum_{m=1}^{\infty}w_m(x,t)\left(\frac{1}{n}\right)^m. \tag{24}$$

If we differentiate the zeroth order deformation Eq. (20) $m-$times with respect to $q$ and we divide by $m!$, respectively, then for $q=0$, we acquire

$$\mathcal{L}_\alpha[w_m(x,t)-k_m w_{m-1}(x,t)]=hH(x,t)\mathcal{R}_m(\vec{w}_{m-1}), \tag{25}$$

where the vectors are described by

$$\vec{w}_m = \{w_0(x,t), w_1(x,t), \dots, w_m(x,t)\}. \quad (26)$$

When Eq. (26) is applied to the inverse conformable Laplace transform, we get

$$w_m(x,t) = k_m w_{m-1}(x,t) + h\mathcal{L}_\alpha{}^{-1}[H(x,t)\mathcal{R}_m(\vec{w}_{m-1})], \quad (27)$$

where

$$\mathcal{R}_m(\vec{w}_{m-1}) = \mathcal{L}_\alpha[w_{m-1}(x,t)] - \left(1-\frac{k_m}{n}\right)\frac{1}{s}w_0(x,t) + \frac{1}{s}\mathcal{L}_\alpha[Mw_{m-1}(\rho_i x, \sigma_i t)$$

$$+H_{m-1}(x,t) - f(x,t)], \quad (28)$$

and

$$k_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \quad (29)$$

Here, $H_m$ is homotopy polynomial and presented as

$$H_m = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}|_{q=0} \text{ and } \varphi(x,t;q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \cdots. \quad (30)$$

Using Eqs. (27) - (28), we get

$$w_m(x,t) = (k_m + h)w_{m-1}(x,t) - \left(1-\frac{k_m}{n}\right)\frac{1}{s}w_0(x,t) + \mathrm{h}\mathcal{L}_\alpha{}^{-1}\left[\left(\frac{1}{s}\mathcal{L}_\alpha[Mw_{m-1}(\rho_i x, \sigma_i t)\right.\right.$$

$$\left.\left.+H_{m-1}(x,t) - f(x,t)]\right)\right]. \quad (31)$$

When q-HATM is used, the series solution is given by

$$w(x,t) = \sum_{i=0}^{\infty} w_m(x,t). \quad (32)$$

**Case (ii) Conformable Shehu Homotopy Perturbation Method (CSHPM)**

Now, to explain the main idea inside CSHPM, we consider the conformable time-fractional order nonlinear partial differential equation with proportional delay:

$$ {}_tT_\alpha w(x,t) + Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) = f(x,t), t > 0, n-1 < \alpha \leq n, \quad (33)$$

with initial condition

$$w(x,0) = a(x), \quad (34)$$

where $M$ is a linear operator, $N$ is a nonlinear operator, $f(x,t)$ is a source term, $\rho_i, \sigma_i \in (0,1)$ and ${}_tT_\alpha$ is a conformable fractional derivative of order $\alpha$.

Applying the conformable fractional Shehu transform to Eq. (33) and using the initial condition, then we get

$$\frac{s}{u}\ {}_cS_\alpha[w(x,t)] - \sum_{m=0}^{k-1} w(x,0) + {}_cS_\alpha[Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) - f(x,t)] = 0. \quad (35)$$

Eq. (35) is simplified, then we have

$${}_cS_\alpha[w(x,t)] - \frac{u}{s} a(x) + \frac{u}{s}\ {}_cS_\alpha[Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) - f(x,t)] = 0. \quad (36)$$

When Eq. (36) is rearranged, it is obtained as

$${}_cS_\alpha[w(x,t)] = \frac{u}{s} a(x) - \frac{u}{s}\ {}_cS_\alpha[Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) - f(x,t)]. \quad (37)$$

When the inverse conformable fractional Shehu transform is implemented to both sides of Eq. (37), we have

$$w(x,t) = A(x,t) - \left({}_cS_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_cS_\alpha[Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t)]\right\}, \quad (38)$$

where the term $A(x,t)$ emerges from the in-homogeneous term and initial conditions.

Applying the homotopy perturbation method yields

$$w(x,t) = \sum_{n=0}^{\infty} p^n w_n(x,t). \tag{39}$$

Now, let the nonlinear term be represented as

$$Nw(x,t) = \sum_{n=0}^{\infty} p^n H_n(w), \tag{40}$$

where $H_n(w)$ is defined by the form

$$H_n(w_0, w_1, \dots, w_n) = \frac{1}{n!}\frac{\partial}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i w_i\right)\right]_{p=0}, n = 0,1,2, \dots \tag{41}$$

Substituting the Eqs. (39)-(40) into Eq. (38) , it is obtained as

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = A(x,t) - p\left\{\left({}_{c}S_{\alpha}\right)^{-1}\left[\frac{u}{s}\ {}_{c}S_{\alpha}\{M\sum_{n=0}^{\infty} p^n w_n(\rho_i x, \sigma_i t) + \sum_{n=0}^{\infty} p^n H_n(w)\}\right]\right\}. \tag{42}$$

Eq. (42) is the combination of the conformable fractional Shehu transform and the homotopy perturbation method. The coefficients of the same power terms of $p$ is compared, then we have the following iterations.

$$p^0: w_0(x,t) = A(x,t), \tag{43}$$

$$p^1: w_1(x,t) = -\left({}_{c}S_{\alpha}\right)^{-1}\left\{\frac{u}{s}\ {}_{c}S_{\alpha}[Mw_0(\rho_i x, \sigma_i t) + H_0(w)]\right\}, \tag{44}$$

$$p^2: w_2(x,t) = -\left({}_{c}S_{\alpha}\right)^{-1}\left\{\frac{u}{s}\ {}_{c}S_{\alpha}[Mw_1(\rho_i x, \sigma_i t) + H_1(w)]\right\}, \tag{45}$$

$$p^3: w_3(x,t) = -\left({}_{c}S_{\alpha}\right)^{-1}\left\{\frac{u}{s}\ {}_{c}S_{\alpha}[Mw_2(\rho_i x, \sigma_i t) + H_2(w)]\right\}, \tag{46}$$

$\vdots$

Thus, the series solution of the equation is obtained in the form

$$w(x,t) = \lim_{p\to 1}\sum_{m=0}^{\infty} p^m w_m(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots \tag{47}$$

## 4. Applications

### Example 4.1

Consider the conformable time-fractional generalized Burgers equation (CTFGBE) with proportional delay (Sakar et al., 2016)

$$\frac{\partial^{\alpha} w(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial w\left(x,\frac{t}{2}\right)}{\partial x} w\left(\frac{x}{2},\frac{t}{2}\right) + \frac{1}{2} w(x,t), \tag{48}$$

where $x, t \in [0,1], 0 < \alpha \leq 1$,
subject to initial condition

$$w(x,0) = x. \tag{49}$$

**Case (i) Cq-HATM solution for CTFGBE**

Applying the conformable Laplace transform to Eq (48) and using Eq. (49), then we have

$$\mathcal{L}_{\alpha}[w(x,t)] = \frac{1}{s} w(x,0) + \frac{1}{s}\mathcal{L}_{\alpha}\left[\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial w\left(x,\frac{t}{2}\right)}{\partial x} w\left(\frac{x}{2},\frac{t}{2}\right) + \frac{1}{2} w(x,t)\right], \tag{50}$$

