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Special Issue Paper

Wave solutions of the time-space fractional complex Ginzburg-Landau equation with Kerr law nonlinearity

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Abstract

In this paper, the bifurcation theory of dynamical system is applied to investigate the time-space fractional complex Ginzburg-Landau equation with Kerr law nonlinearity. We mainly consider the case of $\alpha \neq 2\beta$ which is not discussed in previous work. By overcoming some difficulties aroused by the singular traveling wave system, such as bifurcation analysis of nonanalytic vector field, tracking orbits near the full degenerate equilibrium and calculation of complicated elliptic integrals, we give a total of 20 explicit exact traveling wave solutions of the time-space fractional complex Ginzburg-Landau equation and classify them into 11 categories. Some new traveling wave solutions of this equation are obtained including the compactons and the bounded solutions corresponding to some bounded manifolds.

Mathematics Subject Classification (2020). 34C23, 35C07, 35R11

Keywords. bifurcation, dynamical system, fractional equation, traveling waves

1. Introduction

With the development of some complex engineering and biological applications, many fractional differential equations (FDEs) have emerged. Compared with integer order differential equations, FDEs seem to have more advantages in some fields, such as the rheology, statistical mechanics, biology, circuitry, thermodynamics and electroanalytical chemistry [4, 5, 20, 21, 25, 34].

In fact, in 1832, Liouville proposed the definition of fractional derivative firstly and used it to solve the problem of potential theory, which attracted people's attention to fractional derivative. Subsequently, more definitions of fractional derivatives were proposed, such as the Riemann-Liouville fractional derivative [29], the Caputo fractional derivative [26], the Grünwald-Letnikov fractional derivative [24] and the conformable fractional derivative [16]. Especially, the modified Riemann-Liouville (mR-L) fractional derivative proposed by Jumarie in 2006 is defined by the limit of fractional difference [11]. Compared with the

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definition of Riemann-Liouville fractional derivative, it does not need the condition that a constant whose F-derivative is not zero and removes the effect of the initial value of function [11]. More importantly, the mR-L fractional derivative can handle non-differentiable functions, while the Caputo-Djrbashian fractional derivative [6,7] can only be used for differentiable functions. Due to the advantages of the mR-L fractional derivative mentioned above, people have successfully applied it to fractional thermal wave models [10], stochastic fractional models [12], Fourier transforms of fractional order [13], Laplace problems [14] and fractional epidemic models [33]. How to solve a fractional equation defined by the mR-L fractional derivative becomes an important and interesting problem. In fact, some work to solve these equations has been done. In 2012, Tang used a generalized fractional sub-equation method to solve the space-time fractional Gardner equation and obtained hyperbolic function solutions [37]. In 2014, Bekir employed $(G'/G)-$expansion method to obtain trigonometric function solutions of three kinds of fractional equations [3]. In 2016, Aksoy used exponential rational function method to obtain analytical solutions of the space-time fractional Fokas equation and coupled Burgers' equations [1].

Recently, a new fractional equation with the mR-L fractional derivative was proposed, which is called the time-space fractional complex Ginzburg-Landau (fcGL) equation

$$i\frac{\partial^{\delta}u}{\partial t^{\delta}} + m\frac{\partial^{2\delta}u}{\partial x^{2\delta}} + nF(|u|^2)u = \frac{1}{|u|^2 u^*}\left[\alpha |u|^2 \frac{\partial^{2\delta}|u|^2}{\partial x^{2\delta}} - \beta\left(\frac{\partial^{\delta}|u|^2}{\partial x^{\delta}}\right)^2\right] + \gamma u,$$

where the complex-valued function $u(x,t)$ represents the profile of optical soliton, x and t represent the distance along the fiber and the time in dimensionless form respectively, $m, n, \alpha, \beta, \gamma$ are valued constants. Among them, m, n denote the group velocity dispersion and nonlinearity respectively. The terms with constants α, β, γ come from the perturbation effects. $0 < \delta < 1$ is the order of the fractional derivative. The function $F(|u|^2)$, denoting the nonlinear form of optical fiber, is a real-valued function and must possess the smoothness of the function $F(|u|^2)u : \mathbb{C} \to \mathbb{C}$.

The time-space fcGL equation is a kind of fractional nonlinear Schrödinger equation and controls the pulse propagation dynamics through the optical fibers for transcontinental distances and transoceanic distances [8]. When $\delta = 1$, it degenerates to the classical complex Ginzburg-Landau equation. It can be used to describe phenomena including Bos-Einstein condensation [15], Bénard convection [23] and plane Poiseuille flow [35]. When $F(|u|^2) = u^2$, the time-space fcGL equation has the form

$$i\frac{\partial^{\delta}u}{\partial t^{\delta}} + m\frac{\partial^{2\delta}u}{\partial x^{2\delta}} + nu^3 = \frac{1}{|u|^2 u^*}\left[\alpha |u|^2 \frac{\partial^{2\delta}|u|^2}{\partial x^{2\delta}} - \beta\left(\frac{\partial^{\delta}|u|^2}{\partial x^{\delta}}\right)^2\right] + \gamma u, \tag{1.1}$$

which is called the time-space fcGL equation with Kerr law nonlinearity [22]. Studying the solutions of nonlinear evolution equations has always been a very meaningful thing [31, 32, 39]. In 2018, Sulaiman obtained combined dark-bright and combined singular optical solitons of equation (1.1) by using the extended sinh-Gordon equation expansion method [36]. Arshed obtained solutions of equation (1.1) in the form of hyperbolic, trigonometric and rational functions by using the $\exp(-\phi(\xi))$-expansion method [2]. In 2020, Hussain employed a new extended direct algebraic scheme to construct dark-singular and singular solutions of equation (1.1) [9]. In 2021, Huang obtained four types of soliton solutions via the complete discrimination system method [8]. More recently, Sadaf obtained complexiton, singular and periodic optical solitons of equation (1.1) by using improved $\tan\left(\frac{\psi(\zeta)}{2}\right)$-expansion technique [28]. In addition, the dynamical system of nonlinear evolution equation is also concerned [27,30].

Although there have been relatively rich results in the solitons of the time-space fcGL equation, some problems still need further discussion. Especially, we note that the case of $\alpha \neq 2\beta$ was not discussed in previous work mentioned above. It means that some wave phenomena could not be discovered yet. In addition, when referring to dynamical behaviour of wave solutions, people still do not clearly know how these solutions evolve with variation of parameters. In order to solve these problems, we introduce the dynamical system method to study equation (1.1), which has been shown to be a powerful and efficient method to find traveling wave solutions [17–19,41,42]. By this method, we convert equation (1.1) into corresponding traveling wave system and discuss two cases ($\alpha = 2\beta$ and $\alpha \neq 2\beta$). According to the phase space geometry under different parameter conditions, the existence conditions of various types of traveling wave solutions are given. Finally, by calculating complicated elliptic integrals, we obtain explicit expressions of bounded and unbounded traveling wave solutions of equation (1.1). Some new traveling wave solutions are obtained including the compactons and the bounded solutions corresponding to some bounded manifolds. In fact, to achieve the goal, we need overcome some difficulties and carry on lots of calculations. First of all, when $\alpha \neq 2\beta$, equation (1.1) corresponds to a singular traveling wave system with the nonanalytic vector field. One needs a more detailed analysis to track various orbits. Secondly, when investigating the local bifurcation and distribution of equilibria of this singular traveling wave system, we find that the equilibrium $(0, 0)$ is fully degenerate with a zero linearization. The traditional methods of normal sectors and Z-sectors can not directly applied to judge its type. Thirdly, the degree of the energy function is up to six, which leads to the difficulty of calculating elliptic integrals and giving the explicit exact solutions.

2. Traveling wave systems and bifurcation analysis

In this section, we firstly derive traveling wave systems of equation (1.1) by an appropriate traveling wave transformation. According to different parameter conditions, equation (1.1) can be equivalently converted into two types of traveling wave systems. Then, we carry on bifurcation analysis of the two types of traveling wave systems.

2.1. Two types of traveling wave systems of the time-space fcGL equation (1.1)

Inspired by the properties of mR-L fractional derivative, we assume that the traveling wave solution of equation (1.1) has the form

$$u(x,t) = \phi(\xi)\exp(i\psi(x,t)), \tag{2.1}$$

where $\xi = \dfrac{x^\delta}{\Gamma(1+\delta)} - \dfrac{ct^\delta}{\Gamma(1+\delta)}$, $\psi(x,t) = \dfrac{-kx^\delta}{\Gamma(1+\delta)} + \dfrac{\omega t^\delta}{\Gamma(1+\delta)} + \theta$ and $\Gamma(\cdot)$ is gamma function. Real functions $\phi(\xi)$ and $\psi(x,t)$ represent the portion of the amplitude and the phase component respectively. Parameters c, k, ω and θ are the speed, frequency, wave number and phase constant of the wave respectively.

