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Research Article

Geometry of Mus-Gradient Metric

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Abstract
In this paper, we give some properties of Riemannian curvature tensors of Mus-gradient metric .i.e. we characterize the Riemannian curvature, the sectional curvature, the Ricci tensor, the Ricci curvature and the scalar curvature.

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1. Introduction

On a Riemannian manifold, we can define a deformation metric of Riemannian metric by a conformal deformation or by a non-conformal deformation. Let (M^m, g) be an m-dimensional Riemannian manifold. The conformal deformation of metric g is defined by

$$\tilde{g}(X,Y) = fg(X,Y),$$

for all vector fields X and Y on M, where $f : M \to]0,+\infty[$ is a strictly positive smooth function.

Benkartab and Cherif, in [1] have introduced a new class of Riemannian metric on a Riemannian manifold (M^m, g) defined by

$$\tilde{g}(X,Y) = g(X,Y) + X(f)Y(f),$$

for all vector fields X and Y on M, where f be a smooth function on M. They searched some properties of the Riemannian manifold with this metric. They presented a new example of non-harmonic biharmonic maps, and they characterized the biharmonicity of some curves on the Riemannian manifolds. Here, we also refer to the [2].

In this direction, the authors, Djaa and Zagane, in [3] defined another new class of Riemannian metric by

$$\tilde{g}(X,Y) = fg(X,Y) + X(f)Y(f),$$

for all vector fields X and Y on M, where f be strictly positive smooth function on M. $\tilde{g}$ is called the Mus-gradient metric. They investigated the Levi-Civita connection of this metric and they studied some properties of harmonicity with respect to the Mus-gradient metric. They also considered, in [4] a Riemannian metric defined by

$$G(X,Y) = fg(X,Y) + g(\xi,X)g(\xi,Y),$$

for all vector fields X, Y and ξ on M such that $g(\xi,\xi) = 1$ and $\xi(f) = 0$, where f be strictly positive smooth function on M. $\tilde{g}$ is called the semi-conformal deformation of metric. They investigated the Levi-Civita connection of this metric and investigated the Riemannian curvature, the sectional curvature and the scalar curvature. In the last section, they studied some class of proper biharmonic maps.

In addition to our previous work [3], in this note, we introduce some properties of Riemannian curvature tensors of Mus-gradient metric. In here, we study the curvature tensor (Theorem 6 and Corollary 7) and characterize the sectional curvature (Theorem 8, Corollary 9 and Corollary 12), the Ricci tensor (Proposition 11 and Proposition 13), the Ricci curvature (Proposition 14 and Proposition 16) and the scalar curvature (Theorem 17 and Proposition 19).

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and let $\Im^r_s(M)$ be the module over $C^\infty(M)$ of C^∞ tensor fields of type (r,s).

Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g which is characterized by the Koszul formula

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z]), \tag{1}$$

for all $X,Y,Z \in \Im^1_0(M)$.

By R, Ric and $Ricci$ we denote respectively the Riemannian curvature tensor, the Ricci curvature and the Ricci tensor of (M^m, g). Thus R, Ric and $Ricci$ are defined by

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ Ricci(X) &= \sum_{i=1}^{m} R(X,E_i)E_i, \\ Ric(X,Y) &= g(Ricci(X),Y), \end{aligned}$$

for all $X,Y,Z \in \Im^1_0(M)$, where $(E_1,\ldots,E_m)$ be a local orthonormal frame on M.

Let f be a smooth function on M, the gradient of f, is defined by

$$g(grad f, X) = X(f),$$

the Hessian of f, is defined by

$$Hess_f(X,Y) = g(\nabla_X grad f, Y) = X(Y(f)) - (\nabla_X Y)(f), \tag{2}$$

for all $X,Y \in \Im^1_0(M)$.

$X \in \Im^1_0(M)$ is a Killing vector field if

$$g(\nabla_Y X, Z) + g(\nabla_Y Z, X) = 0,$$

for all $Y,Z \in \Im^1_0(M)$. One can easily see that $f \in C^\infty(M)$ is a Killing potential if $grad f$ is a Killing vector field. Then $grad f$ is called a Killing gradient, [5]. $f \in C^\infty(M)$ is Killing potential if and only if $Hess_f = 0$ or equivalently $\nabla_X grad f = 0$, for all $X \in \Im^1_0(M)$, [5].

3. Mus-gradient metric

Definition 1. *(see [3]) Let (M^m, g) be be a Riemannian manifold and $f : M \to]0,+\infty[$ be a strictly positive smooth function. We define the Mus-gradient metric on M noted $\tilde{g}$ by*

$$\tilde{g}(X,Y)_x = f(x)g(X,Y)_x + X_x(f)Y_x(f),$$

where $x \in M$ and $X,Y \in \Im^1_0(M)$, f is called twisting function.
In the following, we consider $\|grad f\| = 1$, where $\|.\|$ denote the norm with respect to (M^m, g). (There are other works on deformation similar to deformation in the article [3] on tangent beam see for example, [6, 7, 8]).