We define the nonlinear operators by using Eq. (50), as

$$N[\varphi(x,t;q)\,] = \mathcal{L}_{\alpha}[\varphi(x,t;q)\,] - \frac{1}{s}x - \frac{1}{s}\mathcal{L}_{\alpha}\left[\frac{\partial^2 \varphi(x,t)}{\partial x^2} + \frac{\partial \varphi\left(x,\frac{t}{2}\right)}{\partial x} \varphi\left(\frac{x}{2},\frac{t}{2}\right) + \frac{1}{2}\varphi(x,t)\right]. \tag{51}$$

By applying the proposed algorithm, the $m-th$ order deformation equations are defined by

$$\mathcal{L}_{\alpha}[w_m(x,t) - k_m w_{m-1}(x,t)] = h\mathcal{R}_m[\vec{w}_{m-1}], \tag{52}$$

where

$$\mathcal{R}_m[\vec{w}_{m-1}] = \mathcal{L}_{\alpha}[\vec{w}_{m-1}(x,t)] - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}x - \frac{1}{s}\mathcal{L}_{\alpha}\left[\frac{\partial^2 w_{m-1}(x,t)}{\partial x^2} + \sum_{r=0}^{m-1}\frac{\partial w_r\left(x,\frac{t}{2}\right)}{\partial x}\right.$$

$$\left.\times\, w_{m-1-r}\left(\frac{x}{2},\frac{t}{2}\right) + \frac{1}{2} w_{m-1}(x,t)\right]. \tag{53}$$

On applying inverse conformable Laplace transform to Eq. (52), then we have

$$w_m(x,t) = k_m w_{m-1}(x,t) + h\mathcal{L}_{\alpha}^{-1}\{\mathcal{R}_m[\vec{w}_{m-1}]\}. \tag{54}$$

By the use of initial condition, then we get

$$w_0(x,t) = x. \tag{55}$$

To find the value of $w_1(x,t)$, putting $m = 1$ in Eq. (54), then we obtain

$$w_1(x,t) = -hx\frac{t^\alpha}{\alpha}. \tag{56}$$

In the same way, if we put $m = 2$ in Eq. (54), we can find the value of $w_2(x,t)$

$$w_2(x,t) = -(n+h)\left(hx\frac{t^\alpha}{\alpha}\right) + xh^2\left(\frac{2^{\alpha-1}+1}{2^\alpha}\right)\frac{t^{2\alpha}}{2\alpha^2}. \tag{57}$$

Analogously, to find the value of $w_3(x,t)$, putting $m = 3$ in Eq. (54), then we get

$$w_3(x,t) = -\frac{xh}{24\alpha^3}\Big[-18\left(-\alpha 2^{-\alpha} + \frac{2\alpha}{3} + \frac{1}{6}\right)(n+h)ht^{2\alpha} + +(2^{1-\alpha} + 64^{-\alpha} + 48^{-\alpha} + 1)$$

$$\times h^2 t^{3\alpha} + 24t^\alpha\alpha^2(n+h)^2\Big]. \tag{58}$$

In this way, the other terms can be found. So, the Cq-HATM solution of Eq. (48) is given by

$$w(x,t) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t)\left(\frac{1}{n}\right)^m. \tag{59}$$

If we put $\alpha = 1, n = 1,\ h = -1$ in Eq. (59), then the obtained results $\sum_{m=1}^{M} w_m(x,t)\left(\frac{1}{n}\right)^m$ converges to the exact solution $w(x,t) = xe^t$ of CTFGBE when $M \to \infty$.

**Case (ii) CSHPM solution for CTFGBE with proportional delay**

Applying the conformable Shehu transform to Eq (48) and using Eq. (49), then we get

$${}_cS_\alpha[w(x,t)] = x\frac{u}{s} + \frac{u}{s}\ {}_cS_\alpha\left[\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial w\left(x,\frac{t}{2}\right)}{\partial x}w\left(\frac{x}{2},\frac{t}{2}\right) + \frac{1}{2}w(x,t)\right]. \tag{60}$$

Applying the inverse conformable Shehu transform to Eq (60), then we obtain

$$w(x,t) = x + \left( {}_{c}S_{\alpha} \right)^{-1} \left\{ \frac{u}{s} \; {}_{c}S_{\alpha} \left[ \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial w\left(x, \frac{t}{2}\right)}{\partial x} w\left(\frac{x}{2}, \frac{t}{2}\right) + \frac{1}{2} w(x,t) \right] \right\}. \quad (61)$$

Now HPM is applied, then we have

$$\sum_{m=0}^{\infty} p^m w_m(x,t) = x + p \left[ \left( {}_{c}S_{\alpha} \right)^{-1} \left\{ \frac{u}{s} \; {}_{c}S_{\alpha} \left[ \frac{\partial^2}{\partial x^2} \left( \sum_{m=0}^{\infty} p^m w_m(x,t) \right) + \sum_{m=0}^{\infty} p^m H_m(w) \right. \right. \right.$$

$$\left. \left. \left. + \frac{1}{2} \left( \sum_{m=0}^{\infty} p^m w_m(x,t) \right) \right] \right\} \right]. \quad (62)$$

We get to the first few components of $H_m(w)$ by

$$H_0(w) = \frac{\partial w_0\left(x, \frac{t}{2}\right)}{\partial x} w_0\left(\frac{x}{2}, \frac{t}{2}\right), \quad (63)$$

$$H_1(w) = \frac{\partial w_0\left(x, \frac{t}{2}\right)}{\partial x} w_1\left(\frac{x}{2}, \frac{t}{2}\right) + \frac{\partial w_1\left(x, \frac{t}{2}\right)}{\partial x} w_0\left(\frac{x}{2}, \frac{t}{2}\right), \quad (64)$$

$$H_2(w) = \frac{\partial w_0\left(x, \frac{t}{2}\right)}{\partial x} w_2\left(\frac{x}{2}, \frac{t}{2}\right) + \frac{\partial w_1\left(x, \frac{t}{2}\right)}{\partial x} w_1\left(\frac{x}{2}, \frac{t}{2}\right) + \frac{\partial w_2\left(x, \frac{t}{2}\right)}{\partial x} w_0\left(\frac{x}{2}, \frac{t}{2}\right), \quad (65)$$

When we compare the coefficients of the same powers of $p$, we get

$$p^0: w_0(x,t) = x, H_0(w) = \frac{x}{2}, \quad (66)$$

$$p^1: w_1(x,t) = x\frac{t^{\alpha}}{\alpha}, H_1(w) = \frac{x t^{\alpha}}{2\alpha 2^{2\alpha}}, \quad (67)$$

$$p^2: w_2(x,t) = \frac{x(2+2^{\alpha})t^{2\alpha}}{2^{\alpha+2}\alpha^2}, H_2(w) = \frac{x(2+32^{\alpha})t^{2\alpha}}{2^{3\alpha+2}\alpha^2}, \quad (68)$$

$$p^3: w_3(x,t) = \frac{x\left(2^2+3.2^{\alpha+1}+2^{2\alpha+1}+2^{3\alpha}\right)t^{3\alpha}}{3.2^{3\alpha+3}\alpha^3}. \quad (69)$$