Inserting (2.1) into equation (1.1) and separating the real part and imaginary part, we have

$$\begin{cases} \text{real part: } (m-2\alpha)\phi'' - (\omega + mk^2 + \gamma)\phi + n\phi^3 = 2(\alpha - 2\beta)\dfrac{(\phi')^2}{\phi}, \\ \text{imaginary part: } c = -2mk, \end{cases} \tag{2.2}$$

where $'$ denotes $d/d\xi$ and $m \neq 2\alpha$. The first equation of (2.2) is equivalent to the following system

$$\begin{cases} \phi' = y, \\ y' = \frac{1}{m-2\alpha}\left[-n\phi^3 + A\phi + 2(\alpha-2\beta)\frac{y^2}{\phi}\right], \end{cases} \tag{2.3}$$

where $A = \omega + mk^2 + \gamma$.

When $\alpha = 2\beta$, system (2.3) has the equivalent form

$$\begin{cases} \phi' = y, \\ y' = \dfrac{n}{2\alpha - m}\left(\phi^3 - \dfrac{A}{n}\phi\right), \end{cases} \tag{2.4}$$

which has the energy function

$$H_1(\phi, y) = \frac{1}{2}y^2 - \frac{n\phi^2}{4(2\alpha - m)}\left(\phi^2 - \frac{2A}{n}\right). \tag{2.5}$$

When $\alpha \neq 2\beta$, system (2.3) is a singular traveling wave system. It has the energy function

$$H(\phi, y) = \phi^r\left(nB_1\phi^4 - 2AB_2\phi^2 + 2B_1B_2y^2\right),$$

where $r = \frac{4(2\beta-\alpha)}{m-2\alpha}$, $B_1 = m + 4\beta - 4\alpha$ and $B_2 = m + 2\beta - 3\alpha$. Especially, system (2.3) has either a trivial traveling wave solution

$$\phi_1(\xi) = C_1 \exp(C_2\xi)$$

if $A = n = 0$ and $\frac{2(\alpha-2\beta)}{m-2\alpha} = 1$ or another trivial traveling wave solution

$$\phi_2(\xi) = \left[\frac{4\alpha - 4\beta - m}{2\alpha - m}(C_1\xi + C_2)\right]^{\frac{2\alpha-m}{4\alpha-4\beta-m}}$$

if $A = n = 0$ and $\frac{2(\alpha-2\beta)}{m-2\alpha} \neq 1$, where C_1, C_2 are constants. In this article, we always assume $n \neq 0$. Taking into account symmetry, solvability and ability to obtain the explicit expressions of solutions, we only consider the case of $r = 2$. Thus, system (2.3) can be rewritten as the following traveling wave system

$$\begin{cases} \phi' = y, \\ y' = \frac{n}{2\alpha-m}\left(\phi^3 - \frac{A}{n}\phi\right) - \frac{y^2}{\phi}, \end{cases} \tag{2.6}$$

which has the energy function

$$H_2(\phi, y) = 6(m - 2\alpha)\phi^2y^2 + \phi^4\left(2n\phi^2 - 3A\right). \tag{2.7}$$

2.2. Bifurcation analysis

In this section, we discuss the distribution of equilibria of system (2.4) and system (2.6) and give their global phase portraits.

Theorem 2.1. *The equilibria of system (2.4) have the following properties:*

- *When $\frac{A}{n} > 0$, system (2.4) has three equilibria. If $\frac{n}{2\alpha-m} > 0$, they are saddle $E_1(-\sqrt{\frac{A}{n}}, 0)$, center $E_2(0,0)$ and saddle $E_3(\sqrt{\frac{A}{n}}, 0)$. If $\frac{n}{2\alpha-m} < 0$, they are center $E_1(-\sqrt{\frac{A}{n}}, 0)$, saddle $E_2(0,0)$ and center $E_3(\sqrt{\frac{A}{n}}, 0)$.*
- *When $A = 0, n \neq 0$, system (2.4) has a degenerate equilibrium $E_4(0,0)$. If $\frac{n}{2\alpha-m} > 0$, it is a saddle. If $\frac{n}{2\alpha-m} < 0$, it is a center.*
- *When $\frac{A}{n} < 0$, system (2.4) has a simple equilibrium $E_5(0,0)$. If $\frac{n}{2\alpha-m} > 0$, it is a saddle. If $\frac{n}{2\alpha-m} < 0$, it is a center.*

Proof. For simply, we only proof the case of $\frac{n}{2\alpha-m} > 0$ in detail since the proof of the case of $\frac{n}{2\alpha-m} < 0$ is similar to it. Let $f_1(\phi) = \phi^3 - \frac{A}{n}\phi$. A direct calculation shows that when $\frac{A}{n} > 0$, function $f_1(\phi)$ has three real zeros: $0, \pm\sqrt{\frac{A}{n}}$, which means system (2.4) has three equilibria: $(0,0)$ and $(\pm\sqrt{\frac{A}{n}}, 0)$. When $A = 0, n \neq 0$, function $f_1(\phi) = \phi^3$ has only one triple real zero: 0. When $\frac{A}{n} < 0$, function $f_1(\phi)$ is strictly monotone increasing, so $f_1(\phi)$ has only one simple real zero.

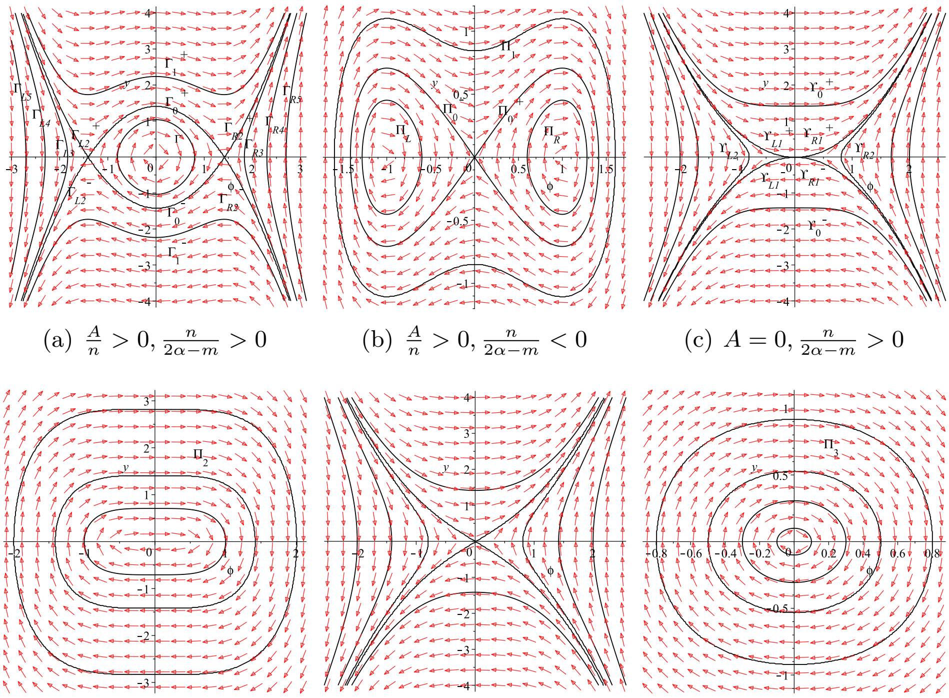
Let $M_1(E_j)$ $(j = 1, ..., 5)$ be the Jacobi matrix of the system (2.4) at the equilibrium E_j

$$M_1(E_j) = \begin{bmatrix} 0 & 1 \\ \frac{n}{2\alpha - m}\left(3\phi^2 - \frac{A}{n}\right) & 0 \end{bmatrix}.$$

We have $J_1(E_j) = det M_1(E_j) = -\frac{n}{2\alpha-m}\left(3\phi^2 - \frac{A}{n}\right)$. So, $J_1(E_1) = J_1(E_3) = -\frac{2A}{2\alpha-m}$, $J_1(E_2) = J_1(E_4) = J_1(E_5) = \frac{A}{2\alpha-m}$. When $\frac{n}{2\alpha-m} > 0$ and $\frac{A}{n} > 0$, we have $J_1(E_1) < 0, J_1(E_2) > 0, J_1(E_3) < 0$. By the theory of planar dynamical system and the properties of Hamiltonian system, it is not difficult for one to check E_1 is a saddle, E_2 is a center and E_3 is a saddle. When $\frac{n}{2\alpha-m} > 0$ and $\frac{A}{n} < 0$, we have $J_1(E_5) < 0$. So, E_5 is a saddle. When $A = 0, n \neq 0$, we have $J_1(E_4) = 0$, which means E_4 is a degenerate equilibrium. In this case, system (2.4) degrades to the following form

$$\begin{cases} \phi' = y := P(\phi, y), \\ y' = \dfrac{n}{2\alpha - m}\phi^3 := Q(\phi, y). \end{cases}$$

According to the qualitative theory of differential equation [40], if $\frac{n}{2\alpha-m} > 0$, E_4 is a saddle. If $\frac{n}{2\alpha-m} < 0$, we have $P(\phi, -y) = -P(\phi, y)$ and $Q(\phi, -y) = Q(\phi, y)$. So, E_4 is a center. □



(a) $\frac{A}{n} > 0, \frac{n}{2\alpha-m} > 0$ (b) $\frac{A}{n} > 0, \frac{n}{2\alpha-m} < 0$ (c) $A = 0, \frac{n}{2\alpha-m} > 0$

(d) $A = 0, \frac{n}{2\alpha-m} < 0$ (e) $\frac{A}{n} < 0, \frac{n}{2\alpha-m} > 0$ (f) $\frac{A}{n} < 0, \frac{n}{2\alpha-m} < 0$

Figure 1. The bifurcations of phase portraits of system (2.4)

According to the above analysis, we have the following results for system (2.4) :

Case I. When $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} > 0$, there are two heteroclinic orbits Γ_0^+ and Γ_0^- connecting saddles E_1 and E_3. Inside them, there exists a family of periodic orbits

$$\Gamma(h) = \left\{ H(\phi, y) = h, h \in \left(0, \frac{A^2}{4n(2\alpha - m)}\right)\right\},$$

which surround center E_2. When h$\to$ 0, orbit $\Gamma(h)$ tends to center E_2. When $h \to \frac{A^2}{4n(2\alpha-m)}$, orbit $\Gamma(h)$ tends to heteroclinic orbits. As shown in figure 1(a), outside of the heteroclinic orbits and the periodic orbits, other orbits of system (2.4) are unbounded.

