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Research Article

Fuzzy rough sets based on Morsi fuzzy hemimetrics

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Abstract

In this paper, we introduce a notion of Morsi fuzzy hemimetrics, a common generalization of hemimetrics and Morsi fuzzy metrics, as the basic structure to define and study fuzzy rough sets. We define a pair of fuzzy upper and lower approximation operators and investigate their properties. It is shown that upper definable sets, lower definable sets and definable sets are equivalent. Definable sets form an Alexandrov fuzzy topology such that the upper and lower approximation operators are the closure and the interior operators respectively.

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1. Introduction

Rough set theory, originally proposed by Pawlak [20], is an effective mathematical tool to deal with vagueness, imprecision and uncertainty characterized by insufficient and incomplete information. It has become a well-established theory in a wide variety of applications related to process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis and other fields.

One central problem of rough set theory is classification or clustering analysis. Indiscernibility is modeled by an equivalence relation which induces a partition of the universe into equivalence classes of indiscernible objects. However, some granularity problems of information tables in the real world can not be dealt with by known knowledge induced by a simple equivalence relation, the tolerance relations [2,25] and similarity relations [26] are also considered. We can call rough sets based on certain kinds of binary relations as relation-based rough sets.

Besides, coverings and neighborhood operators based rough sets are two important extensions of the classical rough sets [1,34]. In the environment of set-valued analysis, neighborhood operators are always associated with coverings. A covering is a generalization of a partition, every member can be considered as an abstract of equivalence classes. Both of

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covering-based and neighborhood operator-based rough sets are indeed the granule-based approaches, where discrete points in the universe are firstly combined into granules and then are used to describe roughness. However, the above mentioned generalizations are no longer equivalent and thus yield different generalized rough set models.

The real world is full of uncertainty, including fuzziness, roughness and probability. Classical rough set theory is based on equivalence relations. However, the definition of equivalence relations is too restrictive for some applications, thus fuzzification becomes another important method to get generalizations of the classical rough set models [6,7,14]. In the framework of relation-based fuzzy rough set theory, various fuzzy generalizations of approximation operators have been proposed and investigated.

Some researchers have tried to develop kinds of fuzzy rough sets based on the unit interval [0,1] and provide axiomatic characterizations of fuzzy rough approximation operators [16, 18, 22, 27–29]. As a matter of fact, the unit interval [0,1] can not be supplied as the truth value table any more in the partial ordering setting. Based on this consideration, Radzikowska et al [21,23] and She et al [24] chose a complete residuated lattice L to investigate L-fuzzy rough approximation operators. In the framework of covering-based fuzzy rough set, a family of fuzzy subsets of the universe is used to define the concept of fuzzy coverings, from which different pairs of upper and lower approximation operators can be constructed [5, 15, 17, 30].

In applications, for both classical and fuzzy settings, although the concrete models are different, all of binary relations, coverings and neighborhood operators have great effects on the development of methods which can be used to describe relations between data objects and play key roles in data mining. Besides these basic structures, some kinds of distance functions are indeed natural structures to describe the relation between data objects. If the distance between two objects is a small number, that is, one object is near to the other, then they should have some similar data; conversely, if the distance is relatively large, that is, one object is far away from the other, then their data have a relatively big difference and are not similar so much. In this sense, the smaller the distance of two objects is, the more similarity they have; and conversely, the larger the distance is, the more difference they have.

By this motivation, Yao et al [32, 33] introduced a new model of fuzzy rough set called the hemimetric-based fuzzy rough set and proposed applications of the hemimetric-based rough set model to fuzzy clustering and contour extraction of digital surfaces, where a hemimetric (or called a pseudo-quasi-metric) is a weak version of the standard metric which allows the distance between different points to be zero.

Different fuzzy metrics, for example, the Morsi fuzzy metrics [19], GV-fuzzy metrics [8] and KM-fuzzy metrics [13], are kinds of important generalizations of the classical metrics, and have important applications in fuzzy mathematics. In this paper, we will introduce a new notion of fuzzy metrics, namely Morsi fuzzy hemimetrics, as the basic structure to construct a fuzzy rough set model in hope of proposing a common framework of the rough set models on hemimetrics and Morsi fuzzy metrics. We will study this model by means of fuzzy upper and lower approximation operators, definability of fuzzy subsets and their topological properties.

2. Preliminaries

2.1. The Morsi fuzzy hemimetrics

In this section, we recall some basic definitions and results on fuzzy set theory and introduce a notion of Morsi fuzzy hemimetrics.

Definition 2.1. Let X be a nonempty set. A mapping $m : X \times X \times (0, +\infty) \longrightarrow [0, 1]$ is called a *Morsi fuzzy hemimetric* if: $\forall x, y, z \in X, \forall s, t > 0$,

(M1) $m(x,x,s) = 0$;
(M2) $m(x,z,s+t) \le m(x,y,s) + m(y,z,t)$.
The pair (X,m) is called a *Morsi fuzzy hemimetric space*.