Consequently, the solution to Eq. (48) for CSHPM is given by

$$w(x,t) = x\left(1 + \frac{t^{\alpha}}{\alpha} + \frac{(2+2^{\alpha})t^{2\alpha}}{2^{\alpha+2}\alpha^2} + \frac{\left(2^2+3.2^{\alpha+1}+2^{2\alpha+1}+2^{3\alpha}\right)t^{3\alpha}}{3.2^{3\alpha+3}\alpha^3}\right). \quad (70)$$

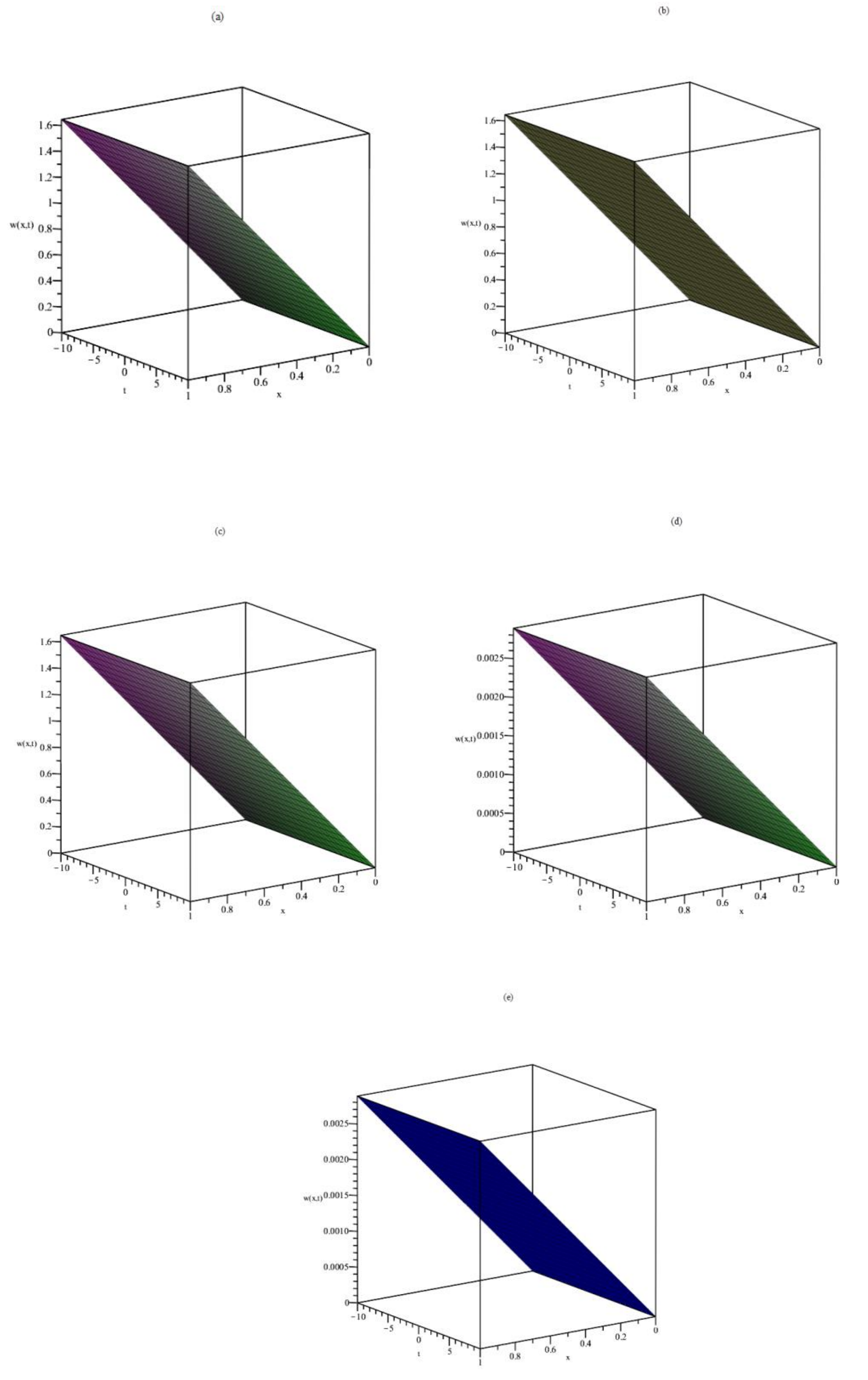



**Figure 1. (a)** Nature of Cq-HATM solution **(b)** Nature of CSHPM solution **(c)** Nature of exact solution **(d)** Nature of absolute error=$|w_{exact} - w_{Cq-HATM}|$ **(e)** Nature of absolute error=$|u_{exact} - u_{CSHPM}|$ for Ex. 4.1 at $\mathrm{h} = -1, \mathrm{n} = 1, \alpha = 1$.

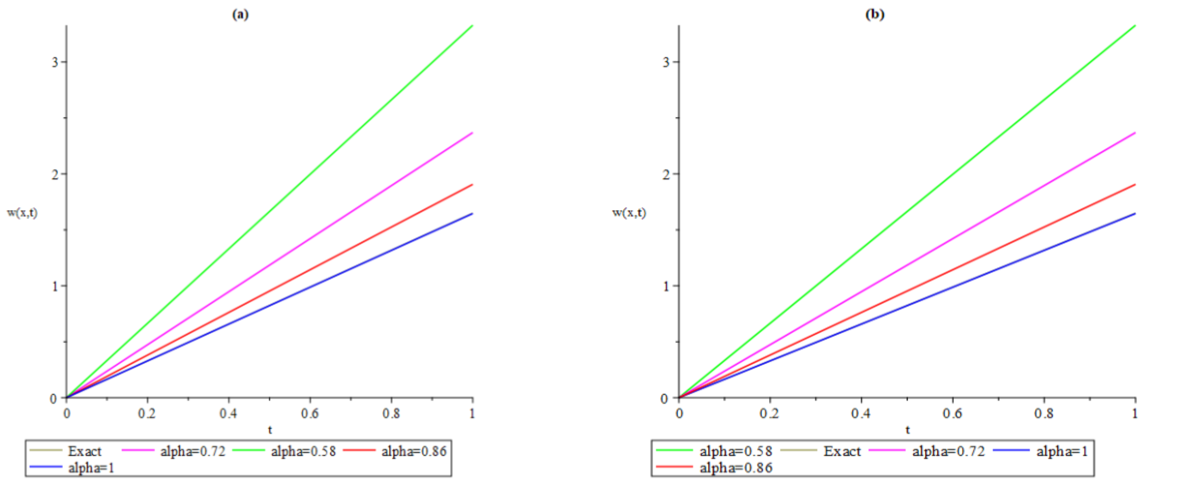



**Figure 2.** **(a)** The comparison of the Cq-HATM solutions and exact solution **(b)** The comparison of the CSHPM solutions and exact solution for Ex. 4.1 at $h=-1, n=1, x=0.5$ with distinct $\alpha$.