Case II. When $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} < 0$, there are two homoclinic orbits Π_0^- and Π_0^+ connecting the saddle E_2. As shown in figure 1(b), center E_1 and center E_3 are surrounded by two families of periodic orbits

$$\Pi_L(h) = \Pi_R(h) = \left\{ H(\phi, y) = h, h \in \left(\frac{A^2}{4n(2\alpha - m)}, 0 \right) \right\}.$$

As $h \to \frac{A^2}{4n(2\alpha-m)}$, $\Pi_L(h)$ and $\Pi_R(h)$ tend to E_1 and E_3 respectively. As $h \to 0$, $\Pi_L(h)$ and $\Pi_R(h)$ tend to Π_0^- and Π_0^+ respectively.

Case III. When $\frac{A}{n} \leq 0$, if $\frac{n}{2\alpha-m} > 0$ ($\frac{n}{2\alpha-m} < 0$), as shown in figure 1(c)-1(f), all the orbits of system (2.4) are unbounded (bounded).

In order to better understand the equilibria and orbits of the system (2.6), we first consider the associated regular system of it. With the transformation $d\xi = \phi d\eta$, it can be converted to the associated regular system

$$\begin{cases} \phi' = \phi y, \\ y' = \frac{n}{2\alpha-m} \left(\phi^4 - \frac{A}{n}\phi^2 \right) - y^2, \end{cases} \tag{2.8}$$

where $'$ denotes $d/d\eta$.

Theorem 2.2. *The equilibria of system (2.8) have the following properties:*

• *When $\frac{A}{n} > 0$, system (2.8) has three equilibria: $E_1(-\sqrt{\frac{A}{n}}, 0), E_2(0,0)$ and $E_3(\sqrt{\frac{A}{n}}, 0)$. Among them, $E_2(0,0)$ is a high degenerate equilibrium. If $\frac{n}{2\alpha-m} > 0$ ($\frac{n}{2\alpha-m} < 0$), E_1 and E_3 are saddles (centers). If $\frac{n}{2\alpha-m} > 0$ ($\frac{n}{2\alpha-m} < 0$), there are two (six) orbits tending to $E_2(0,0)$.*

• *When $A = 0, n \neq 0$, system (2.8) has only one degenerate equilibrium $E_4(0,0)$. If $\frac{n}{2\alpha-m} > 0$ ($\frac{n}{2\alpha-m} < 0$), there are six (two) orbits tending to $E_4(0,0)$.*

• *When $\frac{A}{n} < 0$, system (2.8) also has only one degenerate equilibrium $E_5(0,0)$. If $\frac{n}{2\alpha-m} > 0$ ($\frac{n}{2\alpha-m} < 0$), there are six (two) orbits tending to $E_5(0,0)$.*

Proof. Let $M_2(E_j)$ $(j = 1, ..., 5)$ be the Jacobi matrix of the system (2.8) at the equilibrium E_j

$$M_2(E_j) = \begin{bmatrix} 0 & \phi \\ \frac{n}{2\alpha-m}\left(4\phi^3 - \frac{2A}{n}\phi\right) & 0 \end{bmatrix}.$$

We have $J_2(E_j) = det M_2(E_j) = -\frac{2n}{2\alpha-m}\phi^2\left(2\phi^2 - \frac{A}{n}\right)$. Thus, $J_2(E_2) = J_2(E_4) = J_2(E_5) = 0$ and $J_2(E_1) = J_2(E_3) = -\frac{2A^2}{n(2\alpha-m)}$. When $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} > 0$, we have $J(E_1) < 0$ and $J(E_3) < 0$. So E_1 and E_3 are saddles. When $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} < 0$, we have $J(E_1) > 0$ and $J(E_3) > 0$. So E_1 and E_3 are centers. When $\frac{A}{n} > 0$, for higher degenerate equilibrium $E_2(0,0)$, we need to consider the characteristic equation of system (2.8) with $\phi = \cos\theta$ and $y = \sin\theta$[40]

$$G_1(\theta) = \cos\theta \left[\frac{A}{m - 2\alpha} \cos^2\theta - 2\sin^2\theta \right] = 0, \tag{2.9}$$

and the other function

$$H_1(\theta) = \sin\theta \left[\left(\frac{A}{m - 2\alpha} + 1 \right) \cos^2\theta - \sin^2\theta \right]. \tag{2.10}$$

From $G_1(\theta) = 0$, one has $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \frac{3\pi}{2}$ and when $\frac{A}{m-2\alpha} > 0$, one has $\theta_{3,4} = \pm\arctan\sqrt{\frac{A}{2(m-2\alpha)}}$, $\theta_{5,6} = \pi \pm \arctan\sqrt{\frac{A}{2(m-2\alpha)}}$. Thus, when $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} > 0$, one can check that $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are simple roots of $G_1(\theta) = 0$, and

$$G_1^{'}\left(\frac{\pi}{2}\right)H_1\left(\frac{\pi}{2}\right) = G_1^{'}\left(\frac{3\pi}{2}\right)H_1\left(\frac{3\pi}{2}\right) = -2 < 0,$$

where $'$ denotes $d/d\theta$. According to the work [40], there are only two orbits tending to equilibrium $E_2(0,0)$ in the direction of $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{3\pi}{2}$, respectively. When $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} < 0$, one can check that $G_1(\theta) = 0$ has six single roots. The fact that orbits tend to $(0,0)$ in the direction of θ_1 and θ_2 has the same proof as above. Next, we consider the other roots $\theta_{3,4,5,6}$ of equation (2.9). Due to the symmetry of the phase portrait of (2.8), we only check orbit tends to equilibrium $E_2(0,0)$ in the direction of $\theta_3 = \arctan\sqrt{\frac{A}{2(m-2\alpha)}}$ in the first quadrant. In the subcase, one can easily know that θ_3 is a simple root of the equation (2.9) and

$$G_1^{'}(\theta_3)H_1(\theta_3) = -\frac{(5l+8)(6l+8)(2l+2)}{4(l+4)^3} < 0,$$

where $l = \frac{A}{m-2\alpha} > 0$. According to the work [40], the orbit can only tend to equilibrium $E_2(0,0)$ in the direction of θ_3 in the first quadrant. Thus, for the case of $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} < 0$, there are six orbits tending to $E_2(0,0)$.

When $A = 0, n \neq 0$, the equation (2.9) and function (2.10) become

$$G_2(\theta) = -2\cos\theta\sin^2\theta = 0,$$

and

$$H_2(\theta) = \sin\theta\left(\cos^2\theta - \sin^2\theta\right).$$

From $G_2(\theta) = 0$, one has $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \frac{3\pi}{2}$, $\theta_7 = 0$ and $\theta_8 = \pi$. One can check that $G_2^{'}(\frac{\pi}{2})H_2(\frac{\pi}{2}) = G_2^{'}(\frac{3\pi}{2})H_2(\frac{3\pi}{2}) = -2 < 0$. A similar analysis makes it clear that the orbits can tend to equilibrium $E_4(0,0)$ in the direction of $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{3\pi}{2}$. However, $G_2^{'}(0)H_2(0) = G_2(\pi)^{'}H_2(\pi) = 0$. In order to check that when $\frac{n}{2\alpha-m} > 0$, there are other orbits tending to $E_4(0,0)$, we introduce the method of generalized normal sectors (GNS) [38]. Since the phase portrait of system (2.8) has symmetry when $A = 0$, we only consider the orbit in the first quadrant. From y' in equation (2.8), we have the only horizontal isocline

$$L := \left\{(\phi, y) \in \mathbf{R}_+^2 : y = \sqrt{\frac{n}{2\alpha - m}}\phi^2, 0 < \sqrt{\phi^2 + y^2} < \mathscr{C}\right\},$$

where $\mathscr{C}$ is a sufficiently small constant. Obviously, L is tangent to the ϕ-axis at $O(0,0)$ and it is not a orbit of system (2.8). Let $\Phi := \left\{(\phi, y) \in \mathbf{R}_+^2 : y = 0, 0 < \sqrt{\phi^2 + y^2} < \mathscr{C}\right\}$ and $Y := \left\{(\phi, y) \in \mathbf{R}_+^2 : \phi = 0, 0 < \sqrt{\phi^2 + y^2} < \mathscr{C}\right\}$. One can know that L divides the first quadrant into two parts. Considering the open quasi-sectorial region $\Delta\widehat{LO\Phi}$, we find that expect $(0,0)$, all positive semi-orbits starting from L and Φ enter $\Delta\widehat{LO\Phi}$ (see figure 2) and $\frac{\partial}{\partial y}\left(\frac{y'}{\phi'}\right) = -\frac{y^2 + \frac{n}{2\alpha-m}\phi^4}{\phi y^2} < 0$. According to the work [38], there is only one orbit tending to $(0,0)$ in the direction of $\theta = 0$ in $\Delta\widehat{LO\Phi}$. In another open quasi-sectorial region $\Delta\widehat{YOL}$, we have $\frac{y'}{\phi'} = \frac{\frac{n}{2\alpha-m}\phi^4 - y^2}{\phi y} < 0$. So no orbit tends to $(0,0)$ in $\Delta\widehat{YOL}$. In the first quadrant, there is a unique orbit tending to $(0,0)$ in the direction of $\theta = 0$. In the entire $\phi - y$ plane, there are six orbits tending to $(0,0)$. When $\frac{n}{2\alpha-m} < 0$, in the first quadrant, one can check no orbit tends to $(0,0)$ because of $\frac{y'}{\phi'} < 0$. In the entire $\phi - y$ plane, there are two orbits tending to $(0,0)$ in the direction of $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ respectively.