We shall calculate the Levi-Civita connection $\widetilde{\nabla}$ of $(M^m, \tilde{g})$.

Lemma 2. *(see [3]) Let $grad f$ (resp $\widetilde{grad} f$) denote the gradient of f with respect to g (resp $\tilde{g}$). Then we have*

$$\widetilde{grad} f = \frac{1}{f+1} grad f. \tag{3}$$

Theorem 3. *(see [3]) Let (M^m, g) be a Riemannian manifold. The Levi-Civita connection $\widetilde{\nabla}$ of $(M^m, \tilde{g})$ is given by*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{X(f)}{2f} Y + \frac{Y(f)}{2f} X + \left(\frac{Hess_f(X,Y)}{f+1} - \frac{X(f)Y(f)}{f(f+1)} - \frac{g(X,Y)}{2(f+1)} \right) grad f, \tag{4}$$

for all $X, Y \in \Im_0^1(M)$, where ∇ denote the Levi-Civita connection of (M^m, g) and $Hess_f$ is the Hessian of f with respect to g.

Lemma 4. *(see [3]) Let (M^m, g) be a Riemannian manifold. Then for all $X \in \Im_0^1(M)$, we have*

$$\widetilde{\nabla}_X grad f = \nabla_X grad f + \frac{1}{2f} X - \frac{X(f)}{2f(f+1)} grad f.$$

Lemma 5. *Let (M^m, g) be a Riemannian manifold. Then for all $X, Y \in \Im_0^1(M)$, we have*

$$\widetilde{Hess}_f(X,Y) = \frac{f}{f+1} Hess_f(X,Y) - \frac{1}{f+1} X(f)Y(f) + \frac{1}{2(f+1)} g(X,Y).$$

Proof. Using the Theorem 3, we have

$$\begin{aligned} \widetilde{Hess}_f(X,Y) &= X(Y(f)) - (\widetilde{\nabla}_X Y)(f) \\ &= X(Y(f)) - (\nabla_X Y)(f) - \frac{X(f)Y(f)}{f} - \frac{Hess_f(X,Y)}{f+1} + \frac{X(f)Y(f)}{f(f+1)} + \frac{g(X,Y)}{2(f+1)} \\ &= \frac{f}{f+1} Hess_f(X,Y) - \frac{1}{f+1} X(f)Y(f) + \frac{g(X,Y)}{2(f+1)}. \end{aligned}$$

∎

4. Curvatures of Mus-gradient metric

We shall focus on the Riemannian curvature tensor of $(M^m, \tilde{g})$.

Theorem 6. *Let (M^m, g) be a Riemannian manifold. The Riemannian curvature tensor $\widetilde{R}$ of $(M^m, \tilde{g})$ is given by*

$$\begin{aligned} \widetilde{R}(X,Y)Z = & \left(\frac{Hess_f(Y,Z)}{f+1} - \frac{Y(f)Z(f)}{f(f+1)} - \frac{g(Y,Z)}{2(f+1)} \right) \nabla_X grad f - \left(\frac{Hess_f(X,Z)}{f+1} - \frac{X(f)Z(f)}{f(f+1)} - \frac{g(X,Z)}{2(f+1)} \right) \nabla_Y grad f \\ & - \left(\frac{Hess_f(Y,Z)}{2(f+1)} - \frac{(3f+1)Y(f)Z(f)}{4f^2(f+1)} + \frac{g(Y,Z)}{4f(f+1)} \right) X + \left(\frac{Hess_f(X,Z)}{2(f+1)} - \frac{(3f+1)X(f)Z(f)}{4f^2(f+1)} + \frac{g(X,Z)}{4f(f+1)} \right) Y \\ & + \left(\frac{-X(f)Hess_f(Y,Z)}{2(f+1)^2} + \frac{Y(f)Hess_f(X,Z)}{2(f+1)^2} + \frac{(3f+2)X(f)g(Y,Z)}{4f(f+1)^2} - \frac{(3f+2)Y(f)g(X,Z)}{4f(f+1)^2} \right. \\ & \left. - \frac{g(R(X,Y)Z, grad f)}{f+1} \right) grad f + R(X,Y)Z, \end{aligned} \tag{5}$$

for all $X, Y, Z \in \Im_0^1(M)$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^m, g).