**Table 1.** Numerical solution for CTFGBE equation by Cq-HATM in Ex. 4.1 at $h=-1, n=1$ with distinct values of $x$ and $t$ for diverse $\alpha$.

| $x$ | $t$ | $\alpha = 0.58$ | $\alpha = 0.72$ | $\alpha = 0.86$ | $\alpha = 1$ |
|---|---|---|---|---|---|
| 0.25 | 0.25 | 0.5596460926 | 0.4216721924 | 0.3567107699 | 0.3209635416 |
| | 0.50 | 0.8321510022 | 0.5924122881 | 0.4766549579 | 0.4114583334 |
| | 0.75 | 1.1272272560 | 0.7884373466 | 0.6210788866 | 0.5253906249 |
| | 1.00 | 1.4474996720 | 1.0126616460 | 0.7935251184 | 0.6666666666 |
| 0.50 | 0.25 | 1.1192921850 | 0.8433443848 | 0.7134215400 | 0.6419270833 |
| | 0.50 | 1.6643020030 | 1.1848245760 | 0.9533099160 | 0.8229166666 |
| | 0.75 | 2.2544545150 | 1.5768746930 | 1.2421577730 | 1.0507812500 |
| | 1.00 | 2.8949993420 | 2.0253232910 | 1.5870502380 | 1.3333333340 |
| 0.75 | 0.25 | 1.6789382770 | 1.2650165760 | 1.0701323090 | 0.9628906249 |
| | 0.50 | 2.4964530070 | 1.7772368640 | 1.4299648740 | 1.2343750000 |
| | 0.75 | 3.3816817710 | 2.3653120400 | 1.8632366590 | 1.5761718760 |
| | 1.00 | 4.3424990130 | 3.0379849390 | 2.3805753540 | 2.0000000000 |

**Table 2.** Numerical solution for CTFGBE equation by CSHPM in Ex. 4.1 with distinct values of $x$ and $t$ for diverse $\alpha$.

| $x$ | $t$ | $\alpha = 0.58$ | $\alpha = 0.72$ | $\alpha = 0.86$ | $\alpha = 1$ |
|---|---|---|---|---|---|
| 0.25 | 0.25 | 0.5596460925 | 0.4216721923 | 0.3567107699 | 0.3209635417 |
| | 0.50 | 0.8321510022 | 0.5924122881 | 0.4766549580 | 0.4114583333 |
| | 0.75 | 1.1272272570 | 0.7884373465 | 0.6210788866 | 0.5253906250 |
| | 1.00 | 1.4474996710 | 1.0126616460 | 0.7935251183 | 0.6666666667 |
| 0.50 | 0.25 | 1.1192921850 | 0.8433443848 | 0.7134215399 | 0.6419270833 |
| | 0.50 | 1.6643020040 | 1.1848245760 | 0.9533099160 | 0.8229166667 |
| | 0.75 | 2.2544545150 | 1.5768746930 | 1.242157773 | 1.0507812500 |

| | | | | | |
|---|---|---|---|---|---|
| | 1.00 | 2.8949993430 | 2.0253232920 | 1.587050236 | 1.3333333330 |
| 0.75 | 0.25 | 1.6789382780 | 1.2650165770 | 1.070132310 | 0.9628906250 |
| | 0.50 | 2.4964530060 | 1.7772368640 | 1.429964874 | 1.2343750000 |
| | 0.75 | 3.3816817720 | 2.3653120400 | 1.863236659 | 1.5761718750 |
| | 1.00 | 4.3424990130 | 3.0379849390 | 2.3805753550 | 2.0000000000 |

**Table 3.** Comparison of Cq-HATM solution with HPM (Sakar et al., 2016) and the exact solution, when $\alpha = 1$.

| $x$ | $t$ | Exact solution | Cq-HATM | HPM (Sakar et al., 2016) |
|---|---|---|---|---|
| 0.25 | 0.25 | 0.3210063542 | 0.3209635416 | 0.3209635415 |
| 0.25 | 0.50 | 0.4121803178 | 0.4114583334 | 0.4114583335 |
| 0.25 | 0.75 | 0.5292500042 | 0.5253906249 | 0.5253906250 |
| 0.25 | 1.00 | 0.6795704570 | 0.6666666666 | 0.6666666665 |
| 0.50 | 0.25 | 0.6420127085 | 0.6419270833 | 0.6419270830 |
| 0.50 | 0.50 | 0.8243606355 | 0.8229166666 | 0.8229166670 |
| 0.50 | 0.75 | 1.0585000080 | 1.0507812500 | 1.0507812500 |
| 0.50 | 1.00 | 1.3591409140 | 1.3333333340 | 1.3333333330 |
| 0.75 | 0.25 | 0.9630190628 | 0.9628906249 | 0.9628906245 |
| 0.75 | 0.50 | 1.2365409530 | 1.2343750000 | 1.2343750000 |
| 0.75 | 0.75 | 1.5877500130 | 1.5761718760 | 1.5761718750 |
| 0.75 | 1.00 | 2.0387113710 | 2.0000000000 | 2.0000000000 |

**Table 4.** Comparison of CSHPM solution with HPM (Sakar et al., 2016) and the exact solution, when $\alpha = 1$.

| $x$ | $t$ | Exact solution | CSHPM | HPM (Sakar et al., 2016) |
|---|---|---|---|---|
| 0.25 | 0.25 | 0.3210063542 | 0.3209635417 | 0.3209635415 |
| 0.25 | 0.50 | 0.4121803178 | 0.4114583333 | 0.4114583335 |
| 0.25 | 0.75 | 0.5292500042 | 0.5253906250 | 0.5253906250 |
| 0.25 | 1.00 | 0.6795704570 | 0.6666666667 | 0.6666666665 |
| 0.50 | 0.25 | 0.6420127085 | 0.6419270833 | 0.6419270830 |
| 0.50 | 0.50 | 0.8243606355 | 0.8229166667 | 0.8229166670 |
| 0.50 | 0.75 | 1.0585000080 | 1.0507812500 | 1.0507812500 |
| 0.50 | 1.00 | 1.3591409140 | 1.3333333330 | 1.3333333330 |
| 0.75 | 0.25 | 0.9630190628 | 0.9628906250 | 0.9628906245 |
| 0.75 | 0.50 | 1.2365409530 | 1.2343750000 | 1.2343750000 |
| 0.75 | 0.75 | 1.5877500130 | 1.5761718750 | 1.5761718750 |

| | | | | |
|---|---|---|---|---|
| 0.75 | 1.00 | 2.0387113710 | 2.0000000000 | 2.0000000000 |

**Table 5.** Comparison of absolute error between Cq-HATM, CSHPM, and HPM (Sakar et al., 2016) for Ex. 4.1 with $\alpha = 1$.