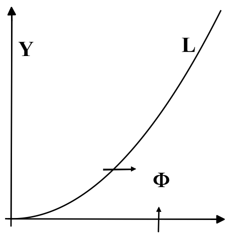
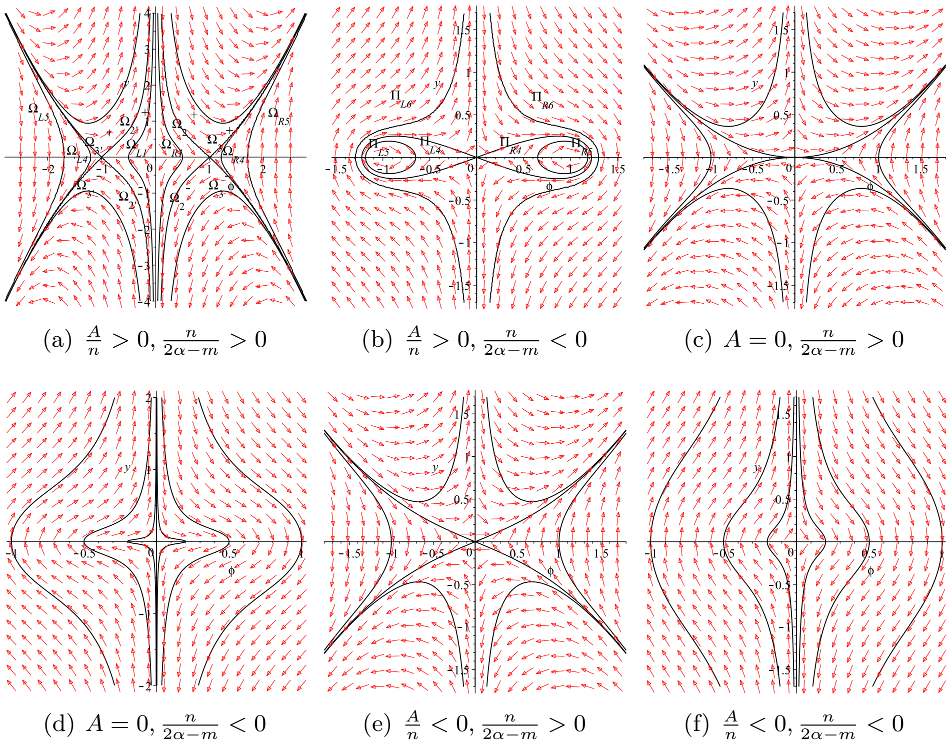


Figure 2. $\phi - y$ plane

When $\frac{A}{n} > 0$, for the degenerate equilibrium $E_5(0,0)$, we have the same analysis as the equilibrium $E_2(0,0)$. We omit this proof for simplicity. □



(a) $\frac{A}{n} > 0, \frac{n}{2\alpha - m} > 0$ (b) $\frac{A}{n} > 0, \frac{n}{2\alpha - m} < 0$ (c) $A = 0, \frac{n}{2\alpha - m} > 0$

(d) $A = 0, \frac{n}{2\alpha - m} < 0$ (e) $\frac{A}{n} < 0, \frac{n}{2\alpha - m} > 0$ (f) $\frac{A}{n} < 0, \frac{n}{2\alpha - m} < 0$

Figure 3. The bifurcations of phase portraits of system (2.6)

Because of the symmetry of system (2.6), we only consider the case of $A > 0$. According to the Theorem 2.2, we have the following results for system (2.6):

Case I. When $\frac{A}{n} > 0$ and $\frac{n}{2\alpha - m} > 0$, there are four stable manifolds and four unstable manifolds connecting the saddles E_1 and E_3. Among them, only manifolds Ω_2^+, Ω_2^-, $\Omega_{2'}^+$ and $\Omega_{2'}^-$ are bounded orbits. Inside them, there exists a family of compact orbits

$$\Gamma(h) = \left\{ H(\phi, y) = h, h \in \left(-\frac{A^3}{n^2}, 0 \right) \right\},$$

for which one $\Gamma(h)$ tends to the singular straight line $\phi = 0$ as $|y| \to \infty$. When $h \to 0$, orbit $\Gamma(h)$ gets closer to the singular straight line. When $h \to -\frac{A^3}{n^2}$, orbit $\Gamma(h)$ tends to the saddle. Outside of the orbits mentioned here, other orbits of system (2.6) are unbounded.

Case II. When $\frac{A}{n} > 0$ and $\frac{n}{2\alpha - m} < 0$, as shown in figure 3(b), there are two homoclinic orbits Π_{L4} and Π_{R4} connecting the equilibrium E_2. Center E_1 and center E_3 are surrounded by two families of periodic orbits

$$\Pi_{L5}(h) = \Pi_{R5}(h) = \left\{ H(\phi, y) = h, h \in \left(-\frac{A^3}{n^2}, 0 \right) \right\}.$$

As $h \to -\frac{A^3}{n^2}$, $\Pi_{L5}(h)$ and $\Pi_{R5}(h)$ tend to E_1 and E_3 respectively. As $h \to 0$, $\Pi_{L5}(h)$ and $\Pi_{L5}(h)$ tend to homoclinic orbits Π_{L4} and Π_{R4} respectively.
Case III. When $\frac{A}{n} \le 0$, if $\frac{n}{2\alpha - m} > 0$ ($\frac{n}{2\alpha - m} < 0$), as shown in figure 3(c)-3(f), all the orbits of system (2.6) are unbounded (bounded).

3. Exact solutions of systems (2.4) and (2.6)

In this section, by calculating the complicated elliptic integrals, we give explicit exact solutions of of system (2.4) and (2.6).

3.1. Explicit exact solutions of system (2.4)

In this part, we will give the bounded and unbounded solutions of system (2.4), which requires us to identify every type of orbits of system (2.4) including homoclinic orbits, heteroclinic orbits, periodic orbits and other unbounded ones.

In order to calculating the solutions of system (2.4), from the first equation of it, we have

$$\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\pm\sqrt{\frac{n}{2(2\alpha - m)}}\sqrt{\phi^4 - \frac{2A}{n}\phi^2 + \frac{4(2\alpha - m)h}{n}}},$$

for $\frac{n}{(2\alpha - m)} > 0$,

$$\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\pm\sqrt{\frac{-n}{2(2\alpha - m)}}\sqrt{-\phi^4 + \frac{2A}{n}\phi^2 - \frac{4(2\alpha - m)h}{n}}},$$

for $\frac{n}{(2\alpha - m)} < 0$.

First of all, we discuss bounded solutions of system (2.4), including the kick wave solutions, smooth solitary wave solutions and periodic solutions. As shown in figure 1(a), 1(b), 1(d), 1(f), four cases need to be discussed.