Proof. For all $X, Y, Z \in \Im_0^1(M)$, the Riemannian curvature tensor is as follows:

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z.$$

From the Theorem 3, we obtain

i.

$$\begin{aligned} \widetilde{\nabla}_X \widetilde{\nabla}_Y Z &= \widetilde{\nabla}_X \left(\nabla_Y Z + \frac{Y(f)}{2f} Z + \frac{Z(f)}{2f} Y + \left(\frac{Hess_f(Y,Z)}{f+1} - \frac{Y(f)Z(f)}{f(f+1)} - \frac{g(Y,Z)}{2(f+1)} \right) grad f \right) \\ &= \widetilde{\nabla}_X (\nabla_Y Z) + \widetilde{\nabla}_X \left(\frac{Y(f)}{2f} Z \right) + \widetilde{\nabla}_X \left(\frac{Z(f)}{f+1} Y \right) + \widetilde{\nabla}_X \left(\frac{Hess_f(Y,Z)}{f+1} grad f \right) - \widetilde{\nabla}_X \left(\frac{Y(f)Z(f)}{f(f+1)} grad f \right) \\ & \quad - \widetilde{\nabla}_X \left(\frac{g(Y,Z)}{2(f+1)} grad f \right). \end{aligned}$$

Direct computations give

$$\widetilde{\nabla}_X(\nabla_Y Z) = \nabla_X\nabla_Y Z + \frac{X(f)}{2f}\nabla_Y Z + \frac{(\nabla_Y Z)(f)}{2f}X + \left(\frac{Hess_f(X,\nabla_Y Z)}{f+1} - \frac{X(f)(\nabla_Y Z)(f)}{f(f+1)} - \frac{g(X,\nabla_Y Z)}{2(f+1)}\right) grad\, f,$$

$$\begin{aligned}\widetilde{\nabla}_X\left(\frac{Y(f)}{2f}Z\right) &= \frac{Y(f)}{2f}\nabla_X Z + \frac{Y(f)Z(f)}{4f^2}X + \left(\frac{X(Y(f))}{2f} - \frac{X(f)Y(f)}{4f^2}\right)Z \\ &+\left(\frac{Y(f)Hess_f(X,Z)}{2f(f+1)} - \frac{X(f)Y(f)Z(f)}{2f^2(f+1)} - \frac{Y(f)g(X,Z)}{4f(f+1)}\right) grad\, f,\end{aligned}$$

$$\begin{aligned}\widetilde{\nabla}_X\left(\frac{Z(f)}{2f}Y\right) &= \frac{Z(f)}{2f}\nabla_X Y + \frac{Z(f)Y(f)}{4f^2}X + \left(\frac{X(Z(f))}{2f} - \frac{X(f)Z(f)}{4f^2}\right)Y \\ &+\left(\frac{Z(f)Hess_f(X,Y)}{2f(f+1)} - \frac{X(f)Z(f)Y(f)}{2f^2(f+1)} - \frac{Z(f)g(X,Y)}{4f(f+1)}\right) grad\, f,\end{aligned}$$

$$\begin{aligned}\widetilde{\nabla}_X\left(\frac{Hess_f(Y,Z)}{f+1}grad\, f\right) &= \frac{Hess_f(Y,Z)}{f+1}\nabla_X grad\, f + \frac{Hess_f(Y,Z)}{2f(f+1)}X \\ &+\left(\frac{g(\nabla_X\nabla_Y grad\, f, Z)}{f+1} + \frac{Hess_f(Y,\nabla_X Z)}{f+1} - \frac{(2f+1)X(f)Hess_f(Y,Z)}{2f(f+1)^2}\right) grad\, f,\end{aligned}$$

$$\begin{aligned}\widetilde{\nabla}_X\left(\frac{Y(f)Z(f)}{f(f+1)}grad\, f\right) &= \frac{Y(f)Z(f)}{f(f+1)}\nabla_X grad\, f + \frac{Y(f)Z(f)}{2f^2(f+1)}X \\ &+\left(\frac{X(Y(f))Z(f)}{f(f+1)} + \frac{Y(f)X(Z(f))}{f(f+1)} - \frac{(4f+3)X(f)Y(f)Z(f)}{2f^2(f+1)^2}\right) grad\, f,\end{aligned}$$

$$\begin{aligned}\widetilde{\nabla}_X\left(\frac{g(Y,Z)}{2(f+1)}grad\, f\right) &= \frac{g(Y,Z)}{2(f+1)}\nabla_X grad\, f + \frac{g(Y,Z)}{4f(f+1)}X \\ &+\left(\frac{g(\nabla_X Y,Z)}{2(f+1)} + \frac{g(Y,\nabla_X Z)}{2(f+1)} - \frac{(2f+1)X(f)g(Y,Z)}{4f(f+1)^2}\right) grad\, f.\end{aligned}$$

ii. With permutation of X by Y, we get $\widetilde{\nabla}_Y\widetilde{\nabla}_X Z$.

iii.

$$\widetilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + \frac{[X,Y](f)}{2f}Z + \frac{Z(f)}{2f}[X,Y] + \left(\frac{Hess_f([X,Y],Z)}{f+1} - \frac{[X,Y](f)Z(f)}{f(f+1)} - \frac{g([X,Y],Z)}{2(f+1)}\right) grad\, f.$$