Remark 2.2. Morsi defined his fuzzy metrics by using Hutton's fuzzy real numbers w.r.t. the maximum s-norm $\vee$ on $[0,1]$. We here rewrite it in the form of the mapping in Definition 2.1. A Morsi fuzzy metric on a nonempty set X is a mapping $m : X \times X \times [0,\infty) \longrightarrow [0,1]$ satisfying (M1) and
(Mor2$_\vee$) $m(x,z,s+t) \le m(x,y,s) \vee m(y,z,t)$ $(\forall s,t > 0)$.
(Mor3) $m(x,y,r) = m(y,x,r)$ $(\forall x,y \in X,\ \forall r > 0)$;
(Mor4) $m(x,y,r) = 0$ $(\forall r > 0)$ implies $x = y$ $(\forall x,y \in X)$;
(Mor5) $m(x,y,\cdot)(\forall x,y \in X)$ is decreasing and left-continuous.

There are two explanations of the value $m(x,y,r)$. The first one is that it can be interpreted as the degree of the distance between x,y being larger than or equal to r. The second explanation can be applied to algorithm implementation, we can consider $m(x,y,r)$ as the ability to distinguish the real distance between different pixels. Assume that we are looking at a digital plane filled up with pixels. Then we can use a Morsi fuzzy hemimetric $m(x,y,r)$ to estimate the distance between x,y. The smaller the distance between the place and the digital plane is, the more clearly we can see how far the two pixels x and y are; and conversely, the larger the distance is, the weaker ability to distinguish the real distance between different pixels we have, and at some points, two different pixels may be merged into one in our eyes.

Example 2.3. Let (X,d) be a hemimetric space [9], that is $d : X \times X \longrightarrow [0,+\infty)$ satisfying: (H1) $d(x,x) = 0$; (H2) $d(x,z) \le d(x,y) + d(y,z)$. Define $m_d, m^d : X \times X \times (0,+\infty) \longrightarrow [0,1]$ by

$$m_d(x,y,t) = d(x,y)/(d(x,y)+t);$$

$$m^d(x,y,t) = d(x,y) * (1/r+1),$$

then both m_d, m^d are Morsi fuzzy hemimetrics.

Proof. For m_d, for every $x,y,z \in X$, by (H1),

$$m_d(x,x,t) = d(x,x)/(d(x,x)+t) = 0;$$

For every $x \in X$ and $a \in (0,\infty)$, let $f(x) = x/(a+x)$, then it is easy to know that $f(x)$ is a monotonically increasing function. Therefore, by (H2),

$$\begin{aligned}
& m_d(x,z,s+t) \\
= \;& \frac{d(x,z)}{d(x,z)+(s+t)} \\
\le \;& \frac{d(x,y)+d(y,z)}{d(x,y)+d(y,z)+(s+t)} \\
\le \;& \frac{d(x,y)}{d(x,y)+d(y,z)+(s+t)} + \frac{d(y,z)}{d(x,y)+d(y,z)+(s+t)} \\
\le \;& \frac{d(x,y)}{d(x,y)+(s+t)} + \frac{d(y,z)}{d(y,z)+(s+t)} \\
\le \;& \frac{d(x,y)}{d(x,y)+s} + \frac{d(y,z)}{d(y,z)+t}.
\end{aligned}$$

Hence, m_d is a Morsi fuzzy hemimetric.

For m^d, for every $x,y,z \in X$, by (H1),

$$m^d(x,x,t) = d(x,x) * (1/r+1) = 0;$$

To prove that m^d satisfies the property (M2), we need to prove that

$$m^d(x,z,s+t) \le m^d(x,y,s) + m^d(y,z,t),$$

which is equivalent to that,

$$\begin{aligned}
&d(x,z) * (\frac{1}{s+t} + 1) \le d(x,y) * (\frac{1}{s} + 1) + d(y,z) * (\frac{1}{t} + 1);\\
&d(x,z) * \frac{s+t+1}{s+t} \le d(x,y) * \frac{s+1}{s} + d(y,z) * \frac{t+1}{t};\\
&d(x,z) \le d(x,y) * \frac{s+1}{s} * \frac{s+t}{s+t+1} + d(y,z) * \frac{t+1}{t} * \frac{s+t}{s+t+1};\\
&d(x,z) \le d(x,y) * \frac{s^2+st+s+t}{s^2+st+s} + d(y,z) * \frac{t^2+st+s+t}{t^2+st+t}.
\end{aligned}$$

The last inequality is valid since both of the two fractions therein are obviously greater than 1. Hence, m^d is a Morsi fuzzy hemimetric. □

2.2. The Lukasiewicz logic system on $[0,1]$

In order to study Morsi fuzzy hemimetric based fuzzy rough sets more deeply, we can give some logical descriptions of definitions and formulas. Before that, we need logical operations on $[0,1]$. The most famous and most useful logical system on $[0,1]$ is the Lukasiewicz system[11,12], which is a concrete MV-algebraic system.

The related four binary operations $\oplus, \ominus, \otimes, \to$ on $[0,1]$ are given by

Table 1. Logical operations on [0,1].

Operation	Logic meaning	Definition
$\oplus$	disjunction	$a \oplus b = \min\{a+b, 1\}$
$\ominus$	difference	$a \ominus b = \max\{a-b, 0\}$
$\otimes$	conjunction	$a \otimes b = \max\{a+b-1, 0\}$
$\to$	implication	$a \to b = \min\{1-a+b, 1\}$

The pair $(\otimes, \to)$ forms an adjoint pair on $[0,1]$, that is, $a \otimes b \le c$ iff $a \le b \to c$ $(\forall a,b,c \in [0,1])$.