(I) For $\frac{A}{n} > 0$ and $\frac{n}{m - 2a} > 0$ in figure 1(a), there are three equilibria: saddle $E_1(-\sqrt{\frac{A}{n}}, 0)$, center $E_2(0,0)$ and saddle $E_3(\sqrt{\frac{A}{n}}, 0)$. We have two subcases in this case.
(i) The periodic orbits Γ in figure 1(a), whose energy is higher than 0, but lower than the energy of saddles E_1 and E_3, can be expressed by

$$y = \pm\sqrt{\frac{n}{2(2\alpha - m)}}\sqrt{(\phi + \phi_1)(\phi + \phi_2)(\phi_1 - \phi)(\phi_2 - \phi)},$$

where ϕ_1 and ϕ_2 are reals and the relation $0 < \phi < \phi_2 < \sqrt{\frac{A}{n}} < \phi_1$ holds. Choosing initial value $\phi(0) = \phi_2$ and assuming the period is $2T_1$, we have

$$\int_{\phi_2}^{\phi} \frac{d\phi}{\sqrt{\frac{n}{2(2\alpha - m)}}\sqrt{(\phi + \phi_1)(\phi + \phi_2)(\phi_1 - \phi)(\phi_2 - \phi)}} = \int_0^{\xi} d\xi, \quad 0 < \xi < T_1,$$

$$\int_{\phi}^{\phi_2} \frac{d\phi}{-\sqrt{\frac{n}{2(2\alpha - m)}}\sqrt{(\phi + \phi_1)(\phi + \phi_2)(\phi_1 - \phi)(\phi_2 - \phi)}} = \int_{\xi}^{0} d\xi, \quad -T_1 < \xi < 0,$$

which can be rewritten as

$$\int_{\phi_2}^{\phi} \sqrt{\frac{2(2\alpha - m)}{n}} \frac{d\phi}{\sqrt{(\phi_1^2 - \phi^2)(\phi_2^2 - \phi^2)}} = \mid \xi \mid, \quad -T_1 < \xi < T_1.$$

By calculating the elliptic integral, we get first type of periodic solution of system (2.4)

$$\phi_{b_1}(\xi) = -\phi_1 + \frac{\phi_1^2 - \phi_2^2}{\phi_1 + \phi_2 - 2\phi_2 sn^2\left(\sqrt{\frac{n}{2\alpha-m}}\xi, k_1\right)}, \quad -T_1 < \xi < T_1,$$

where $k_1^2 = \frac{4\phi_1\phi_2}{(\phi_1+\phi_2)^2}$.
(ii) The heteroclinic orbit Γ_0^+ in figure 1(a), whose energy is equal to the energy of two saddles $h = H(E_1) = H(E_2) = \frac{A^2}{4n(2\alpha-m)}$, can be expressed by

$$y = \sqrt{\frac{n}{2(2\alpha - m)}}(\tfrac{A}{n} - \phi^2),$$

where $-\sqrt{\frac{A}{n}} < \phi < \sqrt{\frac{A}{n}}$. Choosing the initial value $\phi(0) = 0$, we have

$$\int_0^\phi \frac{d\phi}{\sqrt{\frac{n}{2(2\alpha-m)}}(\frac{A}{n} - \phi^2)} = \int_0^\xi d\xi, \quad -\infty < \xi < +\infty.$$

So, we obtain the kink wave solution of system (2.4)

$$\phi_{b_2} = \sqrt{\frac{A}{n}}\left(1 - \frac{2}{1 + \exp\left(\sqrt{\frac{2A}{2\alpha-m}}\xi\right)}\right), \quad -\infty < \xi < +\infty.$$

The another kink wave solution of system (2.4), corresponding heteroclinic orbit Γ_0^- , has similar expression

$$\phi_{b_2'} = \sqrt{\frac{A}{n}}\left(1 - \frac{2}{1 + \exp\left(-\sqrt{\frac{2A}{2\alpha-m}}\xi\right)}\right), \quad -\infty < \xi < +\infty.$$

(II) For $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} < 0$ in figure 1(b), there are three equilibria: center $E_1(-\sqrt{\frac{A}{n}}, 0)$, saddle $E_2(0, 0)$ and center $E_3(\sqrt{\frac{A}{n}}, 0)$. We have three subcases in this case.
(1) The periodic orbit Π_R in figure 1(b), whose energy is higher than the energy of centers E_1 and E_3, but lower than 0, can be expressed by

$$y = \pm\sqrt{-\frac{n}{2(2\alpha - m)}}\sqrt{(\phi + \phi_3)(\phi + \phi_4)(\phi - \phi_4)(\phi_3 - \phi)},$$

where ϕ_3 and ϕ_4 are reals and the relations $0 < \phi_4 < \sqrt{\frac{A}{n}} < \phi_3$ and $\phi_4 < \phi < \phi_3$ hold. Choosing initial value $\phi(0) = \phi_3$ and assuming the period is $2T_2$, we have

$$\int_\phi^{\phi_3} \sqrt{\frac{2(m - 2\alpha)}{n}} \frac{d\phi}{\sqrt{(\phi_3^2 - \phi^2)(\phi^2 - \phi_4^2)}} = |\, \xi \,|, \quad -T_2 < \xi < T_2.$$

By calculating the elliptic integral

$$\int_\phi^{\phi_3} \frac{d\phi}{\sqrt{(\phi_3^2 - \phi^2)(\phi^2 - \phi_4^2)}} = \frac{1}{\phi_3} sn^{-1}\left(\sqrt{\frac{\phi_3^2 - \phi^2}{\phi_3^2 - \phi_4^2}}, k_2\right),$$

where $k_2^2 = \frac{\phi_3^2 - \phi_4^2}{\phi_3^2}$, we get second type of periodic solution of system (2.4)

$$\phi_{b_3} = \sqrt{\phi_3^2 + (\phi_4^2 - \phi_3^2) sn^2\left(\sqrt{\frac{n}{2(m - 2\alpha)}}\phi_3\xi\right)}, \quad -T_2 < \xi < T_2.$$

The another periodic solution of system (2.4), corresponding periodic orbit Π_L and initial value $\phi(0)=-\phi_3$, has similar expression $\phi_{b_3'}=-\phi_{b_3}$.
(2) The homoclinic orbit Π_0^+ shown in figure 1(b) , whose energy is equal to 0. It can be expressed by

$$y=\pm\sqrt{-\frac{n}{2(2\alpha-m)}}\sqrt{\phi^2\left(\phi+\sqrt{\frac{2A}{n}}\right)\left(\sqrt{\frac{2A}{n}}-\phi\right)},$$

where the relation $0<\phi<\sqrt{\frac{2A}{n}}$ holds. Choosing initial value $\phi(0)=\sqrt{\frac{2A}{n}}=\phi_5$, we have

$$\int_{\phi_5}^{\phi}\sqrt{\frac{2(m-2\alpha)}{n}}\frac{d\phi}{\phi\sqrt{\phi_5^2-\phi^2}}=-\mid\xi\mid,\quad -\infty<\xi<+\infty.$$

By a direct calculation, we obtain the solitary wave solution of system (2.4)

$$\phi_{b_4}=\frac{\phi_5}{\cosh(\sqrt{\frac{A}{m-2\alpha}}\xi)},\quad -\infty<\xi<+\infty.$$

The another solitary wave solution of system (2.4) in figure 1(b), corresponding homoclinic orbit Π_0^-, has similar expression $\phi_{b_4'}=-\phi_{b_4}$.
(3) The periodic orbit Π_1 in figure 1(b), whose energy is higher than 0, can be expressed by

$$y=\pm\sqrt{-\frac{n}{2(2\alpha-m)}}\sqrt{(\phi+\phi_6)(\phi_6-\phi)\left(\phi^2+\phi_6^2-\frac{2A}{n}\right)},$$

where ϕ_6 is a real and the relations $\phi_6>\sqrt{\frac{2A}{n}}$ and $0<\phi<\phi_6$ hold. Choosing initial value $\phi(0)=-\phi_6$ and assuming the period is $2T_3$, we have

$$\int_{-\phi_6}^{\phi}\sqrt{\frac{2(m-2\alpha)}{n}}\frac{d\phi}{\sqrt{(\phi_6^2-\phi^2)\left(\phi^2+\left(\phi_6^2-\frac{2A}{n}\right)\right)}}=|\xi|,\quad -T_3<\xi<T_3.$$

Noting

$$\int_{-\phi_6}^{\phi}\frac{d\phi}{\sqrt{(\phi_6^2-\phi^2)\left(\phi^2+\left(\phi_6^2-\frac{2A}{n}\right)\right)}}=\frac{1}{\sqrt{2\left(\phi_6^2-\frac{A}{n}\right)}}cn^{-1}\left(\frac{-\phi}{\phi_6},k_3\right),$$

where $k_3^2=\frac{\phi_6^2}{2(\phi_6^2-\frac{A}{n})}$, we obtain the third type of periodic solution of system (2.4)

$$\phi_{b_5}=-\phi_6 cn\left(\sqrt{\frac{\phi_6^2 n-A}{m-2\alpha}}\xi\right),\quad -T_3<\xi<T_3.$$

(III) For $A=0$ and $\frac{n}{2\alpha-m}<0$ in figure 1(d), there is only one type of bounded solution, corresponding to periodic orbit Π_2, whose energy is higher than the energy of center E_4. So, it can be expressed by

$$y=\pm\sqrt{-\frac{n}{2(2\alpha-m)}}\sqrt{(\phi_7^2-\phi^2)(\phi_7^2+\phi^2)},$$

where ϕ_7 is a real and the relations $\phi_7 > 0$ and $0 < \phi < \phi_7$ hold. Choosing the initial value $\phi(0) = \phi_7$ and assuming the period is $2T_4$, we have

$$\int_{\phi}^{\phi_7} \sqrt{\frac{2(m-2\alpha)}{n}} \frac{d\phi}{\sqrt{\phi_7^4 - \phi^4}} = |\xi|, \quad -T_4 < \xi < T_4.$$

So, we obtain the four type of periodic solution of system (2.4)

$$\phi_{b_6} = \phi_7 cn\left(\sqrt{\frac{n}{m-2\alpha}}\phi_7 \xi\right), \quad -T_4 < \xi < T_4.$$

(IV) For $\frac{A}{n} < 0$ and $\frac{n}{2\alpha - m} < 0$ in figure 1(f), there is only one type of bounded orbit, such as orbit Π_3. The solution of system (2.4), corresponding orbit Π_3, has the similar form as solution ϕ_{b_5}.

Then, we discuss all unbounded solutions of system (2.4). As shown in figure 1(a), 1(c) and 1(e), three cases need to be considered.