■

Corollary 7. *Let (M^m, g) be a Riemannian manifold. If f is Killing potential (i.e. $\nabla grad\, f = 0$), the Riemannian curvature tensor $\widetilde{R}$ of $(M^m, \tilde{g})$ is given by*

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + \left(\frac{(3f+1)Y(f)Z(f)}{4f^2(f+1)} - \frac{g(Y,Z)}{4f(f+1)}\right)X - \left(\frac{(3f+1)X(f)Z(f)}{4f^2(f+1)} - \frac{g(X,Z)}{4f(f+1)}\right)Y \\ &+ \frac{3f+2}{4f(f+1)^2}\Big(X(f)g(Y,Z) - Y(f)g(X,Z)\Big) grad\, f,\end{aligned}$$

for all $X,Y,Z \in \Im_0^1(M)$, where R denotes the Riemannian curvature tensor of (M^m, g).

For $V,W \in \Im_0^1(M)$ and $x \in M$ such that V_x and W_x are linearly independent, then the sectional curvature of the plane spanned by V_x and W_x is given by [9]

$$\widetilde{K}(V,W) = \frac{\tilde{g}(\widetilde{R}(V,W)W,V)}{\tilde{g}(V,V)\tilde{g}(W,W) - \tilde{g}(V,W)^2}. \tag{6}$$

Theorem 8. *Let (M^m, g) be a Riemannian manifold. If K (resp., $\widetilde{K}$) denotes the sectional curvature of (M^m, g) (resp., $(M, \tilde{g})$), then we have*

$$\begin{aligned}\widetilde{K}(X,Y) &= \frac{1}{f+X(f)^2+Y(f)^2}\Big(K(X,Y) + \frac{Hess_f(X,X)Hess_f(Y,Y)}{f+1} - \frac{(f+2X(f)^2)Hess_f(Y,Y)}{f(f+1)} - \frac{Hess_f(X,Y)^2}{f+1} \\ &\quad - \frac{(f+2Y(f)^2)Hess_f(X,X)}{f(f+1)} + \frac{2X(f)Y(f)Hess_f(X,Y)}{f(f+1)} + \frac{(3f+1)(X(f)^2+Y(f)^2)}{4f^2(f+1)} - \frac{1}{4f(f+1)}\Big), \end{aligned} \tag{7}$$

for any $X,Y \in \Im_0^1(M)$. Here X,Y are orthonormal with respect to g.

Proof. From the formula (5), we have

$$\begin{aligned}\tilde{g}(\widetilde{R}(X,Y)Y,X) &= fg(\widetilde{R}(X,Y)Y,X) + g(\widetilde{R}(X,Y)Y, grad\, f)X(f) \\ &= f\Big(\frac{Hess_f(Y,Y)}{f+1} - \frac{Y(f)^2}{f(f+1)} - \frac{1}{2(f+1)}\Big)Hess_f(X,X) - f\Big(\frac{Hess_f(X,Y)}{f+1} - \frac{X(f)Y(f)}{f(f+1)}\Big)Hess_f(X,Y) \\ &\quad - f\Big(\frac{Hess_f(Y,Y)}{2(f+1)} - \frac{(3f+1)Y(f)^2}{4f^2(f+1)} + \frac{1}{4f(f+1)}\Big) + f\Big(\frac{-X(f)Hess_f(Y,Y)}{2(f+1)^2} + \frac{Y(f)Hess_f(X,Y)}{2(f+1)^2} \\ &\quad + \frac{(3f+2)X(f)}{4f(f+1)^2} - \frac{g(R(X,Y)Y, grad\, f)}{f+1}\Big)X(f) + \big(\frac{Hess_f(X,Y)}{2(f+1)} - \frac{(3f+1)X(f)Y(f)}{4f^2(f+1)}\big)X(f)Y(f) \\ &\quad - \Big(\frac{Hess_f(Y,Y)}{2(f+1)} - \frac{(3f+1)Y(f)^2}{4f^2(f+1)} + \frac{1}{4f(f+1)}\Big)X(f)^2 + fg(R(X,Y)Y,X) + g(R(X,Y)Y,X)X(f) \\ &\quad + \Big(\frac{-X(f)Hess_f(Y,Y)}{2(f+1)^2} + \frac{Y(f)Hess_f(X,Y)}{2(f+1)^2} + \frac{(3f+2)X(f)}{4f(f+1)^2} - \frac{g(R(X,Y)Y, grad\, f)}{f+1}\Big)X(f).\end{aligned}$$