Proposition 2.4. *For all $a,b,c \in [0,1]$, we have*
(1) $(a \otimes b) \ominus c = a \otimes (b \ominus c)$;
(2) $(b \ominus a) \ominus c = b \ominus (a \oplus c)$;
(3) $(a \to b) \oplus c = a \to (b \oplus c)$.

2.3. Fuzzy subsets and fuzzy topology

For a nonempty set X, every mapping $A : X \longrightarrow [0,1]$ is called a *fuzzy subset* of X and the value $A(x)$ is called the *membership degree* of x in A. We use a_X to denote the constant fuzzy subset with the value $a \in [0,1]$ and use $\mathcal{F}(X)$ to denote the family of all fuzzy subsets of X. For $\{A_i \mid i \in I\} \subseteq \mathcal{F}(X)$, we define $(\bigvee_i A_i)(x) = \sup\{A_i(x) \mid i \in I\}$, $(\bigwedge_i A_i)(x) = \inf\{A_i(x) \mid i \in I\}$, where sup and inf are respectively the supremum and infimum of subsets in $\mathbb{R}$.

Definition 2.5. Let X be a nonempty set. A subfamily $\delta \subseteq \mathcal{F}(X)$ is called a stratified Alexandrov fuzzy topology on X if
(1) $a_X \in \delta$ for all $a \in [0,1]$;
(2) $\bigvee_i A_i,\ \bigwedge_i A_i \in \delta$ for all $\{A_i \mid i \in I\} \subseteq \delta$.

3. Fuzzy rough set model induced by Morsi fuzzy hemimetrics

In this section, we will define a pair of fuzzy rough approximation operators from Morsi fuzzy hemimetrics, and then study their properties and interrelations.

Definition 3.1. Let (X, m) be a Morsi fuzzy hemimetric space. Define two operators $\overline{\mathrm{Apr}}_m, \underline{\mathrm{Apr}}_m : \mathcal{F}(X) \longrightarrow \mathcal{F}(X)$ respectively by, for every $x \in X$,

$$\overline{\mathrm{Apr}}_m(A)(x) = \bigvee_{y\in X}\bigvee_{r>0} A(y) - m(x,y,r);$$

$$\underline{\mathrm{Apr}}_m(A)(x) = \bigwedge_{y\in X}\bigwedge_{r>0} A(y) + m(y,x,r).$$

The operators $\overline{\mathrm{Apr}}_m, \underline{\mathrm{Apr}}_m : \mathcal{F}(X) \longrightarrow \mathcal{F}(X)$ are called the *fuzzy upper rough approximation operator* and the *fuzzy lower rough approximation operator* on X induced by the Morsi fuzzy hemimetric m, respectively.

Since the Morsi fuzzy hemimetric m need not be symmetric, we should notice the order of x, y after m in the definition of the fuzzy upper and lower approximation operators.

Remark 3.2. In [33], for a hemimetric d on a nonempty set X, we define a pair of fuzzy upper and lower approximation operators $\overline{\mathrm{Apr}}_d, \underline{\mathrm{Apr}}_d : \mathcal{F}(X) \longrightarrow \mathcal{F}(X)$ by, for every $x \in X$,

$$\overline{\mathrm{Apr}}_d(A)(x) = \bigvee_{y\in X} A(y) - d(x,y);$$

$$\underline{\mathrm{Apr}}_d(A)(x) = \bigwedge_{y\in X} A(y) + d(y,x).$$

By Example 2.3, we know that the formula $m^d(x,y,r) = d(x,y)(1/r+1)$ defines a Morsi fuzzy hemimetric space. It is a routine to show that

$$\overline{\mathrm{Apr}}_{m^d} = \overline{\mathrm{Apr}}_d, \underline{\mathrm{Apr}}_{m^d} = \underline{\mathrm{Apr}}_d.$$

Proof. For every $x \in X$,

$$\begin{aligned}
& \overline{\mathrm{Apr}}_{m^d}(A)(x) \\
= & \bigvee_{y\in X}\bigvee_{r>0} A(y) - m^d(x,y,r) \\
= & \bigvee_{y\in X}\bigvee_{r>0} A(y) - d(x,y)(1/r+1) \\
= & \bigvee_{y\in X} A(y) - \bigwedge_{r>0} d(x,y)(1/r+1) \\
= & \bigvee_{y\in X} A(y) - d(x,y) * 1 \\
= & \bigvee_{y\in X} A(y) - d(x,y) \\
= & \overline{\mathrm{Apr}}_d(A)(x).
\end{aligned}$$

Hence, $\overline{\mathrm{Apr}}_{m^d} = \overline{\mathrm{Apr}}_d$.

$$\begin{aligned}
& \underline{\mathrm{Apr}}_{m^d}(A)(x) \\
= & \bigwedge_{y\in X}\bigwedge_{r>0} A(y) + m^d(y,x,r) \\
= & \bigwedge_{y\in X}\bigwedge_{r>0} A(y) + d(y,x)(1/r+1) \\
= & \bigwedge_{y\in X} A(y) + \bigwedge_{r>0} d(y,x)(1/r+1) \\
= & \bigwedge_{y\in X} A(y) + d(y,x) * 1 \\
= & \bigwedge_{y\in X} A(y) + d(y,x) \\
= & \underline{\mathrm{Apr}}_{d}(A)(x).
\end{aligned}$$

Hence, $\underline{\mathrm{Apr}}_{m^d} = \underline{\mathrm{Apr}}_{d}$. □

In this sense, we can say that the fuzzy rough set model of Morsi fuzzy hemimetrics in this paper is a generalization of that of ordinary hemimetrics in [33], and by other kind of Morsi fuzzy hemimetric induced by an ordinary hemimetric (for example m_d in Example 2.3), we may get a new pair of fuzzy upper and lower approximation operators.