(I) For $\frac{A}{n} > 0$ and $\frac{n}{2\alpha - m} > 0$ in figure 1(a), we have five subcases in this case.
(1)Consider a class of unbounded orbits in figure 1(a), whose energy $h = \frac{n}{2(2\alpha-m)} h_1$ are higher than the energy of saddles E_1 and E_3, such as the orbits Γ_1^+ and Γ_1^-. Obviously, $h_1 > 0$. Taking the right-hand side of orbit Γ_1^+ as an example, we have the expression for it

$$y = \sqrt{\frac{n}{2(2\alpha - m)}} \sqrt{\phi^4 - \tfrac{2A}{n}\phi^2 + h_1},$$

where $\phi > 0$. Choosing initial value $\phi(0) = +\infty$, we have

$$\int_{+\infty}^{\phi} \sqrt{\frac{2(2\alpha-m)}{n}} \frac{d\phi}{\sqrt{\phi^4 - \frac{2A}{n}\phi^2 + h_1}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{\phi}^{+\infty} \frac{d\phi}{\sqrt{\phi^4 - \frac{2A}{n}\phi^2 + h_1}} = \frac{1}{2\sqrt[4]{h_1}} cn^{-1}\left(\frac{\phi^2 - \sqrt{h_1}}{\phi^2 + \sqrt{h_1}}, k_4\right),$$

where $k_4^2 = \frac{1}{2} + \frac{A}{2n\sqrt{h_1}}$, we obtain the first type of unbounded solution of system (2.4)

$$\phi_{u_1} = \sqrt{\frac{2\sqrt{h_1}}{1 - cn\left(\sqrt{\frac{2n}{2\alpha-m}}\sqrt[4]{h_1}\xi\right)} - \sqrt{h_1}}, \quad 0 < \xi < \xi_1,$$

where $\xi_1 = \sqrt{\frac{8(2\alpha-m)}{n\sqrt{h_1}}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k_4^2 sin^2\theta}}$.
(2) Consider a class of unbounded orbits in figure 1(a), whose energy are equal to the energy of two saddles. They are Γ_{L2}^+, Γ_{L2}^-, Γ_{R2}^+ and Γ_{R2}^-. Take Γ_{R2}^- as an example. It can be expressed respectively by

$$y = -\sqrt{\frac{n}{2(2\alpha-m)}}\left(\phi^2 - \frac{A}{n}\right),$$

where $\phi > \sqrt{\frac{A}{n}}$. Choosing initial value $\phi(0) = +\infty$, we have

$$\int_{+\infty}^{\phi} \frac{d\phi}{-\sqrt{\frac{n}{2(2\alpha-m)}}(\phi^2 - \frac{A}{n})} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

By a direct calculation, we obtain the second type of unbounded solution of system (2.4)

$$\phi_{u_2}=\sqrt{\frac{A}{n}}\left(\frac{2}{1-\exp\left(-\sqrt{\frac{2A}{2\alpha-m}}\xi\right)}-1\right),\quad \xi>0.$$

(3) Consider a class of unbounded orbits in figure 1(a), whose energy are higher than the energy of center E_2, but lower than the energy of two saddles, such as orbits Γ_{R3} and Γ_{L3}. Take the orbit Γ_{R3} as an example. It can be represented by

$$y=\pm\sqrt{\frac{n}{2(2\alpha-m)}}\sqrt{(\phi^2-\phi_1^2)(\phi^2-\phi_2^2)},$$

where $\phi>\phi_1$. Choosing initial value $\phi(0)=+\infty$, we have

$$\int_{\phi}^{+\infty}\sqrt{\frac{2(2\alpha-m)}{n}}\frac{d\phi}{\sqrt{(\phi^2-\phi_1^2)(\phi^2-\phi_2^2)}}=|\xi|.$$

By calculating the elliptic integral

$$\int_{\phi}^{+\infty}\frac{d\phi}{\sqrt{(\phi^2-\phi_1^2)(\phi^2-\phi_2^2)}}=\frac{1}{\phi_1}sn^{-1}\left(\frac{\phi_1}{\phi},k_5\right),$$

where $k_5=\frac{\phi_2}{\phi_1}$, we obtain the third type of unbounded solution of system (2.4)

$$\phi_{u_3}=\frac{\phi_1}{sn\left(\sqrt{\frac{n}{2(2\alpha-m)}}\phi_1|\xi|\right)},\quad -\xi_2<\xi<\xi_2,$$

where $\xi_2=\frac{1}{\phi_1}\sqrt{\frac{32(2\alpha-m)}{n}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k_5^2sin^2\theta}}$.

(4) Consider orbits Γ_{R4} and Γ_{L4} in figure 1(a), whose energy is equal to the energy of the center E_2. Take the orbit Γ_{R4} as an example. It can be represented by

$$y=\pm\sqrt{\frac{n}{2(2\alpha-m)}}\sqrt{\phi^2\left(\phi^2-\phi_5^2\right)},$$

where $\phi>\phi_5$. Choosing initial value $\phi(0)=+\infty$, we have

$$\int_{\phi}^{+\infty}\frac{d\phi}{\sqrt{\frac{n}{2(2\alpha-m)}}\phi\sqrt{\phi^2-\phi_5^2}}=|\xi|.$$

Noting that

$$\int_{\phi}^{+\infty}\frac{d\phi}{\phi\sqrt{\phi^2-\phi_5^2}}=\frac{1}{\phi_5}\arcsin\frac{\phi_5}{\phi},$$

we obtain the fourth type of unbounded solution of system (2.4)

$$\phi_{u_4}=\phi_5\csc\left(\sqrt{\frac{n}{2(2\alpha-m)}}\phi_5|\xi|\right),\quad -\xi_3<\xi<\xi_3,$$

where $\xi_3=\frac{2\pi}{\phi_5}\sqrt{\frac{2(2\alpha-m)}{n}}$.

(5) Consider a class of unbounded orbits in figure 1(a), whose energy is lower than the energy of center E_2, such as orbits Γ_{R5} and Γ_{L5}. Take the orbit Γ_{R5} as an example. It can be represented by

$$y=\pm\sqrt{\frac{n}{2(2\alpha-m)}}\sqrt{(\phi^2-\phi_6^2)\left(\phi^2+\phi_6^2-\frac{2A}{n}\right)},$$

where $\phi > \phi_6 > \sqrt{\frac{2A}{n}}$. Choosing the initial value $\phi(0) = +\infty$, we have

$$\int_{\phi}^{+\infty} \frac{d\phi}{\sqrt{\frac{n}{2(2\alpha-m)}}\sqrt{(\phi^2-\phi_6^2)\left(\phi^2+\phi_6^2-\frac{2A}{n}\right)}} = |\xi|.$$

By calculating the elliptic integral

$$\int_{\phi}^{+\infty} \frac{d\phi}{\sqrt{(\phi^2-\phi_6^2)\left(\phi^2+\left(\phi_6^2-\frac{2A}{n}\right)\right)}} = \frac{1}{\sqrt{2\phi_6^2-\frac{2A}{n}}} sn^{-1}\left(\sqrt{\frac{2\phi_6^2-\frac{2A}{n}}{\phi_6^2-\frac{2A}{n}+\phi^2}}, k_6\right),$$

where $k_6^2 = 1 - \frac{n\phi_6^2}{2n\phi_6^2-A}$, we obtain the fifth type of unbounded solution of system (2.4)

$$\phi_{u_5} = \sqrt{\frac{2(n\phi_6^2-A)}{nsn^2\left(\sqrt{\frac{n\phi_6^2-A}{2\alpha-m}}\xi\right)} - \phi_6^2 + \frac{2A}{n}}, \quad -\xi_4 < \xi < \xi_4,$$

where $\xi_4 = 4\sqrt{\frac{2\alpha-m)}{n\phi_6^2-2A}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k_6^2\sin^2\theta}}$.

(II) For $A = 0$ and $\frac{n}{2\alpha-m} > 0$ in figure 1(c), system (2.4) has only one equilibrium: saddle E_4. In this part, we have three subcases in this case.

(1) Consider a class of unbounded orbits in figure 1(c), whose energy $h = \frac{n}{4(2\alpha-m)}h_2$ is higher than 0, such as orbits γ_0^+ and γ_0^-. Them can be represented by

$$y = \pm\sqrt{\frac{n}{2(2\alpha-m)}}\sqrt{\phi^4+h_2},$$

where $\phi > 0$. Choosing initial value $\phi(0) = +\infty$, the solution corresponding the right-hand side of orbit γ_0^+ is the same as ϕ_{u_1}

$$\phi_{u_6} = \sqrt{\frac{2\sqrt{h_2}}{1-cn\left(\sqrt{\frac{2n}{2\alpha-m}}\sqrt[4]{h_2}\xi\right)} - \sqrt{h_2}}, \quad 0 < \xi < \xi_5,$$

where $\xi_5 = \sqrt{\frac{8(2\alpha-m)}{n\sqrt{h_2}}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\cdot\sin^2\theta}}$.