| $x$ | | $t$ | | | |
|---|---|---|---|---|---|
| | | $0.025$ | $0.050$ | $0.075$ | $0.1$ |
| Cq-HATM | $0.25$ | $4.10x10^{-9}$ | $6.570x10^{-8}$ | $3.3460x10^{-7}$ | $1.0629x10^{-6}$ |
| CSHPM | | $4.20x10^{-9}$ | $6.570x10^{-8}$ | $3.3470x10^{-7}$ | $1.0628x10^{-6}$ |
| HPM | | $4.20x10^{-9}$ | $6.550x10^{-8}$ | $3.3480x10^{-7}$ | $1.0630x10^{-6}$ |
| Cq-HATM | $0.50$ | $8.40x10^{-9}$ | $1.313x10^{-7}$ | $6.6930x10^{-7}$ | $2.1256x10^{-6}$ |
| CSHPM | | $8.40x10^{-9}$ | $1.313x10^{-7}$ | $6.6930x10^{-7}$ | $2.1257x10^{-6}$ |
| HPM | | $8.50x10^{-9}$ | $1.310x10^{-7}$ | $6.6950x10^{-7}$ | $2.1260x10^{-6}$ |
| Cq-HATM | $0.75$ | $1.27x10^{-8}$ | $1.970x10^{-7}$ | $1.0038x10^{-6}$ | $3.1885x10^{-6}$ |
| CSHPM | | $1.27x10^{-8}$ | $1.970x10^{-7}$ | $1.0038x10^{-6}$ | $3.1885x10^{-6}$ |
| HPM | | $1.28x10^{-8}$ | $1.965x10^{-7}$ | $1.0042x10^{-6}$ | $3.1890x10^{-6}$ |

**Example 4.2**

Consider the conformable time-fractional partial differential equation with proportional delay (Sakar et al., 2016)

$$\frac{\partial^{\alpha} w(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 w\left(x,\frac{t}{2}\right)}{\partial x^2} w\left(x,\frac{t}{2}\right) - w(x,t), \tag{71}$$

where $x, t \in [0,1], 0 < \alpha \leq 1$,

subject to initial condition

$$w(x,0) = x^2. \tag{72}$$

**Case (i) Cq-HATM solution for Eq. (71)**

Applying the conformable Laplace transform to Eq (71) and using Eq. (72), then we have

$$\mathcal{L}_{\alpha}[w(x,t)] = \frac{1}{s} w(x,0) + \frac{1}{s} \mathcal{L}_{\alpha}\left[\frac{\partial^2 w\left(x,\frac{t}{2}\right)}{\partial x^2} w\left(x,\frac{t}{2}\right) - w(x,t)\right], \tag{73}$$

We define the nonlinear operators by using Eq. (73), as

$$N[\varphi(x,t;q)] = \mathcal{L}_\alpha[\varphi(x,t;q)] - \frac{1}{s}x^2 - \frac{1}{s}\mathcal{L}_\alpha\left[\frac{\partial^2 w_r(x,\frac{t}{2})}{\partial x^2}\varphi\left(x,\frac{t}{2}\right) - \varphi(x,t)\right]. \quad (74)$$

By applying the proposed algorithm, the $m-th$ order deformation equations are defined by

$$\mathcal{L}_\alpha[w_m(x,t) - k_m w_{m-1}(x,t)] = h\mathcal{R}_m[\vec{w}_{m-1}], \quad (75)$$

where

$$\mathcal{R}_m[\vec{w}_{m-1}] = \mathcal{L}_\alpha[\vec{w}_{m-1}(x,t)] - \left(1-\frac{k_m}{n}\right)\frac{1}{s}x^2 - \frac{1}{s}\mathcal{L}_\alpha\left[\sum_{r=0}^{m-1}\frac{\partial^2 w_r(x,\frac{t}{2})}{\partial x^2}w_{m-1-r}\left(x,\frac{t}{2}\right)\right.$$

$$\left.-w_{m-1}(x,t)\right]. \quad (76)$$

On applying inverse conformable Laplace transform to Eq. (75), then we have

$$w_m(x,t) = k_m w_{m-1}(x,t) + h{\mathcal{L}_\alpha}^{-1}\{\mathcal{R}_m[\vec{w}_{m-1}]\}. \quad (77)$$

By the use of initial condition, then we get

$$w_0(x,t) = x^2. \quad (78)$$

To find the value of $w_1(x,t)$, putting $m=1$ in Eq. (77), then we obtain

$$w_1(x,t) = -hx^2\frac{t^\alpha}{\alpha}. \quad (79)$$

In the same way, if we put $m=2$ in Eq. (77), we can find the value of $w_2(x,t)$

$$w_2(x,t) = -(n+h)\left(hx^2\frac{t^\alpha}{\alpha}\right) + x^2h^2\left(\frac{4-2^\alpha}{2^\alpha}\right)\frac{t^{2\alpha}}{2\alpha^2}. \quad (80)$$

Analogously, to find the value of $w_3(x,t)$, putting $m=3$ in Eq. (77), then we get

$$w_3(x,t) = -\frac{8x^2h}{3\alpha^3}\left[-\frac{3\left(2^{-\alpha}-\frac{1}{8}\right)(n+h)h\alpha t^{2\alpha}}{2} + 8^{-\alpha}t^{3\alpha}h^2 + \frac{3t^\alpha\alpha^2(n+h)^2}{8}\right]. \quad (81)$$

In this way, the other terms can be found. So, the Cq-HATM solution of Eq. (71) is given by

$$w(x,t) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t)\left(\frac{1}{n}\right)^m. \tag{82}$$

If we put $\alpha = 1, n = 1,\ h = -1$ in Eq. (82), then the obtained results $\sum_{m=1}^{M} w_m(x,t)\left(\frac{1}{n}\right)^m$ converges to the exact solution $w(x,t) = x^2 e^t$ of Eq. (71) when $M \to \infty$.

**Case (ii) CSHPM solution for Eq. (71)**

Applying the conformable Shehu transform to Eq. (71) and using Eq. (72), then we get

$${}_cS_\alpha[w(x,t)] = \frac{u}{s}x^2 + \frac{u}{s}\ {}_cS_\alpha\left[\frac{\partial^2 w\left(x,\frac{t}{2}\right)}{\partial x^2} w\left(x,\frac{t}{2}\right) - w(x,t)\right]. \tag{83}$$

Applying the inverse conformable Shehu transform to Eq (83), then we obtain

$$w(x,t) = x^2 + \left({}_cS_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_cS_\alpha\left[\frac{\partial^2 w\left(x,\frac{t}{2}\right)}{\partial x^2} w\left(x,\frac{t}{2}\right) - w(x,t)\right]\right\}. \tag{84}$$

Now HPM is applied, then we have

$$\sum_{m=0}^{\infty} p^m w_m(x,t) = x^2 + p\left[\left({}_cS_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_cS_\alpha\left[\sum_{m=0}^{\infty} p^m H_m(w) - \left(\sum_{m=0}^{\infty} p^m w_m(x,t)\right)\right]\right\}\right]. \tag{85}$$