Now we study the properties of the fuzzy upper and lower rough approximation operators.

Theorem 3.3. *Let (X,m) be a Morsi fuzzy hemimetric space. Then for all $A\in\mathcal{F}(X), a\in[0,1]$ and $\{A_i \mid i\in I\}\subseteq\mathcal{F}(X)$, it holds that*
(U1) $A\leq\overline{\mathrm{Apr}}_m(A)\leq 1_X$;
(U2) $\overline{\mathrm{Apr}}_m(a_X)=a_X$;
(U3) $\overline{\mathrm{Apr}}_m(\bigvee_i A_i)=\bigvee_i\overline{\mathrm{Apr}}_m(A_i)$;
(U4) $\overline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A))=\overline{\mathrm{Apr}}_m(A)$.

Proof. (U1) Clearly, $\overline{\mathrm{Apr}}_m(A)\leq 1_X$. For every $x\in X$,

$$\begin{aligned}
& \overline{\mathrm{Apr}}_m(A)(x) \\
= & \bigvee_{y\in X}\bigvee_{r>0} A(y) - m(x,y,r) \\
\geq & \bigvee_{r>0} A(x) - m(x,x,r) \\
= & A(x) - 0 \\
= & A(x).
\end{aligned}$$

Hence, $A\leq\overline{\mathrm{Apr}}_m(A)\leq 1_X$.

(U2) By (U1), $\overline{\mathrm{Apr}}_m(a_X)\geq a_X$. For every $x\in X$,

$$\overline{\mathrm{Apr}}_m(a_X)(x)=\bigvee_{y\in X}\bigvee_{r>0} a - m(x,y,r)\leq a.$$

Hence, $\overline{\mathrm{Apr}}_m(a_X)=a_X$.

(U3) For every $x\in X$,

$$\begin{aligned}
& \overline{\mathrm{Apr}}_m(\textstyle\bigvee_i A_i)(x) \\
= & \textstyle\bigvee_{y\in X}\bigvee_{r>0}(\bigvee_i A_i)(y) - m(x,y,r) \\
= & \textstyle\bigvee_{y\in X}\bigvee_{r>0}\bigvee_i A_i(y) - m(x,y,r) \\
= & \textstyle\bigvee_i\bigvee_{y\in X}\bigvee_{r>0} A_i(y) - m(x,y,r) \\
= & \textstyle\bigvee_i[\bigvee_{y\in X}\bigvee_{t>0} A_i(y) - m(x,y,r)] \\
= & \textstyle\bigvee_i\overline{\mathrm{Apr}}_m(A_i)(x).
\end{aligned}$$

Hence, $\overline{\mathrm{Apr}}_m(\bigvee_i A_i)=\bigvee_i\overline{\mathrm{Apr}}_m(A_i)$.

(U4) By (U1), we have $\overline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A)) \geq \overline{\mathrm{Apr}}_m(A)$. For every $x \in X$,

$$\begin{aligned}
& \overline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A))(x) \\
= & \bigvee_{y\in X}\bigvee_{r>0}(\overline{\mathrm{Apr}}_m(A))(y) - m(x,y,r) \\
= & \bigvee_{y\in X}\bigvee_{r>0}[\bigvee_{z\in X}\bigvee_{s>0} A(z) - m(y,z,s)] - m(x,y,r) \\
= & \bigvee_{y\in X}\bigvee_{r>0}\bigvee_{z\in X}\bigvee_{s>0} A(z) - (m(x,y,r) + m(y,z,s)) \\
\leq & \bigvee_{z\in X}\bigvee_{r,s>0} A(z) - m(x,z,r+s) \\
\leq & \bigvee_{z\in X}\bigvee_{m>0} A(z) - m(x,z,m) \\
= & \overline{\mathrm{Apr}}_m(A)(x).
\end{aligned}$$

Hence, $\overline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A)) = \overline{\mathrm{Apr}}_m(A)$. □

Theorem 3.4. *Let (X,m) be a Morsi fuzzy hemimetric space. Then for all $A \in \mathcal{F}(X)$, $a \in [0,1]$ and $\{A_i \mid i \in I\} \subseteq \mathcal{F}(X)$, it holds that*
(L1) $0_X \leq \underline{\mathrm{Apr}}_m(A) \leq A$;
(L2) $\underline{\mathrm{Apr}}_m(a_X) = a_X$;
(L3) $\underline{\mathrm{Apr}}_m(\bigwedge_i A_i) = \bigwedge_i \underline{\mathrm{Apr}}_m(A_i)$;
(L4) $\underline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A)) = \underline{\mathrm{Apr}}_m(A)$.