(2) Consider a class of unbounded orbits in figure 1(c), whose energy is equal to 0. They are orbits γ_{L1}^+, γ_{L1}^-, γ_{R1}^+ and γ_{R1}^-. Take the orbit γ_{R1}^- as an example. It can be represented by

$$y = -\sqrt{\frac{n}{2(2\alpha-m)}}\phi^2,$$

where the relation $\phi > 0$ holds. Choosing the initial value $\phi(0) = +\infty$, we obtain the sixth type of unbounded solution of system (2.4)

$$\phi_{u_7} = \sqrt{\frac{2(2\alpha-m)}{n}}\frac{1}{\xi}, \quad \xi > 0.$$

(3) Consider a class of unbounded orbits in figure 1(c), whose energy is lower than 0, such as orbits γ_{L2} and γ_{R2}. Take orbit γ_{R2} as an example. It can be represented by

$$y = \pm\sqrt{\frac{n}{2(2\alpha-m)}}\sqrt{(\phi^2-\phi_7^2)(\phi^2+\phi_7^2)},$$

where the relation $\phi > \phi_7$ holds. Choosing the initial value $\phi(0) = +\infty$, we obtain the seventh type of unbounded solution of system (2.4)

$$\phi_{u_8} = \phi_7\sqrt{\frac{2}{sn^2\left(\sqrt{\frac{n}{2(2\alpha-m)}}\phi_7\xi\right)} - 1}, \quad -\xi_6 < \xi < \xi_6,$$

where $\xi_6 = \frac{4}{\phi_7}\sqrt{\frac{2(2\alpha-m)}{n}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{1}{2}\cdot\sin^2\theta}}$.

(III) For $\frac{A}{n} < 0$ and $\frac{n}{2\alpha-m} > 0$ in figure 1(e), system (2.4) also has only one equilibrium. There are three different types of unbounded orbits. The solutions corresponding to these orbits are similar to the ones we gave above, and will not be written down in detail.

3.2. Explicit exact solutions of system (2.6)

In this part, we will classify the bounded solutions of system (2.6), which requires us to identify every type of orbits of system (2.6) including periodic orbits, homoclinic orbits, compact orbits and other bounded ones. As shown in figure 3(a), 3(b), 3(d), 3(f), four cases need to be discussed.

In order to calculating the solutions of system (2.6), from the first equation of system (2.6), we have

$$\xi = \int_{\phi_0}^{\phi}\frac{d\phi}{\pm\sqrt{\frac{n}{3(m-2\alpha)}}\sqrt{\frac{-\phi^6+\frac{3A}{2n}\phi^4+\frac{h}{2n}}{\phi^2}}},$$

for $\frac{n}{(2\alpha-m)} < 0$,

$$\xi = \int_{\phi_0}^{\phi}\frac{d\phi}{\pm\sqrt{\frac{-n}{3(m-2\alpha)}}\sqrt{\frac{\phi^6-\frac{3A}{2n}\phi^4-\frac{h}{2n}}{\phi^2}}},$$

for $\frac{n}{(2\alpha-m)} > 0$.

(I) For $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} > 0$ in figure 3(a), There are two types of bounded orbits. Thus, we need to discuss two subcases in this case.

(1) Consider a class of bounded orbits in figure 3(a), whose energy is equal to the energy of the two saddles E_1 and E_3. They are stable manifolds Ω_2^+, $\Omega_{2'}^-$ and unstable manifolds Ω_2^-, $\Omega_{2'}^+$. Take the stable manifold Ω_2^+ as an example. It can be represented by

$$y = \sqrt{\frac{n}{3(2\alpha-m)}}\sqrt{\frac{\left(\sqrt{\frac{A}{n}}-\phi\right)^2\left(\phi+\sqrt{\frac{A}{n}}\right)^2(\phi^2+\frac{A}{2n})}{\phi^2}},$$

where $0 < \phi < \frac{A}{n}$. Taking the initial value $\phi(0) = 0$, as shown in figure 4(a), we obtain first type bounded solution of system (2.6)

$$\phi_{b_7} = \sqrt{\frac{3A}{2n}\left(1-\frac{2}{\exp(5-2\sqrt{6}-\sqrt{\frac{2A}{2\alpha-m}}\xi)}\right)^2 - \frac{A}{2n}}, \quad 0 < \xi < +\infty.$$

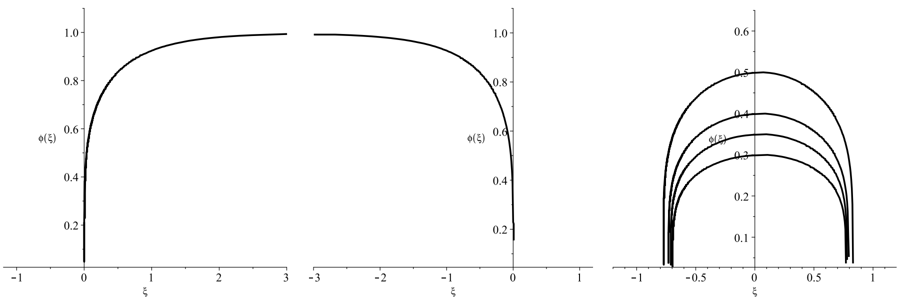
(2) Consider a family of open orbits in figure 3(a), whose energy $H = h, h \in (-\frac{A^3}{n^2}, 0)$, such as Ω_{L1} and Ω_{R1}. Take the bounded orbit Ω_{R1} as an example. It can be represented by

$$y = \pm\sqrt{\frac{n}{3(2\alpha-m)}}\sqrt{\frac{(\phi_1^2-\phi^2)(\phi_2^2-\phi^2)(\phi^2+\phi_1^2+\phi_2^2-\frac{3A}{2n})}{\phi^2}},$$

where $0 < \phi < \phi_2$. Choosing the initial value $\phi(0) = \phi_2$, as shown in figure 4(c), we obtain the first type compacton solution of system (2.6)

$$\phi_{b_8} = \sqrt{\phi_1^2 + \frac{\phi_2^2 - \phi_1^2}{1 - k_7^2 sn^2(g_1\xi, k_7)}}, \quad -\xi_7 < \xi < \xi_7,$$

where $g_1 = \sqrt{\frac{n(2\phi_1^2+\phi_2^2-\frac{3A}{2n})}{3(2\alpha-m)}}$, $k_7^2 = \frac{2\phi_2^2+\phi_1^2-\frac{3A}{2n}}{2\phi_1^2+\phi_2^2-\frac{3A}{2n}}$ and $\xi_7 = \frac{1}{g_1} sn^{-1}\left(\frac{\phi_2}{k_7\phi_1}, k_7\right)$.



(a) (b) (c)

Figure 4. The bounded solutions of system (2.6) defined by ϕ_{b_7} and ϕ_{b_8}

(II) For $\frac{A}{n} > 0$ and $\frac{n}{2\alpha-m} < 0$, we need to discuss three subcases in this case.
(1) Consider a class of bounded orbits in figure 3(b), whose energy is lower than 0 but higher than the energy of the two centers E_1 and E_3, such as orbits Π_{L4} and Π_{R4}. Take the orbit Π_{R4} as an example. It can be expressed by

$$y = \pm\sqrt{\frac{n}{3(m-2\alpha)}}\sqrt{\frac{(\phi_1^2 - \phi^2)(\phi^2 - \phi_2^2)(\phi^2 + \phi_1^2 + \phi_2^2 - \frac{3A}{2n})}{\phi^2}},$$

where $0 < \phi_2 < \phi < \phi_1$. Choosing the initial value $\phi(0) = \phi_2$ and assuming the periodic is $2T_5$, as shown in figure 5(a), we obtain the periodic solution of system (2.6)

$$\phi_{b_9} = \sqrt{\frac{3A}{2n} - \phi_1^2 - \phi_2^2 + \frac{2\phi_2^2 - \frac{3A}{2n} + \phi_1^2}{1 - k_8^2 sn^2(g_2\xi, k_8)}}, \quad -T_5 < \xi < T_5,$$

where $g_2 = \sqrt{\frac{n(2\phi_1^2+\phi_2^2-\frac{3A}{2n})}{3(m-2\alpha)}}$, $k_8^2 = \frac{\phi_1^2-\phi_2^2}{2\phi_1^2+\phi_2^2-\frac{3A}{2n}}$.
(2) Consider the homoclinic orbits Π_{L5} and Π_{R5} in figure 3(b), whose energy is equal to 0. Take the homoclinic orbit Π_{R5} as an example. It can be represented by

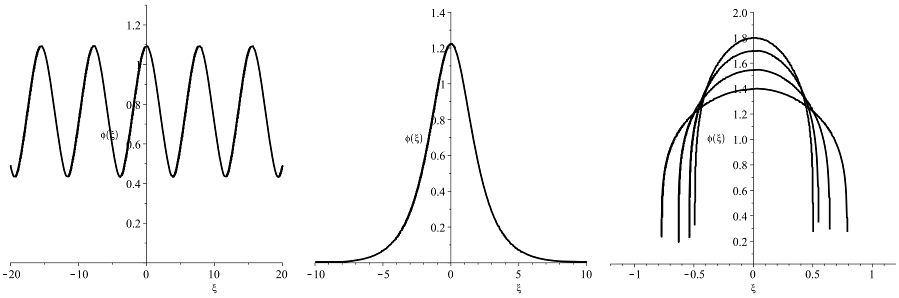
$$y = \pm\sqrt{\frac{n}{3(m-2\alpha)}}\sqrt{\phi^2\left(\frac{3A}{2n} - \phi^2\right)},$$

where $0 < \phi < \sqrt{\frac{3A}{2n}}$. Choosing the initial value $\phi(0) = \sqrt{\frac{3A}{2n}}$, as shown in figure 5(b), we obtain solitary wave solution of system (2.6)

$$\phi_{b_{10}} = \sqrt{\frac{3A}{2n}}\frac{1}{\cosh\left(\sqrt{\frac{A}{2(m-2\alpha)}}\xi\right)}, -\infty < \xi < +\infty.$$