We get to the first few components of $H_m(w)$ by

$$H_0(w) = \frac{\partial^2 w_0\left(x,\frac{t}{2}\right)}{\partial x^2} w_0\left(x,\frac{t}{2}\right), \tag{86}$$

$$H_1(w) = \frac{\partial^2 w_0\left(x,\frac{t}{2}\right)}{\partial x^2} w_1\left(x,\frac{t}{2}\right) + \frac{\partial^2 w_1\left(x,\frac{t}{2}\right)}{\partial x^2} w_0\left(x,\frac{t}{2}\right), \tag{87}$$

$$H_2(w) = \frac{\partial^2 w_0\left(x,\frac{t}{2}\right)}{\partial x^2} w_2\left(x,\frac{t}{2}\right) + \frac{\partial^2 w_1\left(x,\frac{t}{2}\right)}{\partial x^2} w_1\left(x,\frac{t}{2}\right) + \frac{\partial^2 w_2\left(x,\frac{t}{2}\right)}{\partial x^2} w_0\left(x,\frac{t}{2}\right), \tag{88}$$

$$\vdots \tag{89}$$

When we compare the coefficients of the same powers of $p$, we get

$$p^0: w_0(x,t) = x^2, H_0(w) = 2x^2, \tag{90}$$

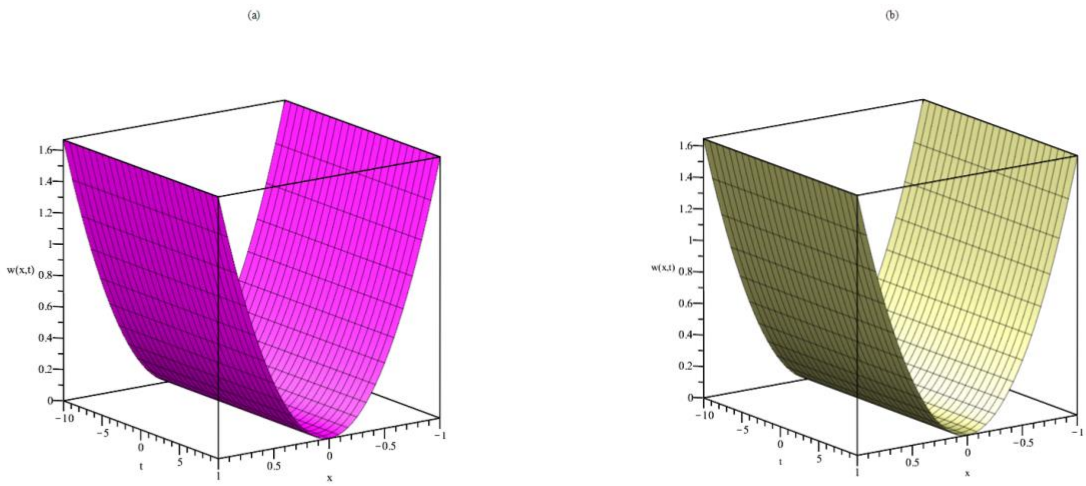
$$p^1: w_1(x,t) = x^2 \frac{t^\alpha}{\alpha}, H_1(w) = \frac{4x^2 t^\alpha}{\alpha 2^\alpha}, \tag{91}$$

$$p^2: w_2(x,t) = \frac{x^2(4-2^\alpha)t^{2\alpha}}{2^{\alpha+1}\alpha^2}, H_2(w) = \frac{16x^2 t^{2\alpha}}{2^{3\alpha+1}\alpha^2}, \tag{92}$$

$$p^3: w_3(x,t) = \frac{x^2\left(2^4-4.2^{2\alpha}+2^{3\alpha}\right)t^{3\alpha}}{3.2^{3\alpha+1}\alpha^3}. \tag{93}$$

Consequently, the solution to Eq. (71) for CSHPM is given by

$$w(x,t) = x^2\left(1 + \frac{t^\alpha}{\alpha} + \frac{(4-2^\alpha)t^{2\alpha}}{2^{\alpha+1}\alpha^2} + \frac{\left(2^4-4.2^{2\alpha}+2^{3\alpha}\right)t^{3\alpha}}{3.2^{3\alpha+1}\alpha^3}\right). \tag{94}$$



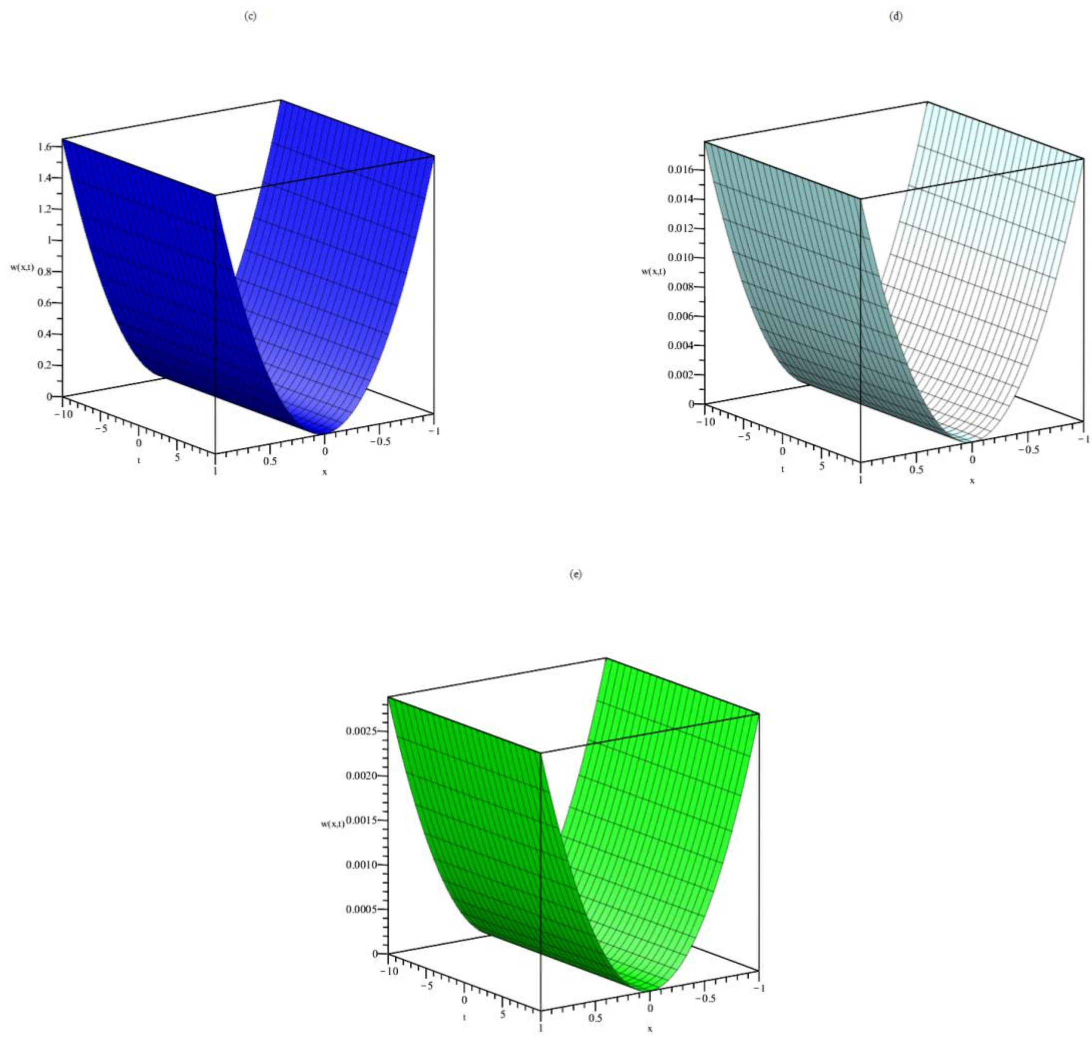



**Figure 3. (a)** Nature of Cq-HATM solution **(b)** Nature of CSHPM solution **(c)** Nature of exact solution **(d)** Nature of absolute error=$|w_{exact} - w_{Cq-HATM}|$ **(e)** Nature of absolute error=$|u_{exact} - u_{CSHPM}|$ for Ex. 4.2 at $\mathrm{h} = -1, \mathrm{n} = 1, \alpha = 1$.