Proof. (L1) Clearly, $0_X \leq \underline{\mathrm{Apr}}_m(A)$. For every $x \in X$,

$$\begin{aligned}
& \underline{\mathrm{Apr}}_m(A)(x) \\
= & \bigwedge_{y\in X}\bigwedge_{r>0} A(y) + m(y,x,r) \\
\leq & \bigwedge_{r>0} A(x) + m(x,x,r) \\
= & A(x) + 0 \\
= & A(x).
\end{aligned}$$

Hence, $\underline{\mathrm{Apr}}_m(A) \leq A$.

(L2) By (L1), $\underline{\mathrm{Apr}}_m(a_X) \leq a_X$. For every $x \in X$,

$$\underline{\mathrm{Apr}}_m(a_X)(x) = \bigwedge_{y\in X}\bigwedge_{r>0} a + m(y,x,r) \geq a.$$

Hence, $\underline{\mathrm{Apr}}_m(a_X) = a_X$.

(L3) For every $x \in X$,

$$\begin{aligned}
& \underline{\mathrm{Apr}}_m(\bigwedge_i A_i)(x) \\
= & \bigwedge_{y\in X}\bigwedge_{r>0}(\bigwedge_i A_i)(y) + m(y,x,r) \\
= & \bigwedge_{y\in X}\bigwedge_{r>0}\bigwedge_i A_i(y) + m(y,x,r) \\
= & \bigwedge_i\bigwedge_{y\in X}\bigwedge_{r>0} A_i(y) + m(y,x,r) \\
= & \bigwedge_i[\bigwedge_{y\in X}\bigwedge_{r>0} A_i(y) + m(y,x,r)] \\
= & \bigwedge_i \underline{\mathrm{Apr}}_m(A_i)(x).
\end{aligned}$$

Hence, $\underline{\mathrm{Apr}}_m(\bigwedge_i A_i) = \bigwedge_i \underline{\mathrm{Apr}}_m(A_i)$.

(L4) By (L1), we have $\underline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A)) \leq \underline{\mathrm{Apr}}_m(A)$. For every $x \in X$,

$$\begin{aligned} & \underline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A))(x) \\ = & \bigwedge_{y\in X} \bigwedge_{r>0} \underline{\mathrm{Apr}}_m(A)(y) + m(y,x,r) \\ = & \bigwedge_{y\in X} \bigwedge_{r>0} [\bigwedge_{z\in X} \bigwedge_{s>0} A(z) + m(z,y,s)] + m(y,x,r) \\ = & \bigwedge_{y\in X} \bigwedge_{r>0} \bigwedge_{z\in X} \bigwedge_{s>0} A(z) + (m(y,x,r) + m(z,y,s)) \\ \geq & \bigwedge_{y,z\in X} \bigwedge_{r,s>0} A(z) + m(z,x,r+s) \\ \geq & \bigwedge_{y,z\in X} \bigwedge_{m>0} A(z) + m(z,x,m) \\ = & \underline{\mathrm{Apr}}_m(A)(x). \end{aligned}$$

Hence, $\underline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A)) = \underline{\mathrm{Apr}}_m(A)$. □

Theorem 3.5. *Let (X,m) be a Morsi fuzzy hemimetric space. Then for every $A \in \mathcal{F}(X)$, it holds that*

(1) $\overline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A)) = \underline{\mathrm{Apr}}_m(A)$;

(2) $\underline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A)) = \overline{\mathrm{Apr}}_m(A)$.

Proof. (1) By (U1), we only need to show $\overline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A)) \leq \underline{\mathrm{Apr}}_m(A)$. Let $x \in X$. Then

$$\begin{aligned} & \overline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A))(x) \\ = & \bigvee_{y\in X} \bigvee_{r>0} \underline{\mathrm{Apr}}_m(A)(y) - m(x,y,r) \\ = & \bigvee_{y\in X} \bigvee_{r>0} [\bigwedge_{z\in X} \bigwedge_{m>0} A(z) + m(z,y,m)] - m(x,y,r) \\ = & \bigvee_{y\in X} \bigvee_{r>0} \bigwedge_{z\in X} \bigwedge_{m>0} A(z) + m(z,y,m) - m(x,y,r). \end{aligned}$$

We need to prove that

$$\bigvee_{y\in X} \bigvee_{r>0} \bigwedge_{z\in X} \bigwedge_{m>0} A(z) + m(z,y,m) - m(x,y,r) \leq \bigwedge_{z\in X} \bigwedge_{s>0} A(z) + m(z,x,s),$$

which is equivalent to that, for every $y, z \in X$ and every $r, s > 0$,

$$\bigwedge_{z\in X} \bigwedge_{m>0} A(z) + m(z,y,m) \leq A(z) + m(z,x,s) + m(x,y,r).$$

In fact,

$$\begin{aligned} & \bigwedge_{z\in X} \bigwedge_{m>0} A(z) + m(z,y,m) \\ \leq & \bigwedge_{z\in X} \bigwedge_{r,s>0} A(z) + m(z,y,r+s) \\ \leq & \bigwedge_{z\in X} \bigwedge_{r,s>0} A(z) + m(z,x,s) + m(x,y,r) \\ \leq & A(z) + m(z,x,s) + m(x,y,r). \end{aligned}$$

Hence, $\overline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A)) = \underline{\mathrm{Apr}}_m(A)$.