(3) Consider a class of orbits in figure 3(b), whose energy is higher than 0, such as orbits Π_{R6} and Π_{L6}. Take the orbit Π_{R6} as an example. It can be represented by

$$y = \pm\sqrt{\frac{n}{3(m-2\alpha)}}\sqrt{\frac{(\phi_6^2 - \phi^2)[\phi^4 + (\phi_6^2 - \frac{3A}{2n})\phi^2 + \phi_6^4 - \frac{3A}{2n}\phi_6^2]}{\phi^2}},$$



(a) (b) (c)

Figure 5. The bounded solutions of system (2.6) defined by ϕ_{b_9}, $\phi_{b_{10}}$ and $\phi_{b_{11}}$

where $0 < \phi < \phi_6$. Choosing the initial value $\phi(0) = \phi_6$, as shown in figure 5(c), we obtain the second type of compacton solution of system (2.6)

$$\phi_{b_{11}} = \sqrt{\phi_6^2 + B_3 - \frac{2B_3}{1 + cn(g_3\xi, k_9)}}, \quad \xi \in (-\xi_8, \xi_8),$$

where $B_3 = \left(-\frac{\phi_6^2}{2} + \frac{3A}{4n} - \phi_6\right)^2 - \frac{(\phi_6^2 - \frac{3A}{2n})^2}{4} - \phi_6^4 + \frac{3A\phi_6^2}{2n}$, $g_3 = 2\sqrt{\frac{nB_3}{3(m-2\alpha)}}$, $k_9^2 = \frac{4B_3 - 3A + 2\phi_6^2 + \phi_6}{8B_3}$ and $\xi_8 = \frac{1}{g_3} cn^{-1}\left(\frac{B_3 - \phi_6}{B_3 + \phi_6}, k_9\right)$.

(III) For $A = 0$ and $\frac{n}{2\alpha - m} < 0$ in figure 3(d), there is one type of bounded orbits, whose energy is higher than 0. Any one of them can be represented by

$$y = \pm\sqrt{\frac{n}{3(m-2\alpha)}}\sqrt{\frac{(\phi_7^2 - \phi^2)(\phi^4 + \phi_7^2\phi^2 + \phi_7^4)}{\phi^2}},$$

where $0 < \phi < \phi_7$. Choosing the initial value $\phi(0) = \phi_7$, we obtain the compacton solution of system (2.6)

$$\phi_{b_{12}} = \phi_7\sqrt{1 + \phi_7 - \frac{2\phi_7}{1 + cn(g_4\xi, k_{10})}}, \quad \xi \in (-\xi_9, \xi_9),$$

where $g_4 = 2\phi_7\sqrt{\frac{n\phi_7}{3(m-2\alpha)}}$, $k_{10}^2 = \frac{4\phi_7^2 + \phi_7 + 1}{8\phi_7^2}$ and $\xi_9 = \frac{1}{g_4} cn^{-1}\left(1 - \frac{2}{\phi_7^2 + 1}, k_{10}\right)$.

(IV) For $\frac{A}{n} < 0$ and $\frac{n}{2\alpha - m} < 0$ in figure 3(f), there is only one type of bounded orbits, whose energy is higher than 0. The corresponding solution for each orbit has the same expression as $\phi_{b_{12}}$.

4. Conclusion

In this paper, we apply the dynamical system method to investigate the traveling wave solutions of the time-space fractional complex Ginzburg-Landau equation with Kerr law nonlinearity. This method allows detailed analysis of the phase space geometry of the traveling wave system (2.4) and the singular traveling wave system (2.6) to clearly observe various orbits which just correspond to different types of traveling wave solutions. Although the singular traveling wave system brought us some difficulties, for example, tracking orbits near the full degenerate equilibrium, we succeed overcoming it by the method of generalized normal sectors. Through a lot of complicated calculations, we give many types of traveling wave solutions of the time-space fcGL equation. The main contributions of our results are threefold.

1. A total of twenty explicit exact traveling wave solutions of the time-space fcGL equation are obtained in this paper. According to the forms of them, we further classify them into 11 categories as shown in table 1.

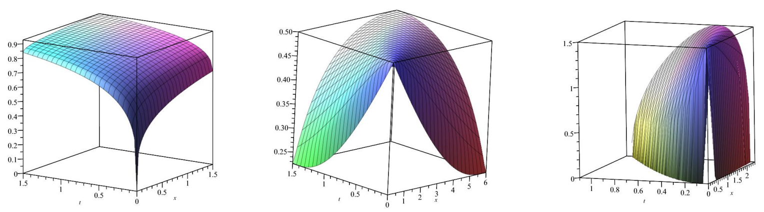
2. We obtain some new solutions $\phi_{b_i}(\xi), (i = 7, ..., 12)$ and $\phi_{u_j}(\xi), (j = 1, ..., 8)$, which are not reported in previous work and will be helpful in understanding the complicated wave phenomena described by equation (1.1). To observe them intuitively, we give the simulations of solutions ϕ_{b_7} and compactons ϕ_{b_8} and $\phi_{b_{11}}$ in figure 6.

3. By the method of dynamical system, we can clearly observe the evolution of traveling waves. For simplicity, we give an example to illustrate it. When $\alpha = 2\beta, \frac{A}{n} > 0$ and $\frac{n}{2\alpha - m} < 0$, as shown in figure 1(b), there exist periodic orbits and homoclinic orbits. Take the initial value $\phi(0) = \phi_3$. When $\sqrt{\frac{A}{n}} < \phi_3 < \sqrt{\frac{2A}{n}}$, system (2.4) has a periodic orbit which corresponds to a periodic traveling wave of the time-space fcGL equation. As ϕ_3 increases from $\sqrt{\frac{A}{n}}$ to $\sqrt{\frac{2A}{n}}$, the amplitude of the periodic wave will gradually increase. When $\phi_3 = \sqrt{\frac{2A}{n}}$, the periodic orbit breaks into the homoclinic orbit which implies the disappearance of periodic wave and the appearance of the solitary wave of the time-space fcGL equation. When $\phi_3 > \sqrt{\frac{2A}{n}}$, periodic wave appears again with the disappearance of the solitary wave. Then, the periodic wave's amplitude will become larger and larger as the initial value ϕ_3 increases.

Table 1. Classification of expression ϕ

Type	Category of solutions	The corresponding expression ϕ
1	$W_1\left(\frac{2}{1\pm\exp(W_2\xi)} - 1\right)$	ϕ_{b_2}, ϕ_{u_2}
2	$\frac{W_1}{\cosh(W_2\xi)}$	$\phi_{b_4}, \phi_{b_{10}}$
3	$W_1 cn(W_2\xi)$	ϕ_{b_5}, ϕ_{b_6}
4	$W_1\sqrt{W_2 \pm \frac{W_3}{1\pm cn(W_4\xi)}}$	$\phi_{u_1}, \phi_{u_6}, \phi_{b_{11}}, \phi_{b_{12}}$
5	$W_1\sqrt{W_2 + \frac{W_3}{W_4+W_5 sn^2(W_6\xi)}}$	$\phi_{b_8}, \phi_{b_9}, \phi_{u_5}, \phi_{u_8}$
6	$W_1 + \frac{W_2}{W_3-W_4 sn^2(W_5\xi)}$	ϕ_{b_1}
7	$\sqrt{W_1 + W_2 sn^2(W_3\xi)}$	ϕ_{b_3}
8	$\sqrt{W_1\left(1 - \frac{2}{\exp(W_2+W_3\xi)}\right)^2 - W_4}$	ϕ_{b_7}
9	$\frac{W_1}{sn(W_2\|\xi\|)}$	ϕ_{u_3}
10	$W_1 \csc(W_2\|\xi\|)$	ϕ_{u_4}
11	$\frac{W_1}{\xi}$	ϕ_{u_7}

[1] ($W_1, W_2, W_3, W_4, W_5, W_6$ are real parameters).



(a) $\phi_{b_7}(x, t)$ (b) $\phi_{b_8}(x, t)$ (c) $\phi_{b_{11}}(x, t)$

Figure 6. The bounded solutions $\phi_{b_i}(x, t)$ $(i = 7, 8, 11)$ of system (2.6) defined by ϕ_{b_i} and ξ

Remark 4.1. Note that the final solution of the time-space fcGL equation has the form $u(x,t) = \phi(\xi)\exp(i\psi(x,t))$, where $\psi(x,t) = \dfrac{-kx^\delta}{\Gamma(1+\delta)} + \dfrac{\omega t^\delta}{\Gamma(1+\delta)} + \theta$ is a given function as shown in expression (2.1). So, the traveling wave solution of the time-space fcGL equation we mentioned in this section refer specifically to the solution $\phi(\xi)$.

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