After simplifying the last equality, we get

$$\begin{aligned}\tilde{g}(\widetilde{R}(X,Y)Y,X) &= fK(X,Y) + \frac{fHess_f(X,X)Hess_f(Y,Y)}{f+1} - \frac{(f+2Y(f)^2)Hess_f(X,X)}{f+1} - \frac{(f+2X(f)^2)Hess_f(Y,Y)}{f+1} \\ &\quad - \frac{fHess_f(X,Y)^2}{f+1} + \frac{2X(f)Y(f)Hess_f(X,Y)}{f+1} + \frac{(3f+1)(X(f)^2+Y(f)^2)}{4f(f+1)} - \frac{1}{4(f+1)},\end{aligned}$$

and we have

$$\begin{aligned}\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2 &= \big(f+X(f)^2\big)\big(f+Y(f)^2\big) - \big(X(f)Y(f)\big)^2 \\ &= f(f+X(f)^2+Y(f)^2).\end{aligned}$$

The division of $\tilde{g}(\widetilde{R}(X,Y)Y,X)$ by $\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2$ gives the formula (7). ■

Corollary 9. *Let (M^m, g) be a Riemannian manifold. If f is Killing potential, then the sectional curvature $\widetilde{K}$ of $(M^m, \tilde{g})$ is given by*

$$\widetilde{K}(X,Y) = \frac{1}{f+X(f)^2+Y(f)^2}\Big(K(X,Y) - \frac{1}{4f(f+1)} + \frac{3f+1}{4f^2(f+1)}(X(f)^2+Y(f)^2)\Big), \tag{8}$$

for any orthonormal vector fields X,Y on (M^m, g).

Remark 10. *Let $\{E_i\}_{i=\overline{1,m}}$, such that $E_1 = grad\, f$ be a local orthonormal frame on (M^m, g), then we define the orthonormal vector fields*

$$\widetilde{E}_1 = \frac{1}{\sqrt{f+1}}E_1, \widetilde{E}_i = \frac{1}{\sqrt{f}}E_i,\ i = \overline{2,m}, \tag{9}$$

then $\{\widetilde{E}_i\}_{i=\overline{1,m}}$ is a local orthonormal frame on $(M^m, \tilde{g})$.

Proposition 11. *Let (M^m, g) be a Riemannian manifold. If Ricci (resp. $\widetilde{Ricci}$) denote the Ricci tensor of (M^m, g) (resp., $(M, \tilde{g})$), then we have*

$$\begin{aligned}
\widetilde{Ricci}(X) &= \frac{1}{f}Ricci(X) - \left(\frac{\Delta(f)}{2f(f+1)} + \frac{(m-4)f+m-3}{4f^2(f+1)^2}\right)X - \left(\frac{\Delta(f)X(f)}{2f(f+1)^2} - \frac{(m-2)(3f+2)X(f)}{4f^2(f+1)^2}\right)grad\, f \\
&\quad - \frac{1}{f(f+1)}\big(R(X, grad\, f)grad\, f + \nabla_{(\nabla_X grad\, f)} grad\, f + Ric(X, grad\, f)grad\, f\big) \\
&\quad + \left(\frac{\Delta(f)}{f(f+1)} - \frac{(m-2)f+m-1}{2f(f+1)^2}\right)\nabla_X grad\, f, \qquad (10)
\end{aligned}$$

for any $X \in \Im_0^1(M)$.

Proof. Let $\{E_i\}_{i=\overline{1,m}}$, such that $E_1 = grad\, f$ be a local orthonormal frame on (M^m, g) and $\{\widetilde{E}_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on $(M^m, \tilde{g})$ defined by (9). By the definition of Ricci tensor, we have

$$\begin{aligned}
\widetilde{Ricci}(X) &= \sum_{i=1}^{m} \widetilde{R}(X, \widetilde{E}_i)\widetilde{E}_i \\
&= \widetilde{R}(X, \frac{1}{\sqrt{f+1}} grad\, f)\frac{1}{\sqrt{f+1}} grad\, f + \sum_{i=2}^{m} \widetilde{R}(X, \frac{1}{\sqrt{f}}E_i)\frac{1}{\sqrt{f}}E_i \\
&= \frac{1}{f+1}\widetilde{R}(X, grad\, f)grad\, f + \frac{1}{f}\sum_{i=2}^{m}\widetilde{R}(X, E_i)E_i \\
&= \frac{1}{f+1}\Big(\big(\frac{-1}{f(f+1)} - \frac{1}{2(f+1)}\big)\nabla_X grad\, f - \big(\frac{-(3f+1)}{4f^2(f+1)} + \frac{1}{4f(f+1)}\big)X + R(X, grad\, f)grad\, f \\
&\quad + \big(\frac{-(3f+1)X(f)}{4f^2(f+1)} + \frac{X(f)}{4f(f+1)}\big)grad\, f\Big) + \frac{1}{f}\sum_{i=2}^{m}\Big(\big(\frac{Hess_f(E_i, E_i)}{f+1} - \frac{1}{2(f+1)}\big)\nabla_X grad\, f \\
&\quad - \big(\frac{Hess_f(X, E_i)}{f+1} - \frac{g(X, E_i)}{2(f+1)}\big)\nabla_{E_i} grad\, f - \big(\frac{Hess_f(E_i, E_i)}{2(f+1)} + \frac{1}{4f(f+1)}\big)X \\
&\quad + \big(\frac{Hess_f(X, E_i)}{2(f+1)} + \frac{g(X, E_i)}{4f(f+1)}\big)E_i + \big(\frac{-X(f)Hess_f(E_i, E_i)}{2(f+1)^2} + \frac{(3f+2)X(f)}{4f(f+1)^2} \\
&\quad - \frac{g(R(X, E_i)E_i, grad\, f)}{f+1}\big)grad\, f + R(X, E_i)E_i\Big),
\end{aligned}$$