(2) By (L1), we only need to show $\underline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A)) \geq \overline{\mathrm{Apr}}_m(A)$. Let $x \in X$. Then

$$\begin{aligned} & \underline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A))(x) \\ = & \bigwedge_{y\in X} \bigwedge_{r>0} \overline{\mathrm{Apr}}_m(A)(y) + m(y,x,r) \\ = & \bigwedge_{y\in X} \bigwedge_{r>0} [\bigvee_{z\in X} \bigvee_{m>0} A(z) - m(y,z,m)] + m(y,x,r) \\ = & \bigwedge_{y\in X} \bigwedge_{r>0} \bigvee_{z\in X} \bigvee_{m>0} A(z) - m(y,z,m) + m(y,x,r). \end{aligned}$$

We need to prove that

$$\bigwedge_{y\in X}\bigwedge_{r>0}\bigvee_{z\in X}\bigvee_{m>0} A(z)-m(y,z,m)+m(y,x,r)\geq \bigvee_{z\in X}\bigvee_{s>0} A(z)-m(x,z,s),$$

which is equivalent to that, for every $y,z\in X$ and every $r,s>0$,

$$\bigvee_{z\in X}\bigvee_{m>0} A(z)-m(y,z,m)\geq A(z)-m(x,z,s)-m(y,x,r).$$

In fact,

$$\begin{array}{ll} & \bigvee_{z\in X}\bigvee_{m>0} A(z)-m(y,z,m) \\ \geq & \bigvee_{z\in X}\bigvee_{r,s>0} A(z)-m(y,z,r+s) \\ \geq & \bigvee_{z\in X}\bigvee_{r,s>0} A(z)-(m(x,z,s)+m(y,x,r)) \\ \geq & A(z)-(m(x,z,s)+m(y,x,r)) \\ = & A(z)-m(x,z,s)-m(y,x,r). \end{array}$$

Hence, $\underline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A))=\overline{\mathrm{Apr}}_m(A)$. □

Since $([0,1],\oplus)$ is a standard MV-algebra, We can give a strict logical description of fuzzy upper and lower rough approximation operators.

Theorem 3.6. *Let (X,m) be a Morsi fuzzy hemimetric space. Then for every $A\in\mathcal{F}(X)$ and every $x\in X$,*

(1) $\overline{\mathrm{Apr}}_m(A)(x)=\bigvee_{y\in X}\bigvee_{r>0}[A(y)\ominus m(x,y,r)]$;

(2) $\underline{\mathrm{Apr}}_m(A)(x)=\bigwedge_{y\in X}\bigwedge_{r>0}[A(y)\oplus m(y,x,r)]$.

Proof. (1) For every $x\in X$,

$$\begin{array}{ll} & \bigvee_{y\in X}\bigvee_{r>0}[A(y)\ominus m(x,y,r)] \\ = & \bigvee_{y\in X}\bigvee_{r>0}\max\{A(y)-m(x,y,r),0\} \\ = & \max\{\bigvee_{y\in X}\bigvee_{r>0} A(y)-m(x,y,r),0\} \\ = & \max\{\overline{\mathrm{Apr}}_m(A)(x),0\} \\ = & \overline{\mathrm{Apr}}_m(A)(x). \end{array}$$

(2) For every $x\in X$,

$$\begin{array}{ll} & \bigwedge_{y\in X}\bigwedge_{r>0}[A(y)\oplus m(y,x,r)] \\ = & \bigwedge_{y\in X}\bigwedge_{r>0}\min\{A(y)+m(y,x,r),1\} \\ = & \min\{\bigwedge_{y\in X}\bigwedge_{r>0} A(y)+m(y,x,r),1\} \\ = & \min\{\underline{\mathrm{Apr}}_m(A)(x),1\} \\ = & \underline{\mathrm{Apr}}_m(A)(x). \end{array}$$

□

Further properties of fuzzy upper and lower rough approximation operators can be studied by the logical meaning of the operations. As is known to us, $(\otimes,\rightarrow)$ forms a Galois adjoint pair on the poset $([0,1],\leq)$. For the operations $\oplus,\ominus$, we have $a\ominus b\leq c\Longleftrightarrow a\leq b\oplus c$ $(\forall a,b,c\in[0,1])$. If we write $a\ominus b=:b\rightsquigarrow a$ and let $\preceq=\leq^{op}$, then $b\oplus c\preceq a\Longleftrightarrow c\preceq b\rightsquigarrow a$ $(\forall a,b,c\in[0,1])$. Then $(\oplus,\rightsquigarrow)$ also forms a Galois adjoint pair on $([0,1],\preceq)$. Then by results in lattice theory [3], for every $a\in[0,1]$, the operator $a\oplus(-)$ preserves arbitrary joins and that $a\rightsquigarrow(-)$ preserves arbitrary meets in $([0,1],\preceq)$, and therefore, $a\oplus(-)$ preserves arbitrary meets and $b\rightsquigarrow(-)$ preserves arbitrary joins in $([0,1],\leq)$.