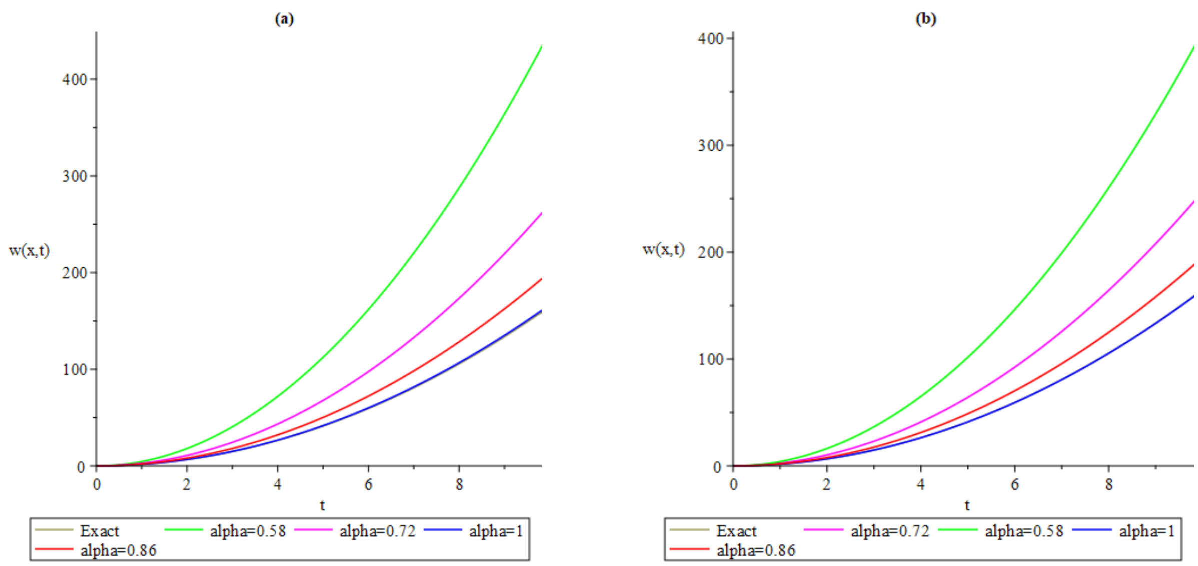



**Figure 4. (a)** The comparison of the Cq-HATM solutions and exact solution **(b)** The comparison of the CSHPM solutions and exact solution for Ex. 4.2 at $\mathrm{h} = -1, \mathrm{n} = 1, x = 0.5$ with distinct $\alpha$.

**Table 6.** Numerical solution for Eq. (71) by Cq-HATM in Ex. 4.2 at $\mathrm{h} = -1, \mathrm{n} = 1$ with distinct values of $x$ and $t$ for diverse $\alpha$.

| $x$ | $t$ | $\alpha = 0.58$ | $\alpha = 0.72$ | $\alpha = 0.86$ | $\alpha = 1$ |
|---|---|---|---|---|---|
| 0.25 | 0.25 | 0.1648194122 | 0.1111928264 | 0.09047262722 | 0.0804036458 |
| | 0.50 | 0.2808111960 | 0.1692915562 | 0.1253087786 | 0.1041666666 |
| | 0.75 | 0.4202222715 | 0.2436397434 | 0.1711188328 | 0.1357421874 |
| | 1.00 | 0.5816624291 | 0.3353252558 | 0.2298608781 | 0.1770833334 |
| 0.50 | 0.25 | 0.6592776490 | 0.4447713057 | 0.3618905088 | 0.3216145834 |
| | 0.50 | 1.1232447840 | 0.6771662248 | 0.5012351148 | 0.4166666666 |
| | 0.75 | 1.6808890850 | 0.9745589741 | 0.6844753311 | 0.5429687500 |
| | 1.00 | 2.3266497160 | 1.3413010230 | 0.9194435124 | 0.7083333334 |
| 0.75 | 0.25 | 1.4833747100 | 1.0007354380 | 0.8142536449 | 0.7236328126 |
| | 0.50 | 2.5273007640 | 1.5236240050 | 1.127779008 | 0.9375000000 |
| | 0.75 | 3.7820004430 | 2.1927576920 | 1.540069495 | 1.2216796870 |
| | 1.00 | 5.2349618610 | 3.0179273020 | 2.068747902 | 1.5937500000 |

**Table 7.** Numerical solution for Eq. (71) by CSHPM in Ex. 4.2 with distinct values of $x$ and $t$ for diverse $\alpha$.

| $x$ | $t$ | $\alpha = 0.58$ | $\alpha = 0.72$ | $\alpha = 0.86$ | $\alpha = 1$ |
|---|---|---|---|---|---|
| 0.25 | 0.25 | 0.1568008432 | 0.1091969740 | 0.08992121464 | 0.0802408854 |
| | 0.50 | 0.2540263556 | 0.1603717952 | 0.1220116639 | 0.1028645833 |
| | 0.75 | 0.3659862465 | 0.2222251252 | 0.1617335037 | 0.1313476562 |
| | 1.00 | 0.4921916428 | 0.2954615189 | 0.2101461204 | 0.1666666667 |
| 0.50 | 0.25 | 0.6272033731 | 0.4367878960 | 0.3596848585 | 0.3209635417 |
| | 0.50 | 1.0161054230 | 0.6414871809 | 0.4880466560 | 0.4114583333 |
| | 0.75 | 1.4639449850 | 0.8889005011 | 0.6469340150 | 0.5253906250 |
| | 1.00 | 1.9687665710 | 1.1818460750 | 0.8405844816 | 0.6666666667 |
| 0.75 | 0.25 | 1.4112075890 | 0.9827727661 | 0.8092909317 | 0.7221679688 |
| | 0.50 | 2.2862372010 | 1.4433461570 | 1.0981049760 | 0.9257812500 |
| | 0.75 | 3.2938762180 | 2.0000261280 | 1.4556015340 | 1.1821289060 |
| | 1.00 | 4.4297247850 | 2.6591536700 | 1.8913150830 | 1.5000000000 |

**Table 8.** Comparison of Cq-HATM solution with HPM (Sakar et al., 2016) and the exact solution, when $\alpha = 1$.

| $x$ | $t$ | Exact solution | $Cq-HATM$ | $HPM$ $(Sakar\ et\ al., 2016)$ |
|---|---|---|---|---|
| 0.25 | 0.25 | 0.0802515885 | 0.0804036458 | 0.0824381510 |
| | 0.50 | 0.1030450794 | 0.1041666666 | 0.1126302084 |
| | 0.75 | 0.1323125011 | 0.1357421874 | 0.1555175781 |
| | 1.00 | 0.1698926142 | 0.1770833334 | 0.2135416666 |

| | | | | |
|---|---|---|---|---|
| 0.50 | 0.25 | 0.3210063542 | 0.3216145834 | 0.3297526040 |
| | 0.50 | 0.4121803178 | 0.4166666666 | 0.4505208335 |
| | 0.75 | 0.5292500042 | 0.5429687500 | 0.6220703125 |
| | 1.00 | 0.6795704570 | 0.7083333334 | 0.8541666665 |
| 0.75 | 0.25 | 0.7222642971 | 0.7236328126 | 0.7419433590 |
| | 0.50 | 0.9274057149 | 0.9375000000 | 1.0136718750 |
| | 0.75 | 1.1908125100 | 1.2216796870 | 1.3996582030 |
| | 1.00 | 1.5290335280 | 1.5937500000 | 1.9218750000 |