and using

$$\sum_{i=2}^{m} Hess_f(E_i, E_i) = \Delta(f),$$

$$\sum_{i=2}^{m} Hess_f(X, E_i)E_i = \nabla_X grad\, f,$$

$$\sum_{i=2}^{m} g(X, E_i)E_i = X - X(f)grad\, f,$$

$$\sum_{i=2}^{m} R(X, E_i)E_i = Ricci(X) - R(X, grad\, f)grad\, f,$$

we get the formula (10). ■

Corollary 12. *Let (M^m, g) be a Riemannian manifold. If f is Killing potential, the Ricci tensor $\widetilde{Ricci}$ of $(M^m, \tilde{g})$ is given by*

$$\widetilde{Ricci}(X) = \frac{1}{f}Ricci(X) - \frac{(m-4)f+m-3}{4f^2(f+1)^2}X + \frac{(m-2)(3f+2)X(f)}{4f^2(f+1)^2}grad\, f, \qquad (11)$$

for any $X \in \Im_0^1(M)$.

Proposition 13. *Let (M^m, g) be a Riemannian manifold of constant sectional curvature λ, the Ricci tensor $\widetilde{Ricci}$ of $(M^m, \tilde{g})$ is given by*

$$\begin{aligned}\widetilde{Ricci}(X) &= \Big(\frac{\lambda((m-1)f+m-2)}{f(f+1)} + \frac{-\Delta(f)}{2f(f+1)} - \frac{(m-4)f+m-3}{4f^2(f+1)^2}\Big)X - \frac{1}{f(f+1)}\nabla_{(\nabla_X grad f)} grad f \\ &\quad -\Big(\frac{(m-2)\lambda}{f(f+1)} + \frac{\Delta(f)}{2f(f+1)^2} + \frac{(m-2)(3f+2)}{4f^2(f+1)^2}\Big)X(f) grad f \\ &\quad +\Big(\frac{\Delta(f)}{f(f+1)} - \frac{(m-2)f+m-1}{2f(f+1)^2}\Big)\nabla_X grad f, \qquad (12)\end{aligned}$$

for any $X \in \Im_0^1(M)$.

Proof. Taking account that for all $X, Y, Z \in \Im_0^1(M)$

$$R(X,Y)Z = \lambda\big(g(Y,Z)X - g(X,Z)Y\big),$$

we obtain

$$\begin{aligned} R(X, grad f) grad f &= \lambda X - \lambda X(f) grad f, \\ Ricci(X) &= (m-1)\lambda X, \\ Ric(X, grad f) &= (m-1)\lambda X(f). \end{aligned}$$

This completes the proof. ∎

Proposition 14. *Let (M^m, g) be a Riemannian manifold. If Ric (resp. $\widetilde{Ric}$) denote the Ricci curvature of (M^m, g) (resp., $(M, \tilde{g})$), then we have*

$$\begin{aligned}\widetilde{Ric}(X,Y) &= Ric(X,Y) - \Big(\frac{\Delta(f)}{2(f+1)} + \frac{(m-4)f+m-3}{4f(f+1)^2}\Big)g(X,Y) + \Big(\frac{\Delta(f)}{f+1} - \frac{(m-2)f+m-1}{2(f+1)^2}\Big)Hess_f(X,Y) \\ &\quad -\frac{1}{f+1}\Big(g(R(X, grad f) grad f, Y) + g(\nabla_X grad f, \nabla_Y grad f)\Big) - \frac{f+2}{2f(f+1)}X(f)Y(f)\Delta(f) \\ &\quad +\frac{(3m-6)f^2 + (4m-6)f + m-1}{4f^2(f+1)^2}X(f)Y(f), \qquad (13)\end{aligned}$$

for any $X, Y \in \Im_0^1(M)$.