Theorem 3.7. *Let (X,m) be a Morsi fuzzy hemimetric space. Then for all $A\in\mathcal{F}(X)$ and $a\in[0,1]$, we have*

(U5) $\overline{\mathrm{Apr}}_m(a\otimes A)=a\otimes\overline{\mathrm{Apr}}_m(A)$;

(U6) $\overline{\mathrm{Apr}}_m(a\rightsquigarrow A)=a\rightsquigarrow\overline{\mathrm{Apr}}_m(A)$;

(L5) $\underline{\mathrm{Apr}}_m(a \oplus A) = a \oplus \underline{\mathrm{Apr}}_m(A)$;
(L6) $\underline{\mathrm{Apr}}_m(a \to A) = a \to \underline{\mathrm{Apr}}_m(A)$.

Proof. The operations $a \otimes (-)$ and $a \rightsquigarrow (-)$ preserve joins, while those of $a \oplus (-)$ and $a \to (-)$ preserve meets. Based on Theorem 3.5, (U5), (U6) and (L6) can be obtained from Proposition 2.4(1–3) respectively, and (L5) is guaranteed by the associative law of the operation $\oplus$. □

4. Definable fuzzy sets

A basic idea of rough set theory is the approximations of undefinable sets by definable sets via certain formulas. Semantically, the formulas in the language are considered the intensions of the concepts. Corresponding to a formula, its meaning set is assigned to the set of objects satisfying the formula, which is considered as the extension of the concept. A set of objects is therefore definable if it is the meaning set of a formula in the descriptive language, otherwise it is undefinable. Another explanation is that definable sets of a formula are exactly its fixed points, which are completely definite (or can be defined) by this formula and then form the basic information of the approximation space [4, 31].

Let (X, m) be a Morsi fuzzy hemimetric space. For $A \in \mathcal{F}(X)$, if $\overline{\mathrm{Apr}}_m(A) = A$ (resp., $\underline{\mathrm{Apr}}_m(A) = A$), then A is called *upper* (resp., *lower*) *definable sets*. We firstly show that upper definability and lower definability are exactly the same thing.

Theorem 4.1. *Let (X, m) be a Morsi fuzzy hemimetric space. For every $A \in \mathcal{F}(X)$, the following statements are equivalent:*
(1) *A is upper definable;*
(2) *$A(x) - A(y) \le m(y, x, r)$ for all $x, y \in X$ and all $r > 0$;*
(3) *A is lower definable.*

Proof. The equivalence between (1) and (2):

$$\begin{aligned} & A \text{ is upper definable} \\ \iff & \overline{\mathrm{Apr}}_m(A) \le A \\ \iff & \textstyle\bigvee_{x\in X} \bigvee_{r>0} A(x) - m(y, x, r) \le A(y) \ (\forall y \in X) \\ \iff & A(x) - A(y) \le m(y, x, r) \ (\forall x, y \in X, \forall r \in (0, +\infty)). \end{aligned}$$

The equivalence between (2) and (3):

$$\begin{aligned} & A \text{ is lower definable} \\ \iff & A \le \underline{\mathrm{Apr}}_m(A), \\ \iff & A(x) \le \textstyle\bigwedge_{y\in X} \bigwedge_{r>0} A(y) + m(y, x, r) \ (\forall x \in X), \\ \iff & A(x) - A(y) \le m(y, x, r) \ (\forall x, y \in X, \forall r \in (0, +\infty)). \end{aligned}$$
□

If $A \in \mathcal{F}(X)$ satisfies the conditions of Theorem 4.1, then we call it a *definable set* of (X, m) and use $D\mathcal{F}(X)$ to denote the family of all definable sets of (X, m).

Remark 4.2. (1) Let (X, m) be a Morsi fuzzy hemimetric space and $A \in \mathcal{F}(X)$. Then $\overline{\mathrm{Apr}}_m(A)$ is the smallest definable set over A, and $\underline{\mathrm{Apr}}_m(A)$ is the largest definable set below A.

(2) Theorem 3.4 can be considered as a corollary of Theorem 4.1. By (U4), we know that $\overline{\mathrm{Apr}}_m(A)$ is definable and then $\underline{\mathrm{Apr}}_m(\overline{\mathrm{Apr}}_m(A)) = \overline{\mathrm{Apr}}_m(A)$; similarly, by (L4), we know that $\underline{\mathrm{Apr}}_m(A)$ is definable and then $\overline{\mathrm{Apr}}_m(\underline{\mathrm{Apr}}_m(A)) = \underline{\mathrm{Apr}}_m(A)$.

Now we observe the topological properties of $D\mathcal{F}(X)$.

Theorem 4.3. *Let (X, m) be a Morsi fuzzy hemimetric space. The family $D\mathcal{F}(X)$ has the following properties:*
(DF1) $a_X \in D\mathcal{F}(X)$ $(\forall a \in [0, 1])$;

(DF2) $\bigvee_i A_i,\ \bigwedge_i A_i \in D\mathcal{F}(X)\ (\forall\{A_i \mid i \in I\} \subseteq D\mathcal{F}(X))$;
(DF3) $a \oplus A, a \rightsquigarrow A, a \otimes A, a \to A \in D\mathcal{F}(X)\ (\forall a \in [0,1], \forall A \in D\mathcal{F}(X))$.