**Table 9.** Comparison of CSHPM solution with HPM (Sakar et al., 2016) and the exact solution, when $\alpha = 1$.

| $x$ | $t$ | Exact solution | $CSHPM$ | $HPM$ ($Sakar$ $et$ $al.$, $2016$) |
|---|---|---|---|---|
| 0.25 | 0.25 | 0.0802515885 | 0.0802408854 | 0.0824381510 |
| | 0.50 | 0.1030450794 | 0.1028645833 | 0.1126302084 |
| | 0.75 | 0.1323125011 | 0.1313476562 | 0.1555175781 |
| | 1.00 | 0.1698926142 | 0.1666666667 | 0.2135416666 |
| 0.50 | 0.25 | 0.3210063542 | 0.3209635417 | 0.3297526040 |
| | 0.50 | 0.4121803178 | 0.4114583333 | 0.4505208335 |
| | 0.75 | 0.5292500042 | 0.5253906250 | 0.6220703125 |
| | 1.00 | 0.6795704570 | 0.6666666667 | 0.8541666665 |
| 0.75 | 0.25 | 0.7222642971 | 0.7221679688 | 0.7419433590 |
| | 0.50 | 0.9274057149 | 0.9257812500 | 1.0136718750 |
| | 0.75 | 1.1908125100 | 1.1821289060 | 1.3996582030 |
| | 1.00 | 1.5290335280 | 1.5000000000 | 1.9218750000 |

**Table 10.** Comparison of absolute error between Cq-HATM, CSHPM, and HPM (Sakar et al., 2016) for Example 4.2 with $\alpha = 1$.

| $x$ | | $t$ | | | |
|---|---|---|---|---|---|
| | | $0.025$ | $0.050$ | $0.075$ | $0.1$ |
| **Cq-HATM** | $0.25$ | $1.61x10^{-7}$ | $1.28x10^{-6}$ | $4.31x10^{-6}$ | $1.01x10^{-5}$ |
| **CSHPM** | | $1.05x10^{-9}$ | $1.64x10^{-8}$ | $8.36x10^{-8}$ | $2.65x10^{-7}$ |
| **HPM** | | $1.97x10^{-5}$ | $8.00x10^{-5}$ | $1.82x10^{-4}$ | $3.27x10^{-4}$ |
| **Cq-HATM** | $0.50$ | $6.46x10^{-7}$ | $5.14x10^{-6}$ | $1.72x10^{-5}$ | $4.06x10^{-5}$ |
| **CSHPM** | | $4.20x10^{-9}$ | $6.57x10^{-8}$ | $3.34x10^{-7}$ | $1.06x10^{-6}$ |
| **HPM** | | $7.90x10^{-5}$ | $3.20x10^{-4}$ | $7.29x10^{-4}$ | $1.31x10^{-3}$ |
| **Cq-HATM** | $0.75$ | $1.27x10^{-8}$ | $1.15x10^{-5}$ | $3.87x10^{-5}$ | $9.13x10^{-4}$ |
| **CSHPM** | | $1.27x10^{-8}$ | $1.47x10^{-7}$ | $7.52x10^{-7}$ | $2.39x10^{-6}$ |
| **HPM** | | $1.28x10^{-8}$ | $7.20x10^{-4}$ | $1.64x10^{-3}$ | $2.95x10^{-3}$ |

## 5. Results and Discussion

For the conformable time-fractional generalized Burgers equation (CTFGBE) with proportional delay, the graphs of the temperature $w(x,t)$ which is obtained by Cq-HATM for different values of $\alpha = 0.58, \alpha = 0.72, \alpha = 0.86,$ and $\alpha = 1$ are drawn in Table 1. Table 2 depicts the graphs of the temperature $w(x,t)$ acquired by CSHPM for different values of $\alpha = 0.58, \alpha = 0.72, \alpha = 0.86,$ and $\alpha = 1$. In Table 3, the Cq-HATM solution for $\alpha = 1$ is compared to the exact solution and the HPM solution in (Sakar et al., 2016). In the same way, Table 4 compares the CSHPM solution for $\alpha = 1$ to the exact solution and the HPM solution in (Sakar et al., 2016). In Table 5, the difference between Cq-HATM, CSHPM, and HPM and the exact solution is calculated as the absolute error. The absolute error for Cq-HATM, CSHPM, and HPM are all almost the same, as shown in Table 5. Figure 1 shows 3D graphs of Cq-HATM solution, CSHPM solution, the exact solution, the absolute error for Cq-HATM, and the absolute error for CSHPM for $\alpha = 1$. The behavior of Cq-HATM and CSHPM solutions of CTFGBE is depicted in Figure 2 via 2D graphs for different $\alpha$ values. For Eq. (71), the graphs of the temperature $w(x,t)$ that Cq-HATM gets for different values of $\alpha = 0.58, \alpha = 0.72, \alpha = 0.86,$ and $\alpha = 1$ are shown in Table 6. In the same way, Table 7 shows graphs of the temperature $w(x,t)$ evaluated by CSHPM for different values of $\alpha = 0.58, \alpha = 0.72, \alpha = 0.86,$ and $\alpha = 1$. In Table 8, the Cq-HATM solution for $\alpha = 1$ is compared to the exact solution and the HPM solution from (Sakar et al., 2016). In the same way, Table 9 compares the CSHPM solution for $\alpha = 1$ to both the exact solution and the HPM solution in (Sakar et al., 2016). In Table 10, the absolute error is the difference between Cq-HATM, CSHPM, and HPM and the exact solution. It has been observed from Table 10 that the suggested methods generate considerably smaller errors than HPM. Figure 3 shows 3D graphs of the exact solution, the Cq-HATM solution, the CSHPM solution, the absolute error for Cq-HATM, and the absolute error for CSHPM of Eq. (71) when $\alpha = 1$. Figure 4 depicts the behavior of Cq-HATM and CSHPM solutions of Eq. (71) for various values of $\alpha$ via 2D graphs.

## 6. Conclusion

In this paper, conformable time-fractional partial differential equations with proportional delay are analyzed by using Cq-HATM and CSHPM. Besides, the graphs of the solutions of these equations for the different $\alpha$ values have been obtained in MAPLE software. It is observed that the general construction of the surface graphs generated by the Maple software, for Ex. 4.1 varies. It is observed that the general structure of the surface graphs generated by the Maple software, for Ex. 4.2, differs. It may be determined that the newly proposed methods for solving nonlinear conformable time-fractional partial differential equations with proportional delay are both beneficial and efficient.

## Authors' Contributions

All authors contributed equally to the study.

## Statement of Conflicts of Interest

There is no conflict of interest between the authors.

## Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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