Proof. From definition of the Ricci curvature, we have

$$\begin{aligned}\widetilde{Ric}(X,Y) &= \tilde{g}(\widetilde{Ricci}(X), Y) \\ &= f g(\widetilde{Ricci}(X), Y) + g(\widetilde{Ricci}(X), grad f)Y(f). \qquad (14)\end{aligned}$$

Using (10), we have

$$\begin{aligned} f g(\widetilde{Ricci}(X), Y) &= Ric(X,Y) - \Big(\frac{\Delta(f)}{2(f+1)} + \frac{(m-4)f+m-3}{4f(f+1)^2}\Big)g(X,Y) - \frac{1}{f+1}g(R(X, grad f) grad f, Y) \\ &\quad -\frac{1}{f+1}g(\nabla_X grad f, \nabla_Y grad f) + \Big(\frac{\Delta(f)}{f+1} - \frac{(m-2)f+m-1}{2(f+1)^2}\Big)Hess_f(X,Y) \\ &\quad -\frac{\Delta(f)}{2(f+1)}X(f)Y(f) + \frac{(m-2)(3f+2)}{4f(f+1)^2}X(f)Y(f) - \frac{1}{f+1}Y(f)Ric(X, grad f), \qquad (15)\end{aligned}$$

and

$$g(\widetilde{Ricci}(X), grad f)Y(f) = \frac{1}{f+1}Y(f)Ric(X, grad f) - \frac{\Delta(f)}{f(f+1)}X(f)Y(f) + \frac{(m-1)(2f+1)}{4f^2(f+1)^2}X(f)Y(f). \qquad (16)$$

Using (15) and (16) and substituting them in (14) we find the formula (13). ∎

Corollary 15. *Let (M^m, g) be a Riemannian manifold. If f is Killing potential, then the Ricci curvature $\widetilde{Ric}$ of $(M^m, \tilde{g})$ is given by*

$$\widetilde{Ric}(X,Y) = Ric(X,Y) - \frac{(m-4)f+m-3}{4f(f+1)^2}g(X,Y) + \frac{(3m-6)f^2+(4m-6)f+m-1}{4f^2(f+1)^2}X(f)Y(f), \tag{17}$$

for any $X,Y \in \Im_0^1(M)$.

Proposition 16. *Let (M^m, g) be a Riemannian manifold of constant sectional curvature λ, the Ricci curvature $\widetilde{Ric}$ of $(M^m, \tilde{g})$ is given by*

$$\begin{aligned}\widetilde{Ric}(X,Y) \quad = \quad & \Big(\frac{((m-1)f+m-2)\lambda}{f+1} - \frac{\Delta(f)}{2(f+1)} + \frac{(m-4)f+m-3}{4f(f+1)^2}\Big)g(X,Y) - \frac{1}{f+1}g(\nabla_X grad\, f, \nabla_Y grad\, f)\\ & + \Big(\frac{\lambda}{f+1} - \frac{(f+2)\Delta(f)}{2f(f+1)} + \frac{(3m-6)f^2+(4m-6)f+m-1}{4f^2(f+1)^2}\Big)X(f)Y(f)\\ & + \left(\frac{\Delta(f)}{f+1} - \frac{(m-2)f+m-1}{2(f+1)^2}\right)Hess_f(X,Y),\end{aligned} \tag{18}$$

for any $X,Y \in \Im_0^1(M)$.

Proof. Taking account that for all $X,Y,Z \in \Im_0^1(M)$

$$R(X,Y)Z = \lambda\big(g(Y,Z)X - g(X,Z)Y\big),$$

we obtain

$$\begin{aligned} R(X, grad\, f)grad\, f &= \lambda X - \lambda X(f) grad\, f,\\ Ric(X,Y) &= (m-1)\lambda\, g(X,Y).\end{aligned}$$

This completes the proof. ∎

Theorem 17. *Let (M^m, g) be a Riemannian manifold. If σ (resp., $\widetilde{\sigma}$) denote the scalar curvature of (M^m, g) (resp., $(M, \tilde{g})$), then we have*

$$\begin{aligned}\widetilde{\sigma} \quad = \quad & \frac{1}{f}\sigma - \frac{2}{f(f+1)}Ric(grad\, f, grad\, f) - \frac{f^2+(2m+2)(f+1)}{2f(f+1)^2}\Delta(f) + \frac{1}{f(f+1)}\Delta^2(f)\\ & - \frac{1}{f(f+1)}trace_g g(\nabla grad\, f, \nabla grad\, f) + \frac{(3m-6)f^2+(-m^2+11m-12)f-m^2+6m-5}{4f^2(f+1)^2},\end{aligned}$$

where, $trace_g g(\nabla grad\, f, \nabla grad\, f) = \sum_{i=1}^{m} g(\nabla_{E_i} grad\, f, \nabla_{E_i} grad\, f)$ and $\{E_i\}_{i=\overline{1,m}}$, such that $E_1 = grad\, f$ be a local orthonormal frame on (M^m, g).