Proof. We will use Theorem 4.1(2) to prove it. Let $x, y \in X$.
(DF1) For every $a \in [0,1]$,

$$a_X(x) - a_X(y) = a - a = 0 \leq m(y,x,r).$$

Hence, $a_X \in D\mathcal{F}(X)$.
(DF2) For every $\{A_i \mid i \in I\} \subseteq D\mathcal{F}(X)$, we have

$$\begin{aligned} & \bigvee_i A_i(x) - \bigvee_i A_i(y) \\ = & \bigvee_i [A_i(x) - \bigvee_j A_j(y)] \\ \leq & \bigvee_i [A_i(x) - A_i(y)] \\ \leq & m(y,x,r) \end{aligned}$$

and

$$\begin{aligned} & \bigwedge_i A_i(x) - \bigwedge_i A_i(y) \\ = & \bigvee_i [(\bigwedge_j A_j(x)) - A_i(y)] \\ \leq & \bigvee_i [A_i(x) - A_i(y)] \\ \leq & m(y,x,r). \end{aligned}$$

Hence, $\bigvee_i A_i,\ \bigwedge_i A_i \in D\mathcal{F}(X)$.
(DF3) Suppose $A \in \mathcal{F}(X)$ is definable, that is, $A(y) + m(y,x,r) \geq A(x)\ (\forall x, y \in X, \forall r \in (0,+\infty))$. Let $a \in [0,1]$, $x, y \in X$ and $r \in (0,+\infty)$. We have

$$\begin{aligned} & a \oplus A(y) + m(y,x,r) \\ = & \min\{a + A(y), 1\} + m(y,x,r) \\ = & \min\{a + A(y) + m(y,x,r), 1 + m(y,x,r)\} \\ \geq & \min\{a + A(x), 1\} \\ = & a \oplus A(x). \end{aligned}$$

$$\begin{aligned} & a \rightsquigarrow A(y) + m(y,x,r) \\ = & \max\{A(y) - a, 0\} + m(y,x,r) \\ = & \max\{A(y) + m(y,x,r) - a, m(y,x,r)\} \\ \geq & \max\{A(x) - a, 0\} \\ = & a \rightsquigarrow A(x). \end{aligned}$$

$$\begin{aligned} & a \otimes A(y) + m(y,x,r) \\ = & \max\{a + A(y) - 1, 0\} + m(y,x,r) \\ = & \max\{a + A(y) + m(y,x,r) - 1, m(y,x,r)\} \\ \geq & \max\{a + A(x) - 1, 0\} \\ = & a \otimes A(x). \end{aligned}$$

$$\begin{aligned} & a \to A(y) + m(y,x,r) \\ = & \min\{1 - a + A(y), 1\} + m(y,x,r) \\ = & \min\{1 - a + A(y) + m(y,x,r), 1 + m(y,x,r)\} \\ \geq & \min\{1 - a + A(x), 1\} \\ = & a \to A(x). \end{aligned}$$

Hence, $a \oplus A, a \rightsquigarrow A, a \otimes A, a \to A \in D\mathcal{F}(X)$. □

Remark 4.4. It is a routine to show that the interior operator and the closure operator of $D\mathcal{F}(X)$ are exactly the fuzzy lower rough approximation operator and the fuzzy upper rough approximation operator respectively.

5. Conclusion and future work

As a common generalization of hemimetrics and Morsi fuzzy metrics, Morsi fuzzy hemimetrics can be used to describe relations between data objects and thus rough set structure is constructed and studied in this paper. The fuzzy rough set model on Morsi fuzzy hemimetrics has some general properties of rough set model. It is shown that upper definable sets, lower definable sets and definable sets are equivalent. Definable sets form an Alexandrov fuzzy topology such that the upper and lower approximation operators are the closure and the interior operators respectively.

The following topics can be some potential work in the future:

1. Following [32, 33], apply the fuzzy rough set model of the Morsi fuzzy hemimetrics in this paper to find some applications in clustering analysis, image processing and data mining.

2. We can replace $(0,\infty)$ by a complete lattice L (with an s-norm $\oplus$) and replace $[0,1]$ by another one M (with an s-norm $\boxplus$). Then we can get a concept of Morsi LM-fuzzy hemimetric as a mapping $m : X \times X \times L \longrightarrow M$ satisfying

(LMM1) $m(x,x,r) = 0$;

(LMM2) $m(x,z,s \oplus t) \leq m(x,y,s) \boxplus m(y,z,t)$.

Then we can use Morsi LM-fuzzy hemimetrics to construct and study LM-fuzzy rough sets. In fact, The ideal of replacing $[0,1]$ by some lattices with a t-norm or an s-norm has been used in [10].

3. In fact, if we assume that $d(x,y) = \bigwedge\limits_{r>0} m(x,y,r)$, then it is easy to prove that d is a hemimetric on X. Since r can be considered as a parameter, we can define two new operators as

$$\overline{\mathrm{Apr}}^r_m(A)(x) = \bigvee_{y\in X} A(y) - m(x,y,r),$$

$$\underline{\mathrm{Apr}}^r_m(A)(x) = \bigwedge_{y\in X} A(y) + m(y,x,r).$$

Then it can be shown that $\overline{\mathrm{Apr}}_m = \bigvee_{r>0} \overline{\mathrm{Apr}}^r_m$, $\underline{\mathrm{Apr}}_m = \bigwedge_{r>0} \underline{\mathrm{Apr}}^r_m$. That is to say, we can study fuzzy rough set model based on Morsi fuzzy hemimetrics and its applications by a level-wise manner in the future.

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