Proof. Let $\{E_i\}_{i=\overline{1,m}}$, such that $E_1 = grad\, f$ be a local orthonormal frame on (M^m, g) and $\{\widetilde{E}_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on $(M^m, \tilde{g})$ defined by (9). By the definition of the scalar curvature, direct calculations give

$$\begin{aligned}
\widetilde{\sigma} &= \sum_{i=1}^{m} \widetilde{Ric}(\widetilde{E}_i, \widetilde{E}_i) \\
&= \frac{1}{f+1}\widetilde{Ric}(grad\, f, grad\, f) + \frac{1}{f}\sum_{i=2}^{m}\widetilde{Ric}(E_i, E_i) \\
&= \frac{1}{f+1}\Big(Ric(grad\, f, grad\, f) - \frac{\Delta(f)}{2(f+1)} - \frac{(m-4)f+m-3}{4f(f+1)^2} - \frac{(f+2)\Delta(f)}{2f(f+1)} \\
&\quad + \frac{(3m-6)f^2+(4m-6)f+m-1}{4f^2(f+1)^2}\Big) + \frac{1}{f}\sum_{i=2}^{m}\Big(Ric(E_i,E_i) - \frac{\Delta(f)}{2(f+1)} - \frac{(m-4)f+m-3}{4f(f+1)^2} \\
&\quad + \Big(\frac{\Delta(f)}{f+1} - \frac{(m-2)f+m-1}{2(f+1)^2}\Big)Hess_f(E_i,E_i) - \frac{1}{f+1}\Big(g(R(E_i, grad\, f)grad\, f, E_i) + g(\nabla_{E_i} grad\, f, \nabla_{E_i} grad\, f)\Big) \\
&\quad - \frac{f+2}{2f(f+1)}\Delta(f)E_i^2(f) + \frac{(3m-6)f^2+(4m-6)f+m-1}{4f^2(f+1)^2}E_i^2(f)\Big) \\
&= \frac{1}{f+1}Ric(grad\, f, grad\, f) - \frac{\Delta(f)}{f(f+1)} + \frac{(m-1)(2f+1)}{4f^2(f+1)^2} + \frac{1}{f}\sigma - \frac{f+2}{f(f+1)}Ric(grad\, f, grad\, f) \\
&\quad - \frac{f^2+2mf+2m}{2f(f+1)^2}\Delta(f) + \frac{\Delta^2(f)}{f(f+1)} - \frac{1}{f(f+1)}trace_g g(\nabla grad\, f, \nabla grad\, f) \\
&\quad + \frac{(3m-6)f^2+(-m^2+9m-10)f-m^2+5m-4}{4f^2(f+1)^2} \\
&= \frac{1}{f}\sigma - \frac{2}{f(f+1)}Ric(grad\, f, grad\, f) - \frac{f^2+(2m+2)(f+1)}{2f(f+1)^2}\Delta(f) + \frac{1}{f(f+1)}\Delta^2(f) \\
&\quad - \frac{1}{f(f+1)}trace_g g(\nabla grad\, f, \nabla grad\, f) + \frac{(3m-6)f^2+(-m^2+11m-12)f-m^2+6m-5}{4f^2(f+1)^2}.
\end{aligned}$$

■

Corollary 18. *Let (M^m, g) be a Riemannian manifold. If f is Killing potential, the scalar curvature $\widetilde{\sigma}$ of $(M^m, \tilde{g})$ is given by*

$$\widetilde{\sigma} = \frac{1}{f}\sigma + \frac{(3m-6)f^2+(-m^2+11m-12)f-m^2+6m-5}{4f^2(f+1)^2}.$$

Proposition 19. *Let (M^m, g) be a Riemannian manifold of constant sectional curvature λ. Then the scalar curvature $\widetilde{\sigma}$ of $(M^m, \tilde{g})$ is given by*

$$\begin{aligned}
\widetilde{\sigma} &= \frac{\lambda(m-1)(mf+m-2)}{f(f+1)} - \frac{f^2+(2m+2)(f+1)}{2f(f+1)^2}\Delta(f) + \frac{1}{f(f+1)}\Delta^2(f) \\
&\quad - \frac{1}{f(f+1)}trace_g g(\nabla grad\, f, \nabla grad\, f) + \frac{(3m-6)f^2+(-m^2+11m-12)f-m^2+6m-5}{4f^2(f+1)^2}. \qquad (19)
\end{aligned}$$

Proof. Taking account that for all $X, Y, Z \in \Im_0^1(M)$,

$$R(X,Y)Z = \lambda\big(g(Y,Z)X - g(X,Z)Y\big),$$

we find

$$\begin{aligned}
Ric(grad\, f, grad\, f) &= (m-1)\lambda, \\
\sigma &= m(m-1)\lambda.
\end{aligned}$$

■

5. Conclusions

In this work, we introduce an other class of metric on Riemannian manifold called Mus-gradient metric, also we studied the some properties of Riemannian curvature tensors of this metric.i.e. we characterize the Riemannian curvature, the sectional curvature and the scalar curvature